Introduction to Scheduling Theory

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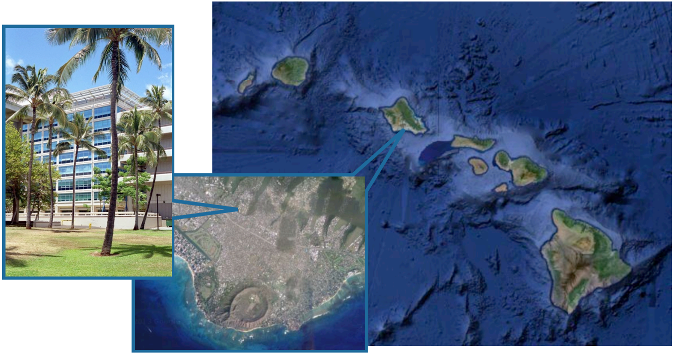
Presentation and thanks



Thanks to NII for inviting me to teach this seminar series!

Some of my research topics in last 5 years

- Scheduling (in a broad sense)
  - Divisible Load Scheduling
  - Scheduling checkpoints for fault-tolerance
  - Resource allocation in virtualized environments
  - Scheduling mixed parallel applications
  - Scheduling applications on volatile resources
  - Scheduling for energy savings
  - ...
- Simulation of distributed systems
  - Simulation tools and methodologies (SIMGRID)
  - Models for network simulation
- Random Network Topologies (with NII researchers)

Seminar topics

- Scheduling
  - A long-studied theoretical subject with practical applications
  - Comes in (too) many flavors
    - We'll explore some of them in this seminar series
- Simulation of distributed platforms and applications
  - Necessary for research on scheduling and other topics
  - Unclear and disappointing state-of-the-art
  - The SIMGRID project

Seminar organization

- Introduction to Scheduling Theory
- Scheduling Case Study: Divisible Load Scheduling
- Scheduling Case Study: Scheduling Checkpoints
- Scheduling Case Study: Scheduling Sensor Data Retrieval
- Fast and Accurate Network Simulations
- Simulating Distributed Applications with SIMGRID

Disclaimer on organization

- There are many possible topics here, especially in the area of scheduling
  - e.g., I picked 3 particular case studies but I'll likely refer to other scheduling domains as well
- I may have too much material for some topics, in which case I'll skip part of it. But my slides will of course be available to all

What is scheduling?

- Scheduling is studied in Computer Science and Operations Research
- Broad definition: *the temporal allocation of activities to resources to achieve some desirable objective*
- Examples:
  - Assign workers to machines in an factory to increase productivity
  - Pick classrooms for classes at a university to maximize the number of free classrooms on Fridays
  - Assign users to a pay-per-hour telescope to maximize profit
  - **Assign computation to processors and communications to network links so as to minimize application execution time**

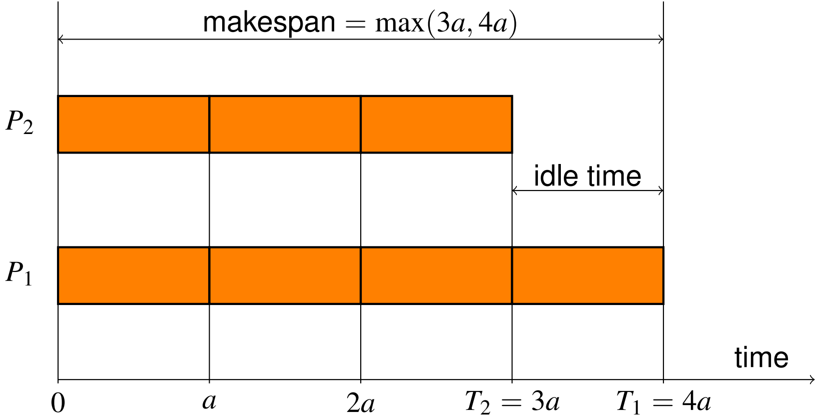
A simple scheduling problem

- A Scheduling Problem is defined by three components:
  1. A description of a set of resources
  2. A description of a set of tasks
  3. A description of a desired objective
- Let us get started with a simple problem: INDEP(2)
  1. Two identical processors, P_1 and P_2
     - Each processor can run only one task at a time
  2. n compute tasks
     - Each task can run on either processor in a seconds
     - Tasks are *independent*: can be computed in any order
  3. objective: minimize $\max(T_1, T_2)$
     - T_i is the time at which processor P_i finishes computing

The easy case

- If all tasks are *identical*, i.e., take the same amount of compute time, then the solution is obvious: Assign $\lceil n/2 \rceil$ tasks to $P1$ and $\lfloor n/2 \rfloor$ tasks to P_2
  - Rule of thumb: try to have both processors finish at the same time
- The problem size is $O(1)$, the “scheduling algorithm” is $O(1)$, therefore we have a polynomial time (in fact linear) algorithm
  - For each task pick one of the two processors by comparing the index of the task with $n/2$
- We declare the problem “solved”

Gantt chart for INDEP(2) with 5 identical tasks



Non-identical tasks

- Task T_i, $i = 1, \dots, n$ takes time $a_i \geq 0$
- We say a problem is "easy" when we have a polynomial-time (p-time) algorithm:
  - Number of elementary operations is $O(f(n))$, where f is a polynomial and n is the problem size
- $\mathcal{P}$ is the set of problems that can be solved with a p-time algorithm
- Question: is there a p-time algorithms to solve INDEP(2)?
- Disclaimer: Some of you may be familiar with algorithms and computational complexity, so bear with me while I review some fundamental background

Decision vs. optimization problem

- Complexity theory is for *decision problems*, i.e., problems that have a yes/no answer
- Scheduling problems are optimization problems
- Decision version of INDEP(2): for an integer k is there a schedule whose makespan is lower than k
- If we have a p-time algorithm for the optimization problem, then we have p-time algorithm for the decision problem
  - Run the optimization algorithm, and check whether the makespan is lower than k

Decision vs. optimization problem

- If the decision problem is in $\mathcal{P}$, then there is often (not always!) a p-time algorithm to solve the optimization problem
  - Binary search for the lowest k ($k \leq n \times \max_i a_i$)
  - Adds a $\log(n \times \max_i a_i)$ complexity factor, still p-time if the a_i's are bounded (reasonable assumption)
- Almost always the case in scheduling, and decision and optimization problems are often thought of as interchangeable

Problem size?

- One has to be careful when defining the problem size
- For INDEP(2):
  - We need to enumerate n integers (the a_i's), so the size is at least polynomial in n
  - Each a_i must be encoded (in binary) in $\lceil \log(a_i) \rceil$ bits
  - The data is $O(f(n) + \sum_{i=1}^{n} \lceil \log(a_i) \rceil)$, where f is a polynomial
- A problem is in P only if an algorithm exist that is polynomial in the data size as defined above

Pseudo-polynomial algorithm

- It is often possible to find algorithms polynomial in a quantity that is exponential in the (real) problem size
- For instance, to solve INDEP(2), one can resort to dynamic programming to obtain and algorithm with complexity $O(n \times \sum_{i=1}^{n} a_i)$
- This is a polynomial algorithm if the a_i are encoded in unary, i.e., polynomial in the numerical value of the a_i's
- But with the a_i encoded in binary, $\sum_{i=1}^{n} a_i$ is exponential in the problem size!
  - To a $\log$, linear is exponential ☺
- We say that this algorithm is *pseudopolynomial*

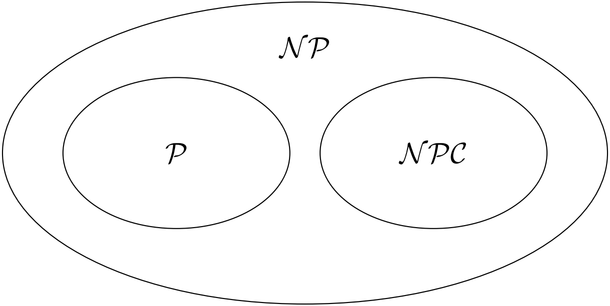
So, is INDEP(2) difficult?

- Nobody knows a p-time algorithm for solving INDEP(2)
- We define a new complexity class, $\mathcal{NP}$
  - Problems for which we can verify a *certificate* in p-time.
  - “Given a possible solution, can we check that the problem’s answer is Yes in p-time?”
- There are problems not in $\mathcal{NP}$, but not frequent
- Obviously $\mathcal{P} \subseteq \mathcal{NP}$
  - empty certificate, just solve the problem
- Big question: is $\mathcal{P} \neq \mathcal{NP}$?
  - Most people believe so, but we have no proof
  - For all the follows, "unless $\mathcal{P} = \mathcal{NP}$" is implied

$\mathcal{NP}$-complete problems

- Some problems in $\mathcal{NP}$ are at least as difficult as all other problems in $\mathcal{NP}$
- They are called $\mathcal{NP}$-complete, and their set is $\mathcal{NPC}$
- Cook's theorem: The SAT problems is in $\mathcal{NPC}$
  - Satisfiability of a boolean conjunction of disjunctions
- How to prove that a problem, P, is $\mathcal{NP}$-complete:
  - Prove that $P \in \mathcal{NP}$ (typically easy)
  - Prove that P *reduces* to Q, where $Q \in \mathcal{NPC}$ (can be hard)
    - For an instance I_Q construct in p-time an instance I_P
    - Prove that I_P has a solution if and only if I_Q has a solution
- By now we know many problems in $\mathcal{NPC}$
- Goal: pick $Q \in \mathcal{NPC}$ so that the reduction is easy

Well-known complexity classes



INDEP(2) is $\mathcal{NP}$-complete

- INDEP(2) (decision version) is in $\mathcal{NP}$
  - Certificate: for each a_i whether it is schedule on P_1 or P_2
  - In linear time, compute the makespan on both processors, and compare to k to answer "Yes"
- Let us consider an instance of 2-PARTITION $\in \mathcal{NPC}$:
  - Given n integers x_i, is there a subset I of $\{1, \ldots, n\}$ such that $\sum_{i \in I} x_i = \sum_{i \notin I} x_i$?
- Let us construct an instance of INDEP(2):
  - Let $k = \frac{1}{2} \sum x_i$, let $a_i = x_i$
- The proof is trivial
  - If k is non-integer, neither instance has a solution
  - Otherwise, each processor corresponds to one subset
- In fact, INDEP(2) is essentially identical to 2-PARTITION

So what?

- This $\mathcal{NP}$-completeness proof is probably the most trivial in the world ☺
- But now we are thus pretty sure that there is no p-time algorithm to solve INDEP(2)
- What we look for now are *approximation algorithms*...

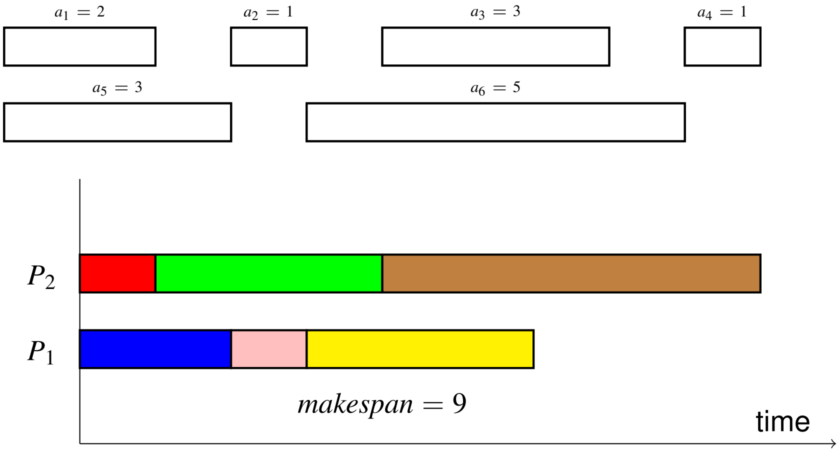
Approximation algorithms

- Consider an optimization problem
- A p-time algorithm is a λ-*approximation algorithm* if it returns a solution that's at most a factor λ from the optimal solution (the closer λ to 1, the better)
  - λ is called the *approximation ratio*
- *Polynomial Time Approximation Scheme* (PTAS): for any ϵ there exists a $(1+\epsilon)$-approximation algorithm (may be non-polynomial is $1/\epsilon$)
- *Fully Polynomial Time Approximation Scheme* (FPTAS): for any ϵ there exists a $(1+\epsilon)$-approximation algorithm polynomial in $1/\epsilon$
- Typical goal: find a FPTAS, if not find a PTAS, if not find a λ-approximation for a low value of λ

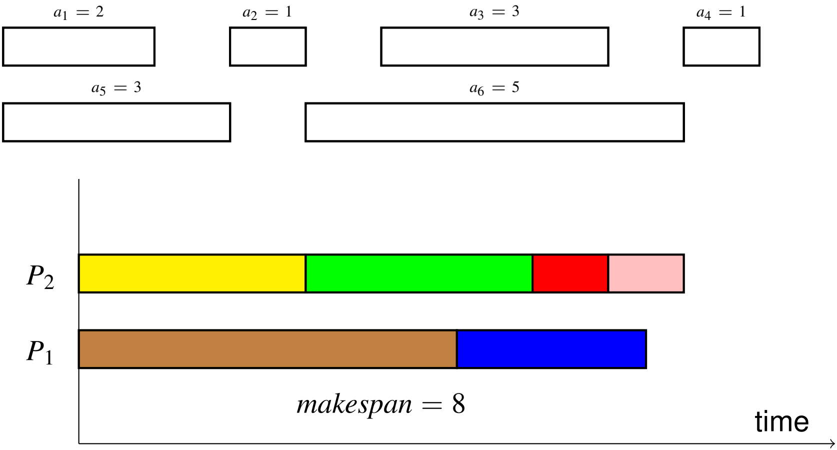
Greedy algorithms

- A greedy algorithm is one that builds a solution step-by-step, via local incremental decisions
- It turns out that several greedy scheduling algorithms are approximation algorithms
  - Informally, they're not as "bad" as one may think
- Two natural greedy algorithms for INDEP(2):
  - **greedy-online**: take the tasks in arbitrary order and assign each task to the least loaded processor
    - We don't know which tasks are coming
  - **greedy-offline**: sort the tasks by decreasing a_i, and assign each task in that order to the least loaded processor
    - We know all the tasks ahead of time

Example with 6 tasks: Online



Example with 6 tasks: Offline



Greedy-online for INDEP(2)

Theorem

Greedy-online is a $\frac{3}{2}$-approximation

- Proof:
  - P_i finishes computing at time M_i (M stands for makespan)
  - Let us assume $M_1 \geq M_2$ ($M_{greedy} = M_1$)
  - Let T_j the last task to execute on P_1
  - Since the greedy algorithm put T_j on P_1, then $M_1 - a_j \leq M_2$
  - We have $M_1 + M_2 = \sum_i a_i = S$
  - $M_{greedy} = M_1 = \frac{1}{2}(M_1 + (M_1 - a_j) + a_j) \leq \frac{1}{2}(M_1 + M_2 + a_j) = \frac{1}{2}(S + a_j)$
  - but $M_{opt} \geq S/2$ (ideal lower bound on optimal)
  - and $M_{opt} \geq a_j$ (at least one task is executed)
  - Therefore: $M_{greedy} \leq \frac{1}{2}(2M_{opt} + M_{opt}) = \frac{3}{2}M_{opt}$ $\square$

Greedy-offline for INDEP(2)

Theorem

Greedy-offline is a $\frac{7}{6}$-approximation

- Proof:
  - If $a_j \leq \frac{1}{3}M_{opt}$, the previous proof can be used
    - $M_{greedy} \leq \frac{1}{2}(2M_{opt} + \frac{1}{3}M_{opt}) = \frac{7}{6}M_{opt}$
  - If $a_j > \frac{1}{3}M_{opt}$, then $j \leq 4$
    - if T_j was the 5th task, then, due to the task ordering, there would be 5 tasks with $a_i > \frac{1}{3}M_{opt}$
    - There would be at least 3 tasks on the same processor in the optimal schedule
    - Therefore $M_{opt} > 3 \times \frac{1}{3}M_{opt}$, a contradiction
  - One can check all possible scenarios for 4 tasks and show optimality □

Bounds are tight

- Greedy-online:
  - a_i's = $\{$1,1,2$\}$
  - $M_{greedy} = 3; M_{opt} = 2$
  - $ratio = \frac{3}{2}$
- Greedy-offline:
  - a_i's = $\{$3, 3, 2, 2, 2$\}$
  - $M_{greedy} = 7; M_{opt} = 6$
  - $ratio = \frac{7}{6}$

PTAS and FPTAS for INDEP(2)

Theorem

There is a PTAS ($(1+\epsilon)$-approximation) for INDEP(2)

- Proof Sketch:
  - Classify tasks as either "small" or "large"
    - Very common technique
  - Replace all small tasks by same-size tasks
  - Compute an optimal schedule of the modified problem in p-time (not polynomial in $1/\epsilon$)
  - Show that the cost is $\leq 1+\epsilon$ away from the optimal cost
  - The proof is a couple of pages, but not terribly difficult

Theorem

There is a FPTAS ($(1+\epsilon)$-approx pol. in $1/\epsilon$) for INDEP(2)

We know a lot about INDEP(2)

- INDEP(2) is NP-complete
- We have simple greedy algorithms with guarantees on result quality
- We have a simple PTAS
- We even have a (less simple) FPTAS
- INDEP(2) is basically "solved"

- Sadly, not many scheduling problems are this well-understood...

INDEP(P) is much harder

- INDEP(P) is $\mathcal{NP}$-complete by trivial reduction to 3-PARTITION:
  - Give $3n$ integers $a_1, \ldots, a_{3n}$ and an integer B, can we partition the $3n$ integers into n sets, each of sum B? (assuming that $\sum_i a_i = nB$)
- 3-PARTITION is $\mathcal{NP}$-complete "in the strong sense", unlike 2-PARTITION
  - Even when encoding the input in unary (i.e., no logarithmic numbers of bits), one cannot find and algorithm polynomial in the size of the input!
  - Informally, a problem is $\mathcal{NP}$-complete "in the weak sense" if it is hard only if the numbers in the input are unbounded
- INDEP(P) is thus fundamentally harder than INDEP(2)

Approximation algorithm for INDEP(P)

Theorem

Greedy-online is a $(2 - \frac{1}{p})$-approximation

- Proof (usual reasoning):
  - Let $M_{greedy} = \max_{1 \leq i \leq p} M_i$, and j be such that $M_j = M_{greedy}$
  - Let T_k be the last task assigned to processor P_j
  - $\forall i, \quad M_i \geq M_j - a_k$ (greedy algorithm)
  - $S = \sum_i^p M_i = M_j + \sum_{i \neq j} M_i \geq M_j + (p-1)(M_j - a_k) = pM_j + (p-1)a_k$
  - Therefore, $M_{greedy} = M_j \leq \frac{S}{p} + (1 - \frac{1}{p})a_k$
  - But $M_{opt} \geq a_k$ and $M_{opt} \geq S/p$
  - So $M_{greedy} \leq M_{opt} + (1 - \frac{1}{p}M_{opt}) \quad \square$
- This ratio is “tight” (e.g., an instance with $p(p-1)$ tasks of size 1 and one task of size p has this ratio)

Approximation algorithm for INDEP(P)

Theorem

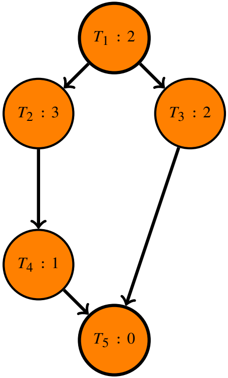
Greedy-offline is a $(\frac{4}{3} - \frac{1}{3p})$*-approximation*

- The proof is more involved, but follows the spirit of the proof for INDEP(2)
- This ratio is tight

- There is a PTAS for INDEP(P), a $(1+\epsilon)$-approximation (massively exponential in $1/\epsilon$)
- There is no known FPTAS, unlike for INDEP(2)

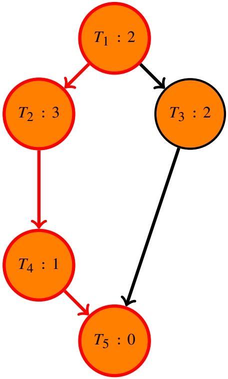
Task dependencies

- In practice tasks often have *dependencies*
- A general model of computation is the Acyclic Directed Graph (DAG), $G = (V, E)$
- Each task has a *weight* (i.e., execution time in seconds), a *parent*, and *children*
- The first task is the *source*, the last task the *sink*
- Topological (partial) order of the tasks



Critical path

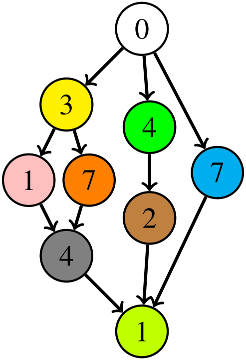
- Assume that the DAG executes on p processors
- The longest path (in seconds) is called the *critical path*
- The length of the critical path (CP) is a *lower bound on* M_{opt}, regardless of the number of processors
- In this example, the CP length is 6 (the other path has length 4)



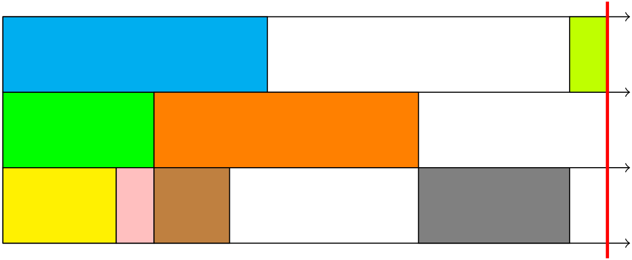
Complexity

- Unsurprisingly, DAG scheduling is $\mathcal{NP}$-complete
  - Independent tasks is a special case of DAG scheduling
- Typical greedy algorithm skeleton:
  - Maintain a list of *ready* tasks (with cleared dependencies)
  - Greedily assign a ready task to an available processor as early as possible (don't leave a processor idle unnecessarily)
  - Update the list of ready tasks
  - Repeat until all tasks have been scheduled
- This is called List Scheduling
- Many list scheduling algorithms are possible
  - Depending on how to select the ready task to schedule next

List scheduling example



3 Processors



Makespan = 16; CP Length = 15

Idle Time = 1+5+5+8 = 19

List scheduling

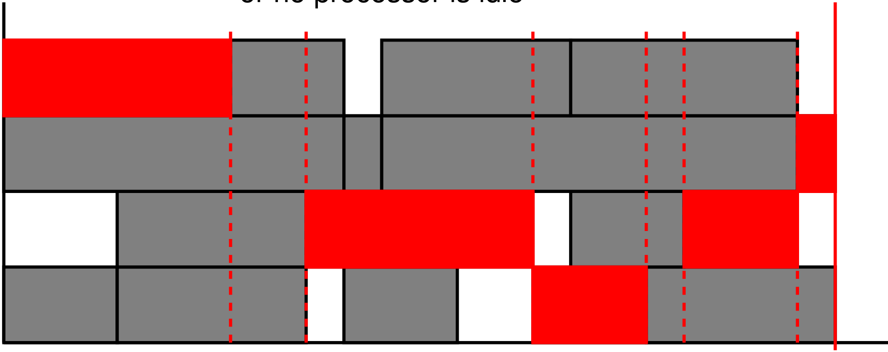
Theorem (fundamental)

List scheduling is a $(2 - \frac{1}{p})$*-approximation*

- Doesn’t matter how the next ready task is selected
- Let’s prove this theorem informally
  - Really simple proof if one doesn’t use the typical notations for schedules
  - I never use these notations in public ☺

Approximation ratio

At any point in time either a task on the red path is running
or no processor is idle



Approximation ratio

- Let L be the length of the red path (in seconds), p the number of processors, I the total idle time, M the makespan, and S the sum of all task weights
- $I \leq (p-1)L$
  - processors can be idle only when a red task is running
- $L \leq M_{opt}$
  - The optimal makespan is longer than any path in the DAG
- $M_{opt} \geq S/p$
  - S/p is the makespan with zero idle time
- $p \times M = I + S$
  - rectangle's area = white boxes + non-white boxes

$\Rightarrow$ $p \times M \leq (p-1)M_{opt} + pM_{opt} \;\; \Rightarrow \;\; M \leq (2 - \frac{1}{p})M_{opt} \quad \square$

Good list scheduling?

- All list scheduling algorithms thus have the same approximation ratio
- But there are many options for list scheduling
  - Many ways of sorting the ready tasks...
- In practice, some may be better than others
- One well-known option, *Critical path scheduling*

Critical path scheduling

- When given a set of ready tasks, which one do we pick to schedule?
- Idea: pick a task on the CP
  - If we prioritize tasks on the CP, then the CP length is reduced
  - The CP length is a lower bound on the makespan
  - So intuitively it's good for it to be low
- For each (ready) task, compute its *bottom level*, the length of the path from the task to the sink
- Pick the task with the *largest* bottom level

Graham's notation

- There are SO many variations on the scheduling problem that Graham has proposed a standard notation: $\alpha|\beta|\gamma$
  - $alpha$: processors
  - $beta$: tasks
  - $gamma$: objective function
- Let's see some examples for each

α: processors

- 1: one processor
- Pn: n identical processors (if n not fixed, not given)
- Qn: n uniform processors (if n not fixed, not given)
  - Each processor has a (different) compute speed
- Rn: n unrelated processors (if n not fixed, not given)
  - Each processor has a (different) compute speed for each (different) task (e.g., P_1 can be faster than P_2 for T_1, but slower for T_2)

β: tasks

- r_j: tasks have *release dates*
- d_j: tasks have *deadlines*
- $p_j = x$: all tasks have weight x
- $prec$: general precedence constraints (DAG)
- $tree$: tree precedence constraints
- $chains$: chains precedence constraints (multiple independent paths)
- $pmtn$: tasks can be preempted and restarted (on other processors)
  - Makes scheduling easier, and can often be done in practice
- ...

γ: objective function

- C_{max}: makespan
- $\sum C_i$: mean flow-time (completion time minus release date if any)
- $\sum w_i C_i$: average weighted flow-time
- L_{max}: maximum lateness ($\max(0, C_i - d_i)$)
- ...

Example scheduling problems

- The classification is not perfect and variations among authors are common
- Some examples:
  - $P2||C_{max}$, which we called INDEP(2)
  - $P||C_{max}$, which we called INDEP(P)
  - $P|prec|C_{max}$, which we called DAG scheduling
  - $R2|chains|\sum C_i$
    - Two related processors, chains, minimize sum-flow
  - $P|r_j; p_j \in \{1, 2\}; d_j; pmtn|L_{max}$
    - Identical processors, tasks with release dates and deadlines, task weights either 1 or 2, preemption, minimize maximum lateness

Where to find known results

- Luckily, the body of knowledge is well-documented (and Graham's notation widely used)
- Several books on scheduling that list known results
  - *Handbook of Scheduling*, Leung and Anderson
  - *Scheduling Algorithms*, Brucker
  - *Scheduling: Theory, Algorithms, and Systems*, Pinedo
  - ...
- Many published survey articles

Example list of known results

- Excerpt from *Scheduling Algorithm*, P. Brucker

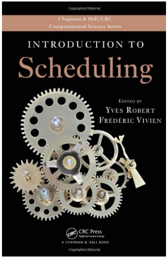
	$P2 \parallel C_{max}$	Lenstra et al. [155]
*	$P \parallel C_{max}$	Garey & Johnson [98]
*	$P \mid p_i = 1; intree; r_i \mid C_{max}$	Brucker et al. [35]
*	$P \mid p_i = 1; prec \mid C_{max}$	Ullman [203]
*	$P2 \mid chains \mid C_{max}$	Du et al. [86]
*	$Q \mid p_i = 1; chains \mid C_{max}$	Kubiak [129]
*	$P \mid p_i = 1; outtree \mid L_{max}$	Brucker et al. [35]
*	$P \mid p_i = 1; intree; r_i \mid \sum C_i$	Lenstra [150]
*	$P \mid p_i = 1; prec \mid \sum C_i$	Lenstra & Rinnooy Kan [152]
*	$P2 \mid chains \mid \sum C_i$	Du et al. [86]
*	$P2 \mid r_i \mid \sum C_i$	Single-machine problem
	$P2 \parallel \sum w_i C_i$	Bruno et al. [58]
*	$P \parallel \sum w_i C_i$	Lenstra [150]
*	$P2 \mid p_i = 1; chains \mid \sum w_i C_i$	Timkovsky [201]
*	$P2 \mid p_i = 1; chains \mid \sum U_i$	Single-machine problem
*	$P2 \mid p_i = 1; chains \mid \sum T_i$	Single-machine problem

Table 5.3: $\mathcal{NP}$-hard parallel machine problems without preemption.

Conclusion

- Scheduling problems are diverse and often difficult
- Relevant theoretical questions:
  - Is it in $\mathcal{P}$?
  - Is it $\mathcal{NP}$-complete?
    - Are there approximation algorithms?
    - Are there PTAS or FTPAS?
    - Are there are least decent non-guaranteed heuristics?
- Luckily, scheduling problems have been studied a lot
- Come up with the Graham notation for your problem and check what is known about it!

Sources and acknowledgments



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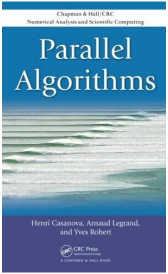

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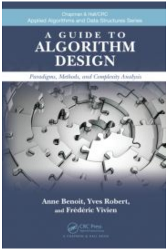

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