Modulus Iterative Methods for Nonnegative Constrained Least Squares Problems Arising from Image Restoration

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Least squares problem with nonnegative constraints

$$\min \|Ax-b\|_2 \quad \text{subject to} \quad x \ge 0$$

- A is a $m \times n$ matrix; may rank-deficient
- Applications: image restoration, reconstruction problem in geodesy and tomography, etc.
- Previous methods: gradient projection, interior point, etc.

How to Solve?

Least squares problem (quadratic programming)

$$\min_{x \in \mathrm{R}^n} \left(x^{\mathrm{T}}(A^{\mathrm{T}}A)x - 2(A^{\mathrm{T}}b)^{\mathrm{T}}x + b^{\mathrm{T}}b \right), \quad x \ge 0$$

Equivalent Kurush-Kuhn-Tucker conditions

$$A^{\mathrm{T}}Ax - A^{\mathrm{T}}b \ge 0, \quad x \ge 0, \quad x^{\mathrm{T}}(A^{\mathrm{T}}Ax - A^{\mathrm{T}}b) = 0$$

$x = z + |z|$

Implicit fixed-point equation

$$(\Omega + A^{\mathrm{T}}A)z = (\Omega - A^{\mathrm{T}}A)|z| + A^{\mathrm{T}}b$$

Ω is a positive diagonal matrix

$\Omega = \omega I$ or $\Omega = \omega \mathrm{diag}(A^{\mathrm{T}}A)$

A series of **unconstrained** least squares problems

$$\min_{z^{k+1} \in \mathrm{R}^n} \left\| \begin{bmatrix} A \\ \Omega^{1/2} \end{bmatrix} z^{k+1} - \begin{bmatrix} b - A|z^k| \\ \Omega^{1/2}|z^k| \end{bmatrix} \right\|_2$$

Normal equation

$$\begin{bmatrix} A \\ \Omega^{1/2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A \\ \Omega^{1/2} \end{bmatrix} z^{k+1} = \begin{bmatrix} A \\ \Omega^{1/2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} b - A|z^k| \\ \Omega^{1/2}|z^k| \end{bmatrix}$$

Using conjugate gradient (CG) method

$$(\Omega + A^{\mathrm{T}}A)z^{k+1} = (\Omega - A^{\mathrm{T}}A)|z^k| + A^{\mathrm{T}}b$$

Application: Image Restoration

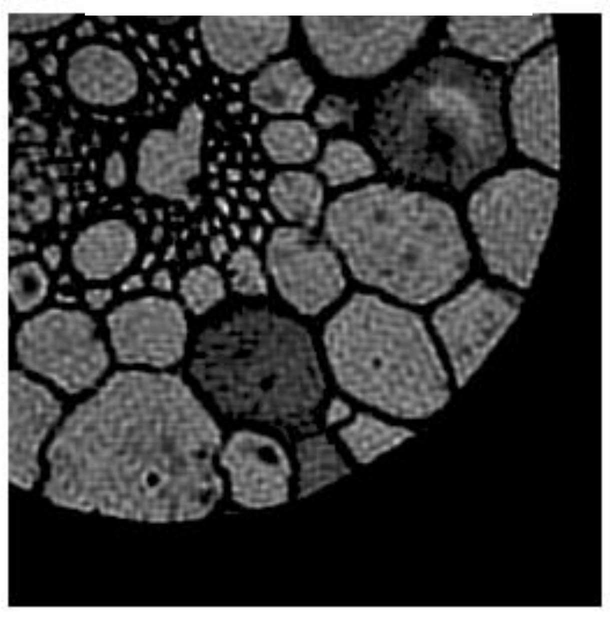
Blurring operator × Exact image + Noise = Observed image

$$Ax^* + e = b$$

Nonnegative constrained least squares problem with **Tikhonov regularization**

$$\min \|Ax-b\|_2^2 + \alpha\|x\|_2^2 = \left\| \begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|_2^2 \quad \text{subject to} \quad x \ge 0$$

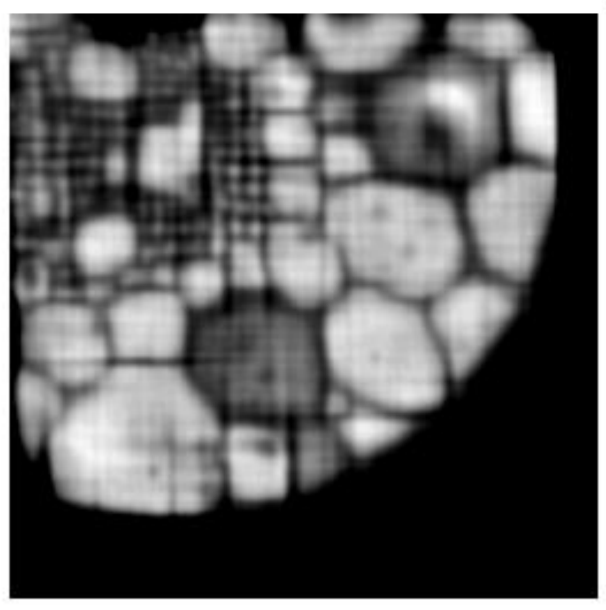
Exact Image



Blur Image

Blur and Noisy Image

Restored Image



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Algorithms

Modulus-type inner outer iteration method

1. Choose an initial approximate solution z^0 and Ω;
2. Compute $x^0 = z^0 + |z^0|$ and $r^0 = b - Ax^0$;
3. Set
$$\tilde{A} = \begin{bmatrix} A \\ \Omega^{1/2} \end{bmatrix} \text{ and } \tilde{r}^0 = \tilde{b}^0 - \tilde{A}z^0 = \begin{bmatrix} r^0 \\ \Omega^{1/2}(|z^0| - z^0) \end{bmatrix}$$
4. For $k = 0, 1, 2, \ldots$ until convergence
5. Compute an approximate solution ω^k by solving
$$\min_{\omega \in \mathbb{R}^n} \|\tilde{A}\omega - \tilde{r}^k\|_2$$
6. Compute $z^{k+1} = z^k + \omega^k$ and $x^{k+1} = z^{k+1} + |z^{k+1}|$
7. Compute $r^{k+1} = b - Ax^{k+1}$
8. Set
$$\tilde{r}^{k+1} = \begin{bmatrix} r^{k+1} \\ \Omega^{1/2}(|z^{k+1}| - z^{k+1}) \end{bmatrix}$$
9. End

Two-stage hybrid modulus method

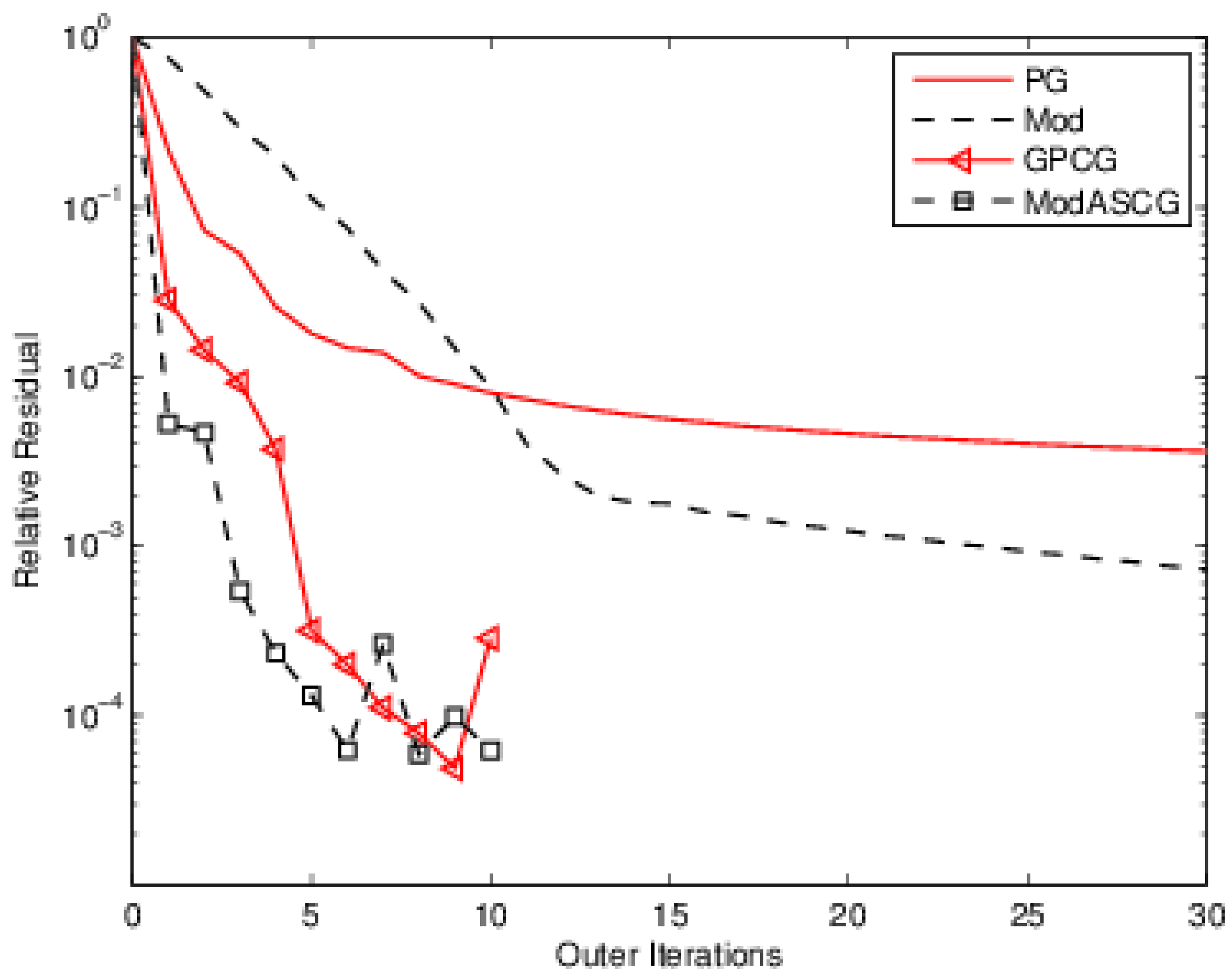
1. Choose an initial approximate solution x^0 and $r^0 = b - Ax^0$;
2. For $k = 0, 1, 2, \ldots$ until convergence
3. **First stage**: choose $y^0 = x^k$ and generate $\{y^j\}_{j=0}^{\infty}$ by modulus inner outer iterations.
4. Set $x^k = y^j$ and compute $r^k = b - Ax^k$
5. **Second stage:** update active set $\mathbf{A}(x^k)$ and free variable set $\mathbf{F}(x^k)$ and ten solve the reduced subproblem
$$\min l_k(w) = \|A_{\mathbf{F}} w - r^k\|_2$$
6. Compute $x^{k+1} = P(x^k + \beta^m Z_k w^{k+1})$ with
$$\|b - Ax^{k+1}\|_2^2 \le \|b - Ax^k\|_2^2 - 2\mu(s^k)^T(x^{k+1} - x^k)$$
7. If $\mathbf{B}(x^{k+1}) = \mathbf{A}(x^{k+1})$, set $x^k = x^{k+1}$ and resume the **Second stage**; otherwise go to the **First stage**.
8. End

Numerical Experiments

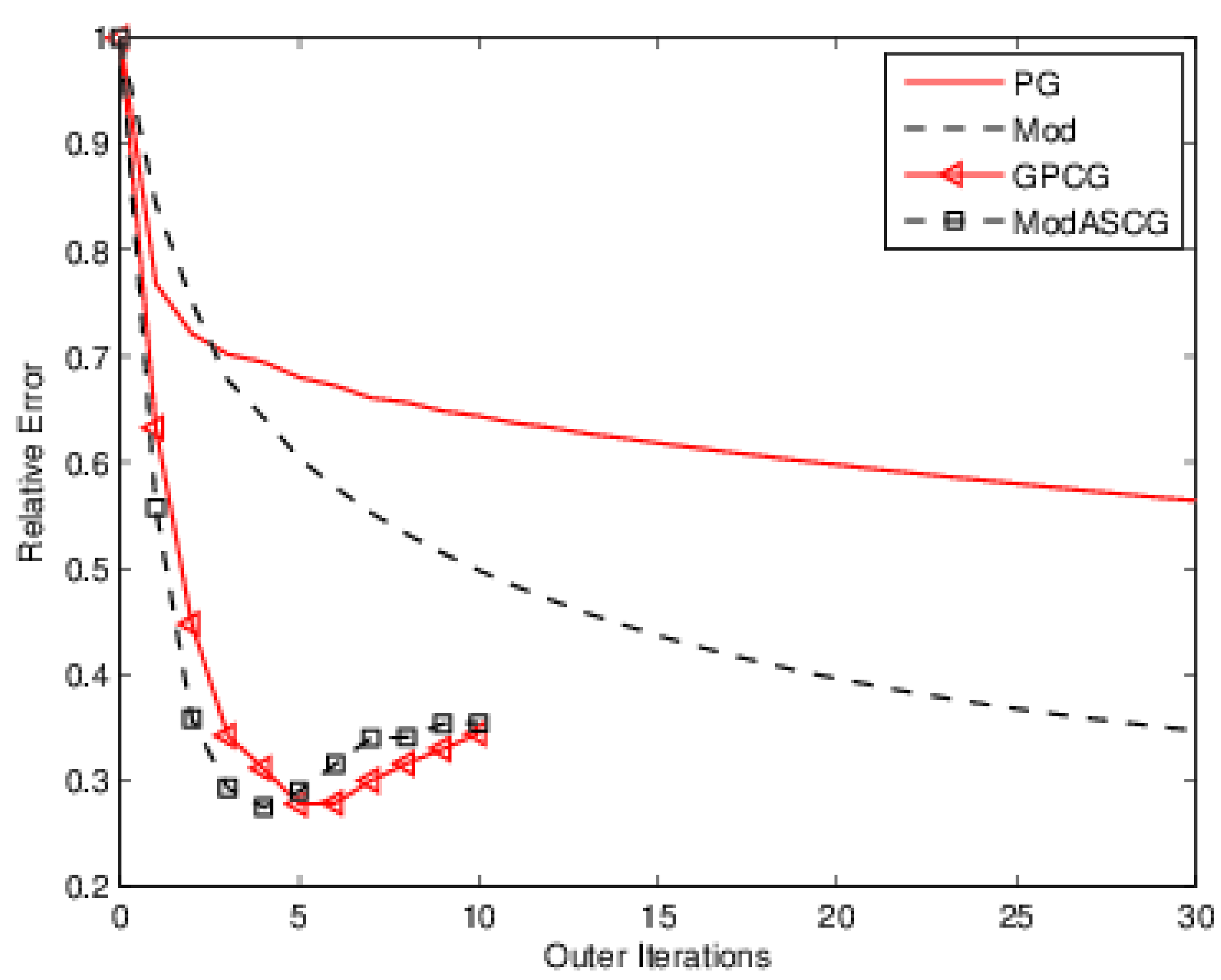
$$\text{Residual} = \frac{\|\min(x^k, -s^k)\|_2}{\|\min(x^0, -s^0)\|_2} \qquad s^k = A^T(b - Ax^k)$$

$$\text{Error} = \frac{\|x^k - x^*\|_2}{\|x^*\|_2} \qquad \text{Noise level } \gamma = \frac{\|e\|_2}{\|Ax^*\|_2} = 10\%$$

Relative residual vs. outer iterations for test problem ``AtmosphericBlur"



Relative error vs. outer iterations for test problem `` AtmosphericBlur"



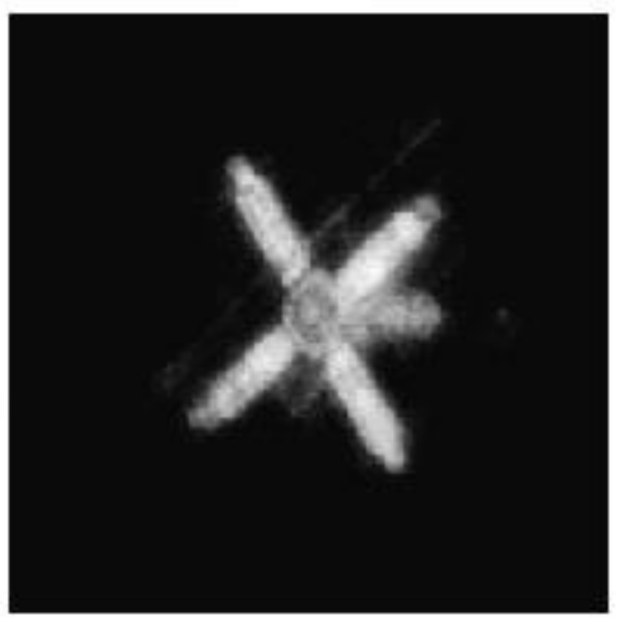
PG: projection gradient method;

Mod: modulus inner outer iteration method;

GPCG: two-stage hybrid PG method;

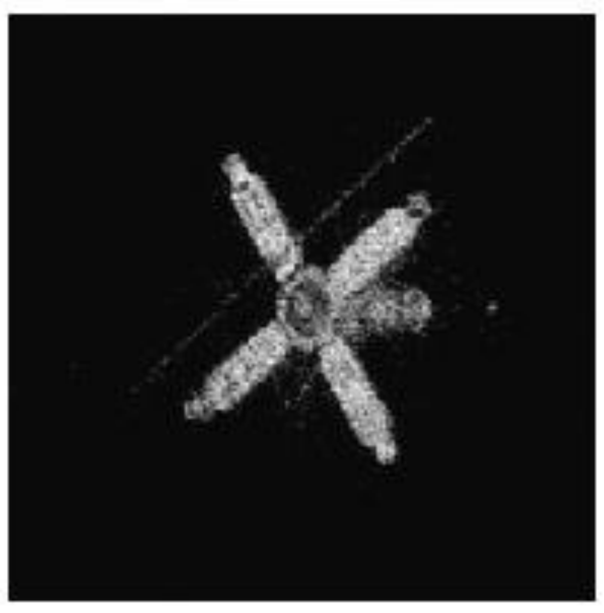
ModASCG: two-stage hybrid Mod method

PG



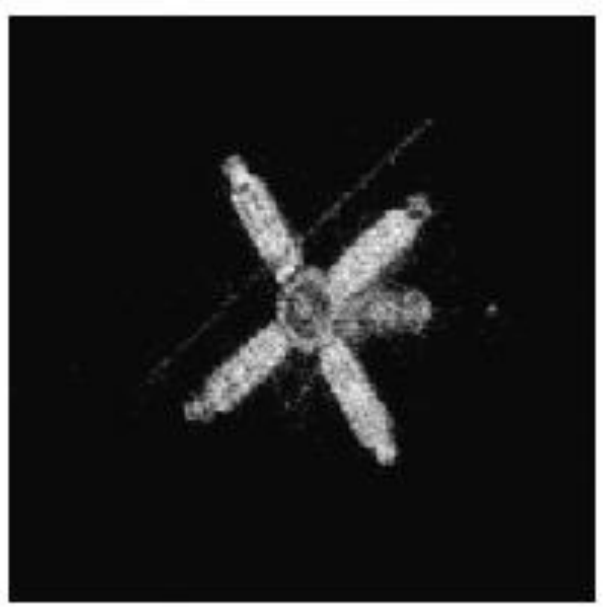
Error= 0.3814

Mod



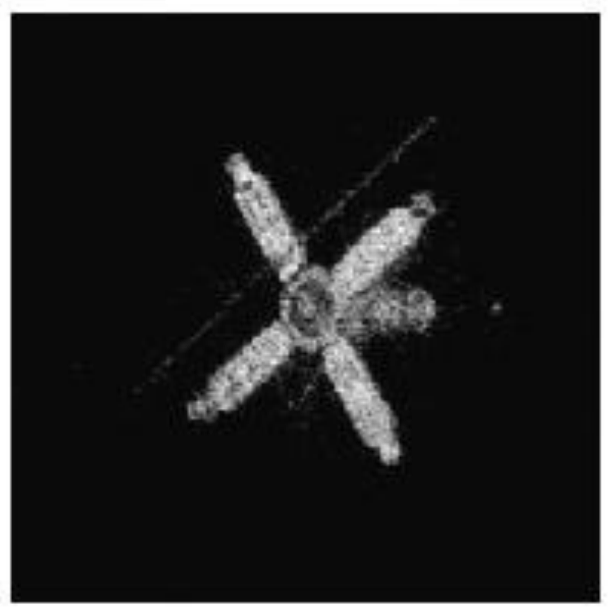
Error= 0.2783

GPCG

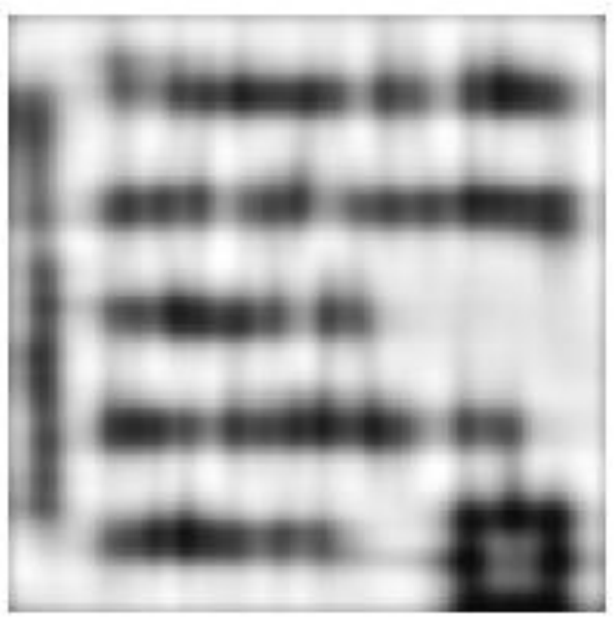


Error= 0.2775

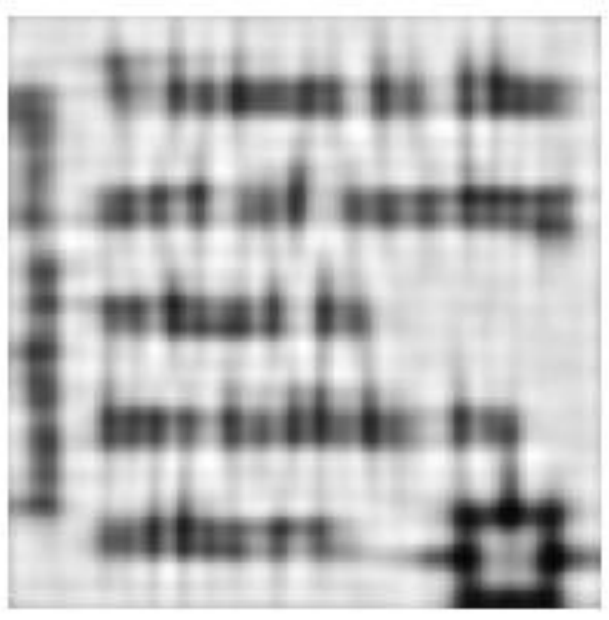
ModASCG



Error= 0.2747



Error= 0.3198



Error= 0.2945

Error= 0.2763

Error= 0.2762

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