6. Quantum error correcting codes

Error correcting codes (A classical repetition code)

Preserving the superposition

- Parity check

Phase errors

- CSS 7-qubit code (Steane code)

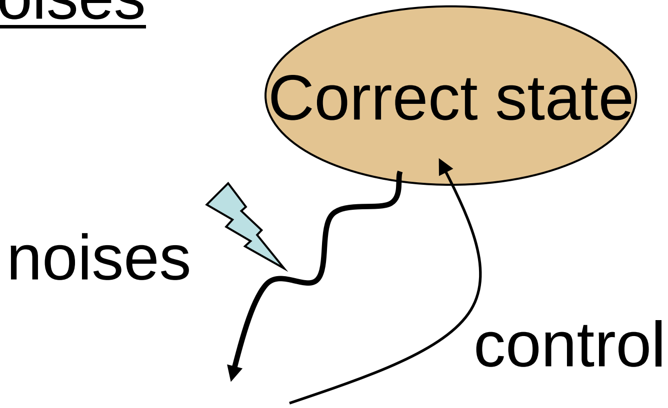
Too many error patterns?

- Syndrome measurement digitizes the error

Description of encoded states

Similarity to classical repetition codes

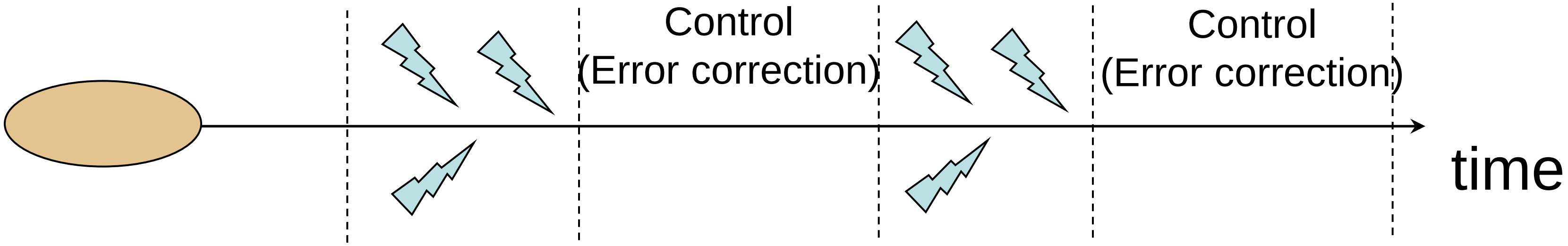
Fighting against noises


Correct state

noises

control

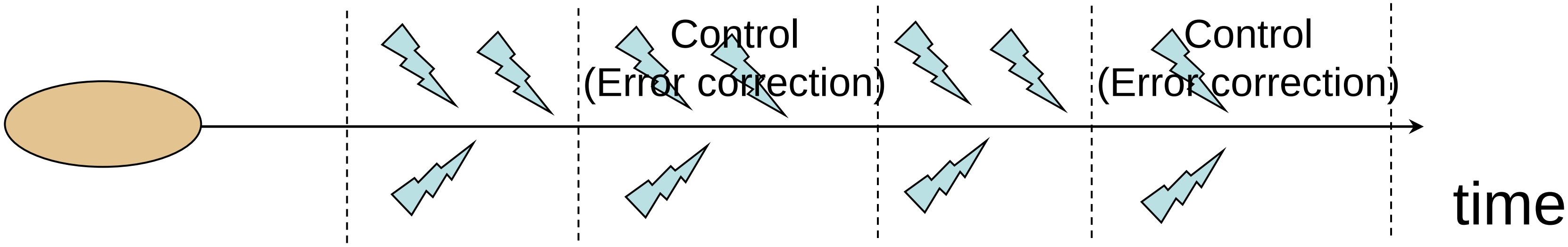
Error correcting codes


Control
(Error correction)

Control
(Error correction)

time

Fault-tolerant computation


Control
(Error correction)

Control
(Error correction)

time

Error correcting codes (Classical)

3-bit repetition code

bit value		codeword
0	→	000
1	→	111

Error model:

Bit error 0 ↔ 1

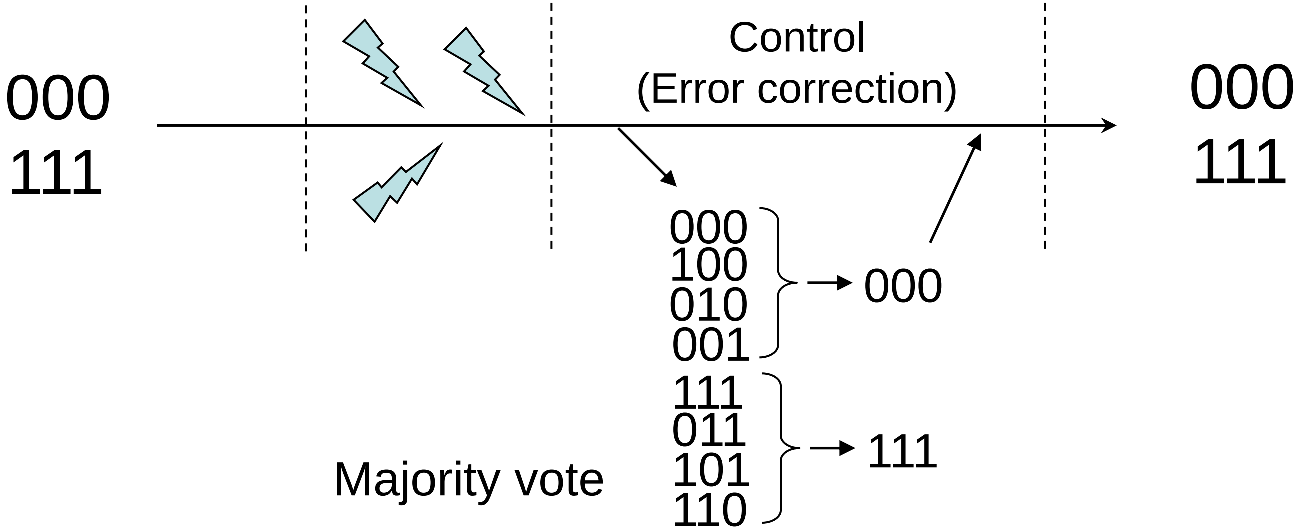
Error propability: ϵ

Independent for each bit

No error	1st bit	2nd bit	3rd bit
000	100	010	001
111	011	101	110

correctible

No flips	$(1-\epsilon)^3$
1 bit	$3\epsilon(1-\epsilon)^2$
2 bits	$3\epsilon^2(1-\epsilon)$
3 bits	ϵ^3



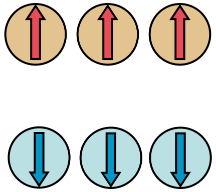
Error rate after the correction

$\sim 3\epsilon^2$

Error correcting codes (Quantum)?

$$|0\rangle \longrightarrow |000\rangle$$

$$|1\rangle \longrightarrow |111\rangle$$



"Bit error"

$$|0\rangle \xleftrightarrow[\hat{\sigma}_x]{} |1\rangle$$

Error in observable $\hat{\sigma}_z$

Error caused by unitary $\hat{\sigma}_x$

Problems:

- If we measure the system for the correction, the superposition may collapse.
- Can we correct the phase error?

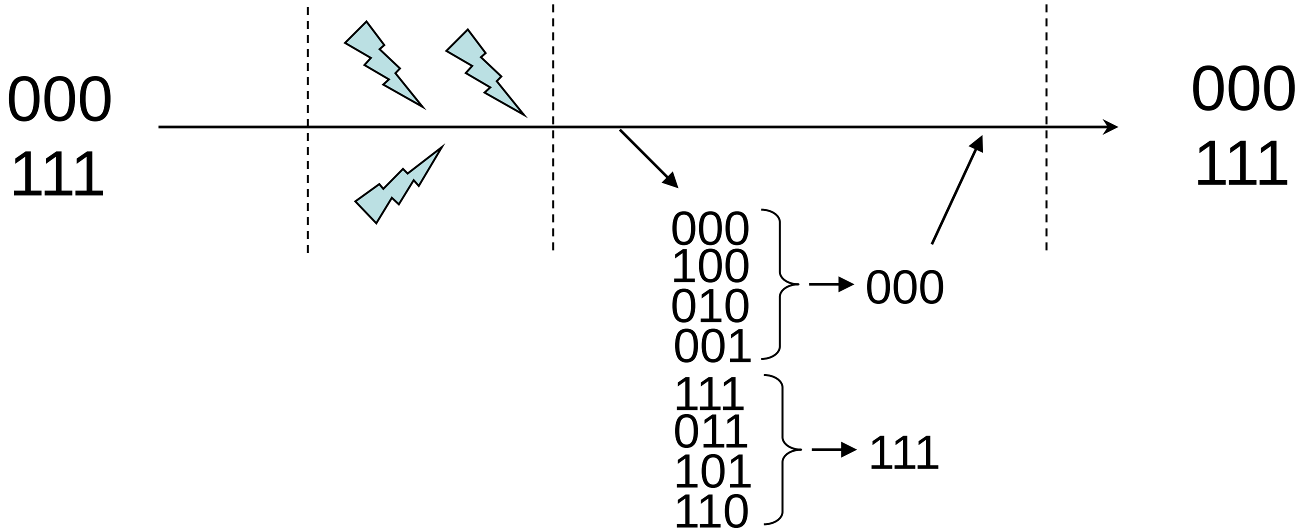
Error in observable $\hat{\sigma}_x$

Error caused by unitary $\hat{\sigma}_z$

- There are infinite number of error patterns. Can we handle all of them?

Does the majority vote work?

- If we measure the system for the correction, the superposition may collapse.



000
111

000
111

000
100
010
001
→ 000

111
011
101
110
→ 111

No error	1st bit	2nd bit	3rd bit
000	100	010	001
111	011	101	110

Distinguish here

States such as $|000\rangle + |111\rangle$ and $|000\rangle - |111\rangle$ will collapse.

(Classical mixture of state $|000\rangle$ and $|111\rangle$)

Parity check

Parity of a subset of bits

XOR

$$s_1 = b_1 \oplus b_2$$
$$s_2 = b_2 \oplus b_3$$

Parity check matrix

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Codewords: All the syndrome bits are zero. $(s_1 = s_2 = 0)$

	No error	1st bit	2nd bit	3rd bit	
	000	100	010	001	
	111	011	101	110	
$s_1 s_2$	00	10	11	01	(syndrome)

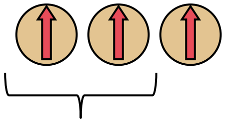
Distinguish the columns

Correction operation $\oplus 000$ $\oplus 100$ $\oplus 010$ $\oplus 001$

Measurement of a syndrome bit

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$s_1 \equiv b_1 \oplus b_2$$



$$s_1 = 0 : \quad |0\rangle_1|0\rangle_2 \qquad (\hat{\sigma}_z^{[1]} \otimes \hat{\sigma}_z^{[2]})|0\rangle_1|0\rangle_2 = (1 \times 1)|0\rangle_1|0\rangle_2$$

$$|1\rangle_1|1\rangle_2 \qquad (\hat{\sigma}_z^{[1]} \otimes \hat{\sigma}_z^{[2]})|1\rangle_1|1\rangle_2 = (-1 \times -1)|1\rangle_1|1\rangle_2$$

Eigenspace of $\hat{\sigma}_z^{[1]}\hat{\sigma}_z^{[2]}$ with eigenvalue 1

$$s_1 = 1 : \quad |0\rangle_1|1\rangle_2 \qquad (\hat{\sigma}_z^{[1]} \otimes \hat{\sigma}_z^{[2]})|0\rangle_1|1\rangle_2 = (1 \times -1)|0\rangle_1|1\rangle_2$$

$$|1\rangle_1|0\rangle_2 \qquad (\hat{\sigma}_z^{[1]} \otimes \hat{\sigma}_z^{[2]})|1\rangle_1|0\rangle_2 = (-1 \times 1)|1\rangle_1|0\rangle_2$$

Eigenspace of $\hat{\sigma}_z^{[1]}\hat{\sigma}_z^{[2]}$ with eigenvalue -1

Measurement of $s_1 \equiv b_1 \oplus b_2$

= Measurement of observable $\hat{\sigma}_z^{[1]}\hat{\sigma}_z^{[2]}$

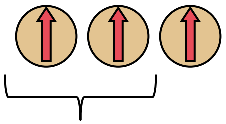
Codeword state: $s_1 = 0$

It should be in the eigenspace of $\hat{\sigma}_z^{[1]}\hat{\sigma}_z^{[2]} = 1$

Measurement of a syndrome bit

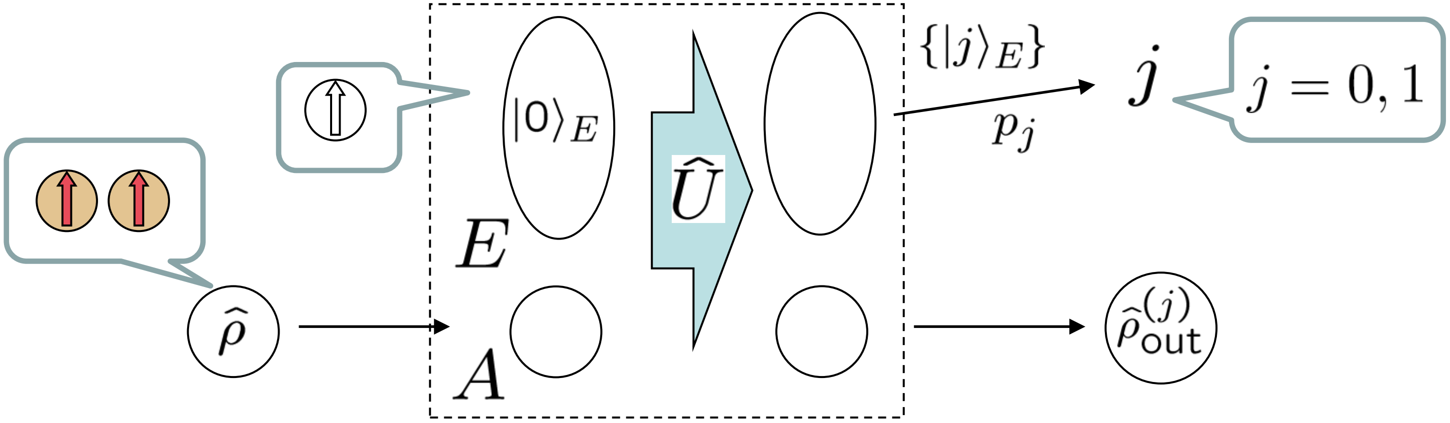
We want to learn s_1 , but not the value of each bit b_1, b_2

$$s_1 \equiv b_1 \oplus b_2$$



$$s_1 = 0 : \quad |0\rangle_1 |0\rangle_2$$
$$|1\rangle_1 |1\rangle_2$$

$$s_1 = 1 : \quad |0\rangle_1 |1\rangle_2$$
$$|1\rangle_1 |0\rangle_2$$



$$p_j \widehat{\rho}^{(j)}_{\mathrm{out}} = {}_E\langle j|\widehat{U}(\widehat{\rho} \otimes |0\rangle_{EE}\langle 0|)\widehat{U}^\dagger |j\rangle_E$$
$$= \widehat{M}^{(j)} \widehat{\rho} \widehat{M}^{(j)\dagger}$$

$$\widehat{M}^{(j)} \equiv {}_E\langle j|\widehat{U}|0\rangle_E$$

$$\boxed{\begin{aligned} \hat{M}^{(0)} &= |00\rangle\langle 00| + |11\rangle\langle 11| \\ \hat{M}^{(1)} &= |01\rangle\langle 01| + |10\rangle\langle 10| \end{aligned}}$$

$$\hat{M}^{(0)}(|00\rangle + |11\rangle) = |00\rangle + |11\rangle$$
$$\hat{M}^{(0)} + \hat{M}^{(1)} = \hat{1}_{1,2}$$

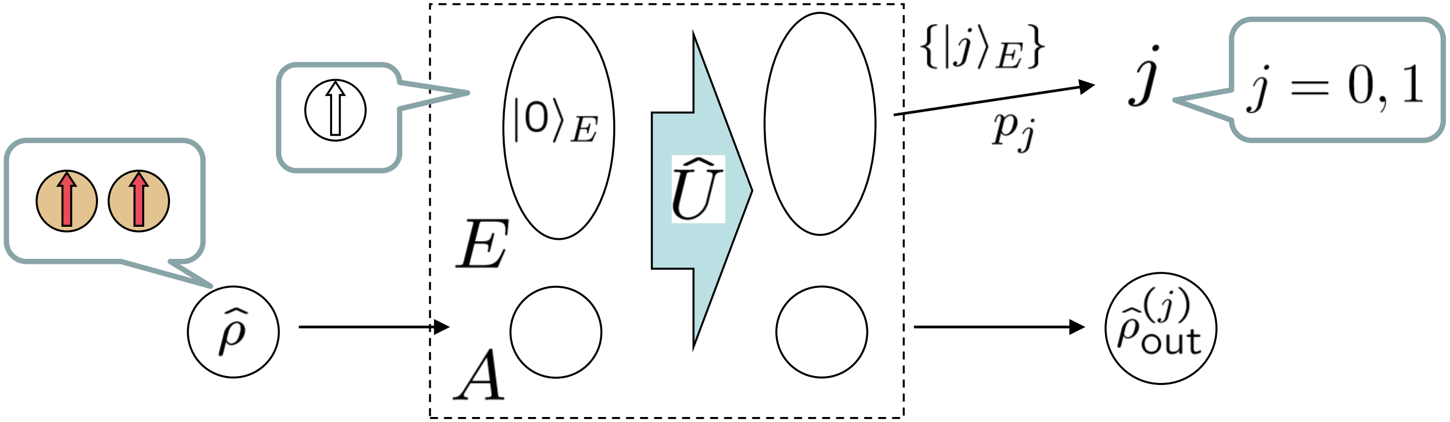
Measurement of a syndrome bit

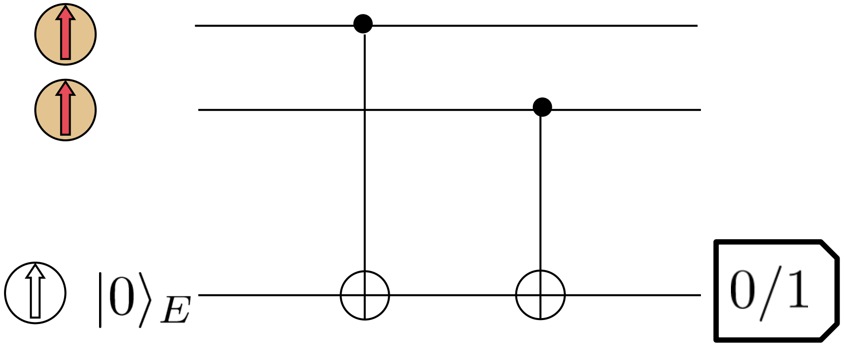
We want to learn s_1 , but not the value of each bit b_1, b_2

$$\hat{M}^{(0)} = |00\rangle\langle 00| + |11\rangle\langle 11| = {}_E\langle 0|\hat{U}|0\rangle_E$$

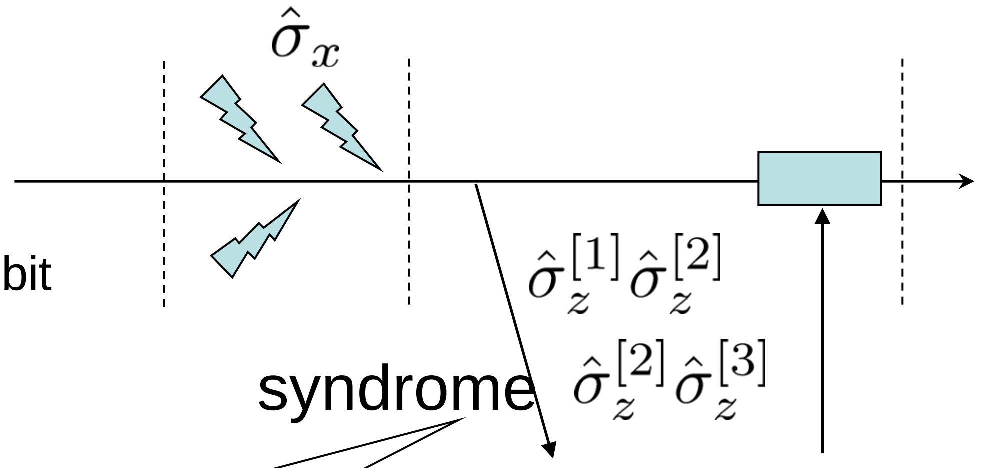
$$\hat{M}^{(1)} = |01\rangle\langle 01| + |10\rangle\langle 10| = {}_E\langle 1|\hat{U}|0\rangle_E$$

$$\begin{aligned}\hat{U} = {} & |0\rangle_{EE}\langle 0| \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|) \\ & + |1\rangle_{EE}\langle 0| \otimes (|01\rangle\langle 01| + |10\rangle\langle 10|) \ + (\cdots)_E\langle 1|\end{aligned}$$





Superposition will survive



$|000\rangle + |111\rangle$

$\hat{\sigma}_x$

$\hat{\sigma}_z^{[1]}\hat{\sigma}_z^{[2]}$

$\hat{\sigma}_z^{[2]}\hat{\sigma}_z^{[3]}$

syndrome

$|000\rangle + |111\rangle$

Encode the logical qubit on the eigenspace of

$$\hat{\sigma}_z^{[1]}\hat{\sigma}_z^{[2]} = 1$$
$$\hat{\sigma}_z^{[2]}\hat{\sigma}_z^{[3]} = 1$$

diagnose the error pattern without seeing the contents.

000 111 →	$\hat{1}^{[1]}\hat{1}^{[2]}\hat{1}^{[3]}$
100 011 →	$\hat{\sigma}_x^{[1]}\hat{1}^{[2]}\hat{1}^{[3]}$
010 101 →	$\hat{1}^{[1]}\hat{\sigma}_x^{[2]}\hat{1}^{[3]}$
001 110 →	$\hat{1}^{[1]}\hat{1}^{[2]}\hat{\sigma}_x^{[3]}$

Any single bit error can be corrected.

Can we correct the phase error?

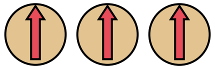
Problems:

- If we measure the system for the correction, the superposition may collapse. OK

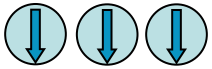
- Can we correct the phase error? $|0\rangle + |1\rangle \underset{\hat{\sigma}_z}{\leftrightarrow} |0\rangle - |1\rangle$

- There are infinite number of error patterns. Can we handle all of them?

$$|0\rangle \longrightarrow |000\rangle$$



$$|1\rangle \longrightarrow |111\rangle$$



Dimension:

8 in total.

2 for data.

4 different bit-error patterns.

We need more space to correct other errors.

7-bit code

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix}$$

$$\hat{\sigma}_z^{[1]} \hat{1}^{[2]} \hat{\sigma}_z^{[3]} \hat{1}^{[4]} \hat{\sigma}_z^{[5]} \hat{1}^{[6]} \hat{\sigma}_z^{[7]}$$

$$\hat{1}^{[1]} \hat{\sigma}_z^{[2]} \hat{\sigma}_z^{[3]} \hat{1}^{[4]} \hat{1}^{[5]} \hat{\sigma}_z^{[6]} \hat{\sigma}_z^{[7]}$$

$$\hat{1}^{[1]} \hat{1}^{[2]} \hat{1}^{[3]} \hat{\sigma}_z^{[4]} \hat{\sigma}_z^{[5]} \hat{\sigma}_z^{[6]} \hat{\sigma}_z^{[7]}$$

Dimension: $2^7 = 128$ in total.

8 different bit-error patterns.

$$128/8 = 16 = 2^4$$

We can encode 4 qubits of data if only the bit errors occur.

If we use only one qubit of data, we can accommodate 8 more errors.

$$\begin{pmatrix} s_4 \\ s_5 \\ s_6 \end{pmatrix}$$

$$\hat{\sigma}_x^{[1]} \hat{1}^{[2]} \hat{\sigma}_x^{[3]} \hat{1}^{[4]} \hat{\sigma}_x^{[5]} \hat{1}^{[6]} \hat{\sigma}_x^{[7]}$$

$$\hat{1}^{[1]} \hat{\sigma}_x^{[2]} \hat{\sigma}_x^{[3]} \hat{1}^{[4]} \hat{1}^{[5]} \hat{\sigma}_x^{[6]} \hat{\sigma}_x^{[7]}$$

$$\hat{1}^{[1]} \hat{1}^{[2]} \hat{1}^{[3]} \hat{\sigma}_x^{[4]} \hat{\sigma}_x^{[5]} \hat{\sigma}_x^{[6]} \hat{\sigma}_x^{[7]}$$

CSS 7-qubit code (Steane code)

$$\hat{\sigma}_x\hat{\sigma}_z = (-1)\hat{\sigma}_z\hat{\sigma}_x$$

$$\hat{\sigma}_z^{[1]}\hat{1}^{[2]}\hat{\sigma}_z^{[3]}\hat{1}^{[4]}\hat{\sigma}_z^{[5]}\hat{1}^{[6]}\hat{\sigma}_z^{[7]} = 1$$

$$\hat{1}^{[1]}\hat{\sigma}_z^{[2]}\hat{\sigma}_z^{[3]}\hat{1}^{[4]}\hat{1}^{[5]}\hat{\sigma}_z^{[6]}\hat{\sigma}_z^{[7]} = 1$$

$$\hat{1}^{[1]}\hat{1}^{[2]}\hat{1}^{[3]}\hat{\sigma}_z^{[4]}\hat{\sigma}_z^{[5]}\hat{\sigma}_z^{[6]}\hat{\sigma}_z^{[7]} = 1$$

commute ⟺

$$\hat{\sigma}_x^{[1]}\hat{1}^{[2]}\hat{\sigma}_x^{[3]}\hat{1}^{[4]}\hat{\sigma}_x^{[5]}\hat{1}^{[6]}\hat{\sigma}_x^{[7]} = 1$$

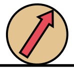
$$\hat{1}^{[1]}\hat{\sigma}_x^{[2]}\hat{\sigma}_x^{[3]}\hat{1}^{[4]}\hat{1}^{[5]}\hat{\sigma}_x^{[6]}\hat{\sigma}_x^{[7]} = 1$$

$$\hat{1}^{[1]}\hat{1}^{[2]}\hat{1}^{[3]}\hat{\sigma}_x^{[4]}\hat{\sigma}_x^{[5]}\hat{\sigma}_x^{[6]}\hat{\sigma}_x^{[7]} = 1$$

Dimension: $2^7 = 128$ in total.

6 observables (binary)
$2^6 = 64$ patterns

Each eigenspace has dimension 2.

		Bit error							
		no	1	2	3	4	5	6	7
Phase error	no								
	1								
	2								
	3								
	4								
	5								
	6								
	7								

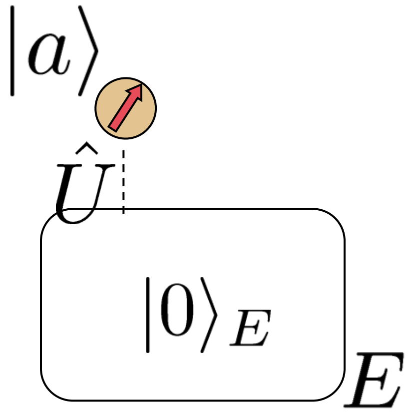
Any single bit error, plus any single phase error can be corrected.

Too many error patterns?

Problems:

- If we measure the system for the correction, the superposition may collapse. OK
- Can we correct the phase error? OK
- There are infinite number of error patterns. Can we handle all of them?

General errors on a single qubit

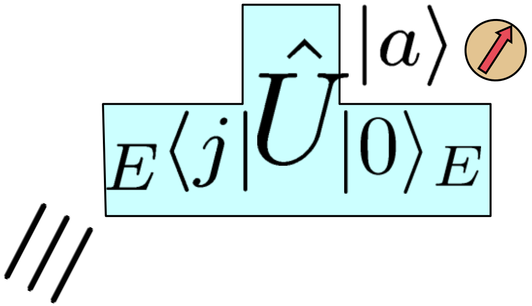


$$\hat{U}\left(|a\rangle \otimes |0\rangle_E\right)$$

Interaction with environment

General errors

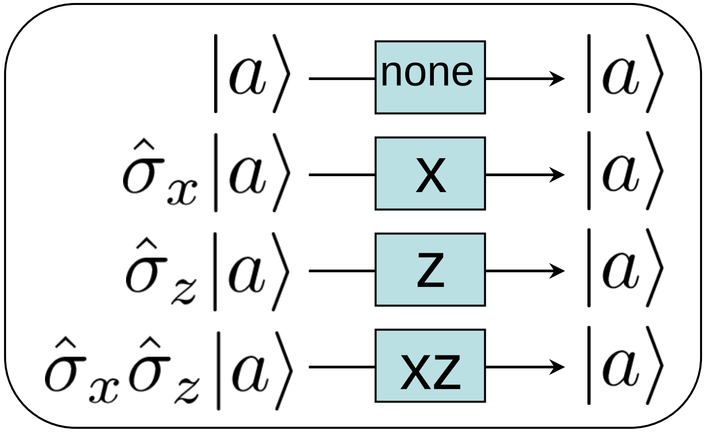
$$\begin{aligned}
&\hat U(|a\rangle \otimes |0\rangle_E) \\
&= \sum_j |j\rangle_{EE}\langle j|\hat U(|a\rangle \otimes |0\rangle_E) \\
&= \sum_j \hat M^{(j)}(|a\rangle \otimes |j\rangle_E) \\
&= |a\rangle \otimes |u_0\rangle_E + \hat\sigma_x|a\rangle \otimes |u_1\rangle_E \\
&\quad +\hat\sigma_z|a\rangle \otimes |u_2\rangle_E + \hat\sigma_x\hat\sigma_z|a\rangle \otimes |u_3\rangle_E
\end{aligned}$$



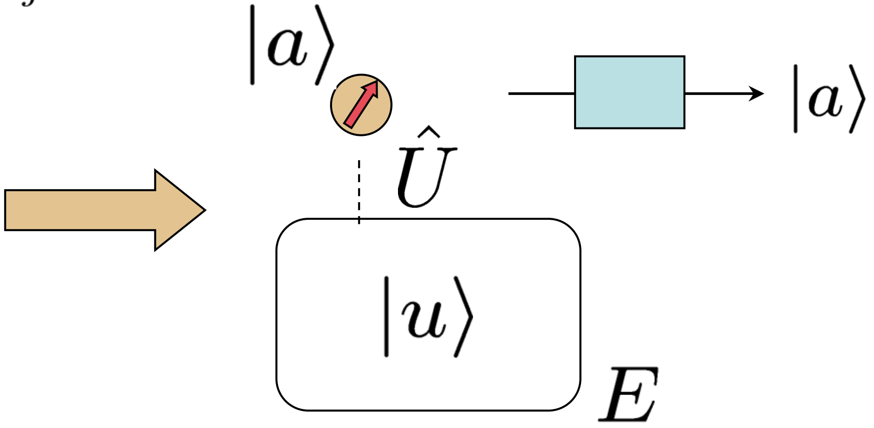
$$\hat M^{(j)}$$

$$= c_0^{(j)}\hat 1 + c_1^{(j)}\hat\sigma_x + c_2^{(j)}\hat\sigma_z + c_3^{(j)}\hat\sigma_x\hat\sigma_z$$

$|u_i\rangle_E \equiv \sum_j c_i^{(j)}|j\rangle_E$:unnormalized, nonorthogonal



Any scheme that can correct bit and phase errors



Any error should be corrected.

Too many error patterns?

Problems:

- If we measure the system for the correction, the superposition may collapse. OK

- Can we correct the phase error? σ_z OK

- There are infinite number of error patterns. Can we handle all of them? OK

Correcting bit and phase errors is enough.

Syndrome measurement projects general errors onto one of these errors.

Syndrome measurement digitizes the error

$$\hat\sigma_z^{[1]}\hat 1^{[2]}\hat\sigma_z^{[3]}\hat 1^{[4]}\hat\sigma_z^{[5]}\hat 1^{[6]}\hat\sigma_z^{[7]}$$
$$\hat 1^{[1]}\hat\sigma_z^{[2]}\hat\sigma_z^{[3]}\hat 1^{[4]}\hat 1^{[5]}\hat\sigma_z^{[6]}\hat\sigma_z^{[7]}$$
$$\hat 1^{[1]}\hat 1^{[2]}\hat 1^{[3]}\hat\sigma_z^{[4]}\hat\sigma_z^{[5]}\hat\sigma_z^{[6]}\hat\sigma_z^{[7]}$$

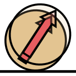
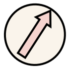
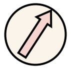
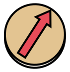
commute

⟷

$$\hat\sigma_x^{[1]}\hat 1^{[2]}\hat\sigma_x^{[3]}\hat 1^{[4]}\hat\sigma_x^{[5]}\hat 1^{[6]}\hat\sigma_x^{[7]}$$
$$\hat 1^{[1]}\hat\sigma_x^{[2]}\hat\sigma_x^{[3]}\hat 1^{[4]}\hat 1^{[5]}\hat\sigma_x^{[6]}\hat\sigma_x^{[7]}$$
$$\hat 1^{[1]}\hat 1^{[2]}\hat 1^{[3]}\hat\sigma_x^{[4]}\hat\sigma_x^{[5]}\hat\sigma_x^{[6]}\hat\sigma_x^{[7]}$$

Bit error

Phase error

	no	1	2	3	4	5	6	7
no								
1								
2								
3								
4								
5								
6								
7								

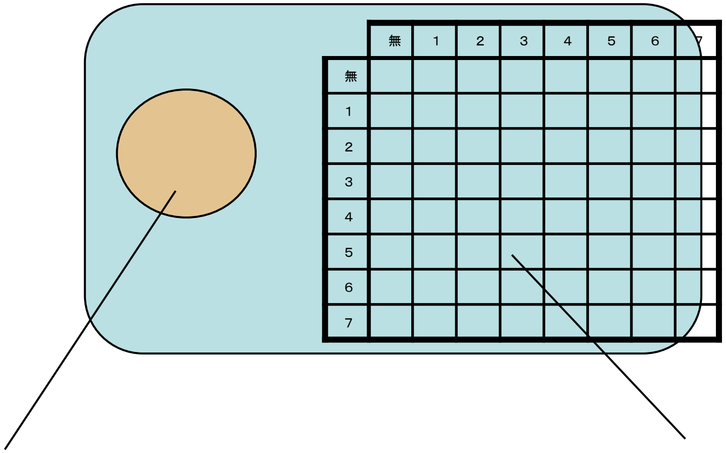
Any error on a single qubit can be corrected.

CSS QECC

Calderbank & Shor (1996)
Steane (1996)

Quantum error correcting codes

Special state with quantum correlation



Data

Quantum

Do not touch!

Error patterns

Changes are allowed, as long as we can keep track of them.

Measurement is OK.

It makes infinite error patterns shrink to finite ones.

Codeword states

A logical qubit should be encoded onto the 2-dimensional eigenspace with the 6 eigenvalues all 1.

$$\hat{\sigma}_z^{[1]}\hat{1}^{[2]}\hat{\sigma}_z^{[3]}\hat{1}^{[4]}\hat{\sigma}_z^{[5]}\hat{1}^{[6]}\hat{\sigma}_z^{[7]} = 1$$

$$\hat{1}^{[1]}\hat{\sigma}_z^{[2]}\hat{\sigma}_z^{[3]}\hat{1}^{[4]}\hat{1}^{[5]}\hat{\sigma}_z^{[6]}\hat{\sigma}_z^{[7]} = 1$$

$$\hat{1}^{[1]}\hat{1}^{[2]}\hat{1}^{[3]}\hat{\sigma}_z^{[4]}\hat{\sigma}_z^{[5]}\hat{\sigma}_z^{[6]}\hat{\sigma}_z^{[7]} = 1$$

commute ⟺

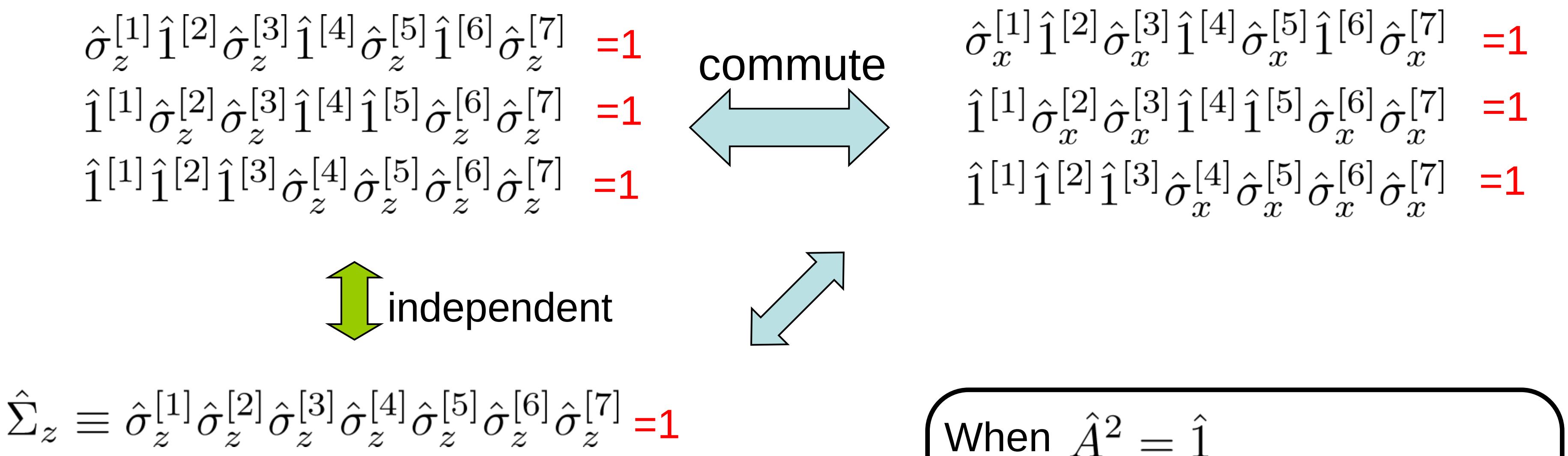
$$\hat{\sigma}_x^{[1]}\hat{1}^{[2]}\hat{\sigma}_x^{[3]}\hat{1}^{[4]}\hat{\sigma}_x^{[5]}\hat{1}^{[6]}\hat{\sigma}_x^{[7]} = 1$$

$$\hat{1}^{[1]}\hat{\sigma}_x^{[2]}\hat{\sigma}_x^{[3]}\hat{1}^{[4]}\hat{1}^{[5]}\hat{\sigma}_x^{[6]}\hat{\sigma}_x^{[7]} = 1$$

$$\hat{1}^{[1]}\hat{1}^{[2]}\hat{1}^{[3]}\hat{\sigma}_x^{[4]}\hat{\sigma}_x^{[5]}\hat{\sigma}_x^{[6]}\hat{\sigma}_x^{[7]} = 1$$

Codeword states

There should be a single eigenstate for which the 7 eigenvalues are all 1.



$$\hat\sigma_z^{[1]}\hat 1^{[2]}\hat\sigma_z^{[3]}\hat 1^{[4]}\hat\sigma_z^{[5]}\hat 1^{[6]}\hat\sigma_z^{[7]} = 1$$
$$\hat 1^{[1]}\hat\sigma_z^{[2]}\hat\sigma_z^{[3]}\hat 1^{[4]}\hat 1^{[5]}\hat\sigma_z^{[6]}\hat\sigma_z^{[7]} = 1$$
$$\hat 1^{[1]}\hat 1^{[2]}\hat 1^{[3]}\hat\sigma_z^{[4]}\hat\sigma_z^{[5]}\hat\sigma_z^{[6]}\hat\sigma_z^{[7]} = 1$$

commute

$$\hat\sigma_x^{[1]}\hat 1^{[2]}\hat\sigma_x^{[3]}\hat 1^{[4]}\hat\sigma_x^{[5]}\hat 1^{[6]}\hat\sigma_x^{[7]} = 1$$
$$\hat 1^{[1]}\hat\sigma_x^{[2]}\hat\sigma_x^{[3]}\hat 1^{[4]}\hat 1^{[5]}\hat\sigma_x^{[6]}\hat\sigma_x^{[7]} = 1$$
$$\hat 1^{[1]}\hat 1^{[2]}\hat 1^{[3]}\hat\sigma_x^{[4]}\hat\sigma_x^{[5]}\hat\sigma_x^{[6]}\hat\sigma_x^{[7]} = 1$$

independent

$$\hat\Sigma_z \equiv \hat\sigma_z^{[1]}\hat\sigma_z^{[2]}\hat\sigma_z^{[3]}\hat\sigma_z^{[4]}\hat\sigma_z^{[5]}\hat\sigma_z^{[6]}\hat\sigma_z^{[7]} = 1$$

When $\hat A^2 = \hat 1$

$$\hat A(|u\rangle + \hat A|u\rangle) = \hat A|u\rangle + \hat A^2|u\rangle$$
$$= |u\rangle + \hat A|u\rangle$$

$|0000000\rangle$ (All $\sigma_z^{[j]} = 1$)

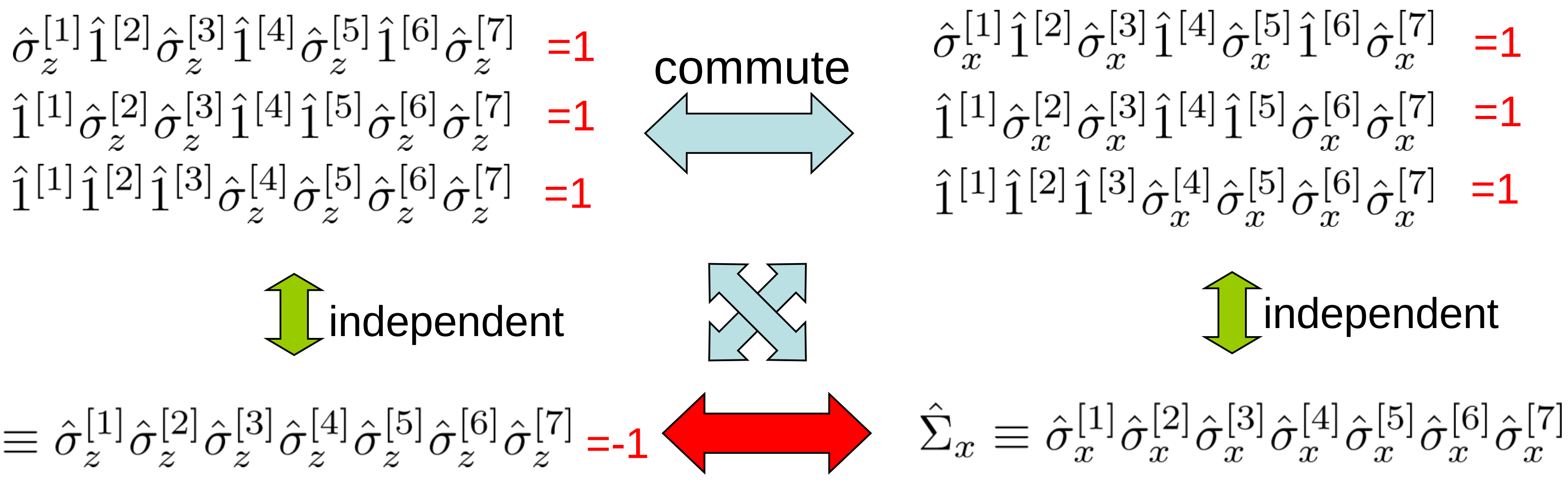
$$|0000000\rangle + |1010101\rangle$$

$$|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$$

$$|\mathbf{0}\rangle = |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$$
$$+|0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle$$

Codeword states

Find a codeword state that is orthogonal to $|\mathbf{0}\rangle$



$\hat\sigma_z^{[1]}\hat1^{[2]}\hat\sigma_z^{[3]}\hat1^{[4]}\hat\sigma_z^{[5]}\hat1^{[6]}\hat\sigma_z^{[7]}$ =1

$\hat1^{[1]}\hat\sigma_z^{[2]}\hat\sigma_z^{[3]}\hat1^{[4]}\hat1^{[5]}\hat\sigma_z^{[6]}\hat\sigma_z^{[7]}$ =1

$\hat1^{[1]}\hat1^{[2]}\hat1^{[3]}\hat\sigma_z^{[4]}\hat\sigma_z^{[5]}\hat\sigma_z^{[6]}\hat\sigma_z^{[7]}$ =1

commute

$\hat\sigma_x^{[1]}\hat1^{[2]}\hat\sigma_x^{[3]}\hat1^{[4]}\hat\sigma_x^{[5]}\hat1^{[6]}\hat\sigma_x^{[7]}$ =1

$\hat1^{[1]}\hat\sigma_x^{[2]}\hat\sigma_x^{[3]}\hat1^{[4]}\hat1^{[5]}\hat\sigma_x^{[6]}\hat\sigma_x^{[7]}$ =1

$\hat1^{[1]}\hat1^{[2]}\hat1^{[3]}\hat\sigma_x^{[4]}\hat\sigma_x^{[5]}\hat\sigma_x^{[6]}\hat\sigma_x^{[7]}$ =1

independent

independent

$\hat\Sigma_z \equiv \hat\sigma_z^{[1]}\hat\sigma_z^{[2]}\hat\sigma_z^{[3]}\hat\sigma_z^{[4]}\hat\sigma_z^{[5]}\hat\sigma_z^{[6]}\hat\sigma_z^{[7]}$ =-1

$\hat\Sigma_x \equiv \hat\sigma_x^{[1]}\hat\sigma_x^{[2]}\hat\sigma_x^{[3]}\hat\sigma_x^{[4]}\hat\sigma_x^{[5]}\hat\sigma_x^{[6]}\hat\sigma_x^{[7]}$

Anti-commute

$$\hat\Sigma_z\hat\Sigma_x = -\hat\Sigma_x\hat\Sigma_z$$

$$\hat\Sigma_z\hat\Sigma_x|\mathbf{0}\rangle = -\hat\Sigma_x\hat\Sigma_z|\mathbf{0}\rangle = -\hat\Sigma_x|\mathbf{0}\rangle$$

$$\begin{aligned}|\mathbf{1}\rangle \equiv \hat\Sigma_x|\mathbf{0}\rangle = |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle \\ +|1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle\end{aligned}$$

Description of the encoded states

$$|\psi_{\rm logical}\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |\psi_{\rm physical}\rangle = \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle$$

where

$$|\mathbf{0}\rangle = |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$$
$$+|0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle$$
$$|\mathbf{1}\rangle = |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle$$
$$+|1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle$$

Do we have to use these complicated descriptions of states?

Not necessarily, if the state is already assured to be in the code space.

Matrix representation on the basis $\{|\mathbf{0}\rangle, |\mathbf{1}\rangle\}$

$$\hat{\Sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\Sigma}_x = \begin{pmatrix} \langle\mathbf{0}|\hat{\Sigma}_x|\mathbf{0}\rangle & \langle\mathbf{0}|\hat{\Sigma}_x|\mathbf{1}\rangle \\ \langle\mathbf{1}|\hat{\Sigma}_x|\mathbf{0}\rangle & \langle\mathbf{1}|\hat{\Sigma}_x|\mathbf{1}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\Sigma}_y \equiv i\hat{\Sigma}_x\hat{\Sigma}_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{\mathbf{\Sigma}} \equiv (\hat{\Sigma}_x, \hat{\Sigma}_y, \hat{\Sigma}_z)$$

$$\hat{\Sigma}_z \equiv \hat{\sigma}_z^{[1]}\hat{\sigma}_z^{[2]}\hat{\sigma}_z^{[3]}\hat{\sigma}_z^{[4]}\hat{\sigma}_z^{[5]}\hat{\sigma}_z^{[6]}\hat{\sigma}_z^{[7]}$$
$$\hat{\Sigma}_x \equiv \hat{\sigma}_x^{[1]}\hat{\sigma}_x^{[2]}\hat{\sigma}_x^{[3]}\hat{\sigma}_x^{[4]}\hat{\sigma}_x^{[5]}\hat{\sigma}_x^{[6]}\hat{\sigma}_x^{[7]}$$

$$\hat{\Sigma}_z\hat{\Sigma}_x = -\hat{\Sigma}_x\hat{\Sigma}_z$$
$$\hat{\Sigma}_z^2 = \hat{\Sigma}_x^2 = 1$$
$$\hat{\Sigma}_z|\mathbf{0}\rangle = |\mathbf{0}\rangle$$
$$|\mathbf{1}\rangle \equiv \hat{\Sigma}_x|\mathbf{0}\rangle$$
$$\hat{\Sigma}_z|\mathbf{1}\rangle = -|\mathbf{1}\rangle$$

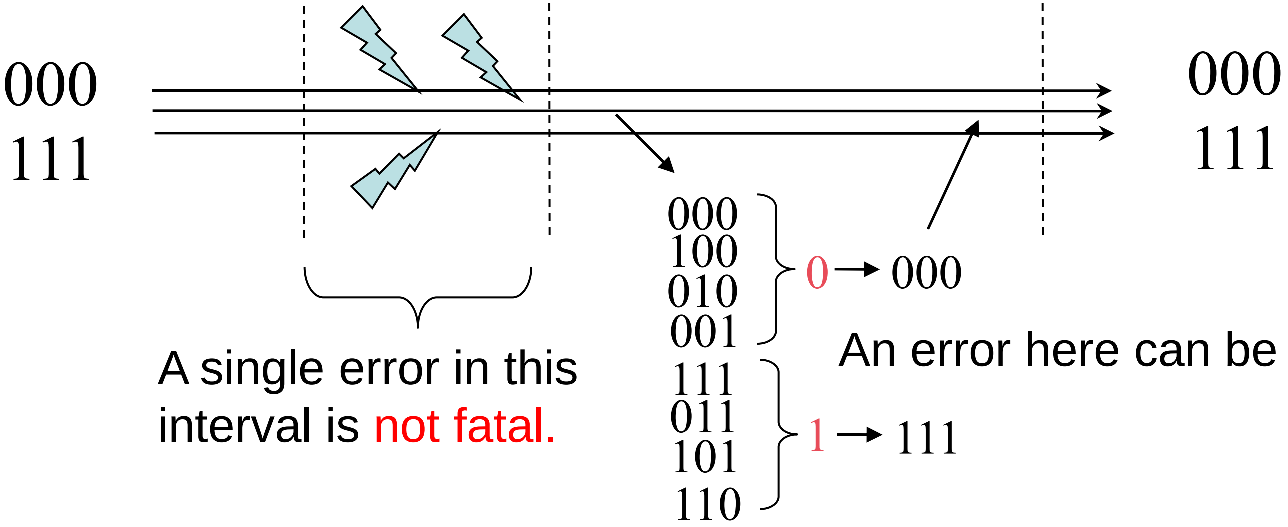
$$\hat{\Sigma}_y = -\hat{\sigma}_y^{[1]}\hat{\sigma}_y^{[2]}\hat{\sigma}_y^{[3]}\hat{\sigma}_y^{[4]}\hat{\sigma}_y^{[5]}\hat{\sigma}_y^{[6]}\hat{\sigma}_y^{[7]}$$

$i^7 = -i$

$$\hat{\rho}_{\rm logical} = \frac{1}{2}(\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}) \longrightarrow \hat{\rho}_{\rm physical} = \frac{1}{2}(\hat{1} + \boldsymbol{P}\cdot\hat{\mathbf{\Sigma}})$$

What happens if we are careless?

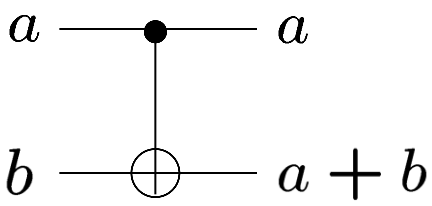
Classical repetition cord

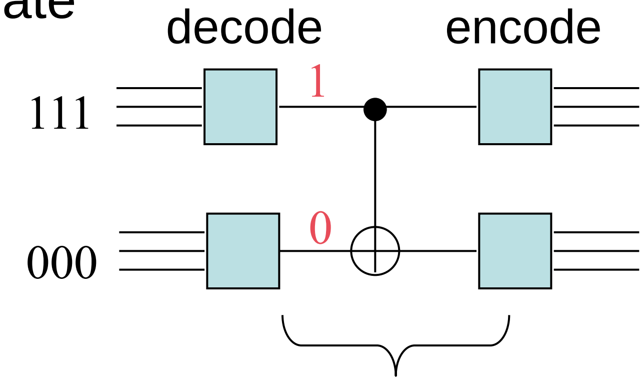


A single error in this interval is not fatal.

An error here can be fatal.

We want to apply a two-bit gate





A single error in this interval is fatal.

Fault-tolerant scheme

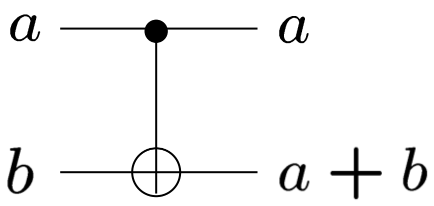
Tolerance against a single error at any place.

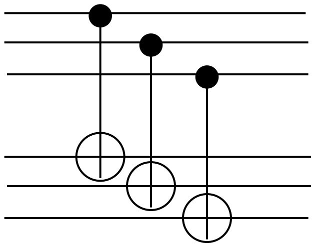
We should not decode. Operate on the encoded data.

A single error should not spread over many physical bits.

Classical repetition cord

A solution:





This looks trivial because this is just a simple repetition code.

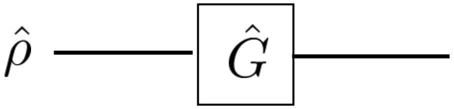
Can we do the same thing with more complex quantum codes?

YES, at least for a special set of gates.

Similarity to the classical repetition codes

$$\hat\Sigma_z \equiv \hat\sigma_z^{[1]}\hat\sigma_z^{[2]}\hat\sigma_z^{[3]}\hat\sigma_z^{[4]}\hat\sigma_z^{[5]}\hat\sigma_z^{[6]}\hat\sigma_z^{[7]}$$
$$\hat\Sigma_x \equiv \hat\sigma_x^{[1]}\hat\sigma_x^{[2]}\hat\sigma_x^{[3]}\hat\sigma_x^{[4]}\hat\sigma_x^{[5]}\hat\sigma_x^{[6]}\hat\sigma_x^{[7]}$$
$$\hat\Sigma_y = -\hat\sigma_y^{[1]}\hat\sigma_y^{[2]}\hat\sigma_y^{[3]}\hat\sigma_y^{[4]}\hat\sigma_y^{[5]}\hat\sigma_y^{[6]}\hat\sigma_y^{[7]}$$

1-qubit gate $\hat G$



$$g(\hat\rho) = \hat G\hat\rho\hat G^\dagger$$

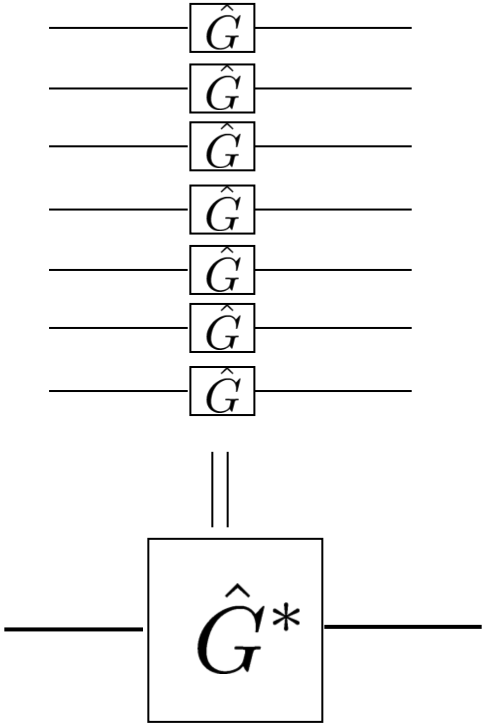
(mirror image)

$\hat G^*$:“complex conjugate of $\hat G$”

$$\langle j|\hat G^*|j'\rangle = \overline{\langle j|\hat G|j'\rangle}$$ on the basis $\{|0\rangle, |1\rangle\}$

$$\hat\sigma_z^* = \hat\sigma_z,\ \hat\sigma_x^* = \hat\sigma_x,\ \hat\sigma_y^* = -\hat\sigma_y$$
$$g^*(\hat\rho) \equiv \hat G^*\hat\rho\hat G^{*\dagger} = (\hat G\hat\rho^*\hat G^\dagger)^* = g(\hat\rho^*)^*$$
$$(c\hat\sigma_\mu)^{\otimes 7} = (c\hat\Sigma_\mu)^* \quad c = \pm 1, \pm i$$

Pauli group: $V \equiv \{\pm 1, \pm i\} \times \{\hat 1, \hat\sigma_z, \hat\sigma_x, \hat\sigma_z\hat\sigma_x\}$



Suppose that $\boxed{\hat G\hat v\hat G^\dagger \in V \quad \text{for all} \quad \hat v \in V}$

(Pauli operators are mapped to Pauli operators)

$$\hat\Sigma_z \mapsto g(\hat\sigma_z^{[1]})g(\hat\sigma_z^{[2]})g(\hat\sigma_z^{[3]})g(\hat\sigma_z^{[4]})g(\hat\sigma_z^{[5]})g(\hat\sigma_z^{[6]})g(\hat\sigma_z^{[7]})$$
$$= g(\hat\Sigma_z)^* = g(\hat\Sigma_z^*)^* = g^*(\hat\Sigma_z)$$

Similarly, $\hat\Sigma_x \mapsto g^*(\hat\Sigma_x)$

$$\hat\Sigma_y \mapsto -g(\hat\sigma_y^{[1]})g(\hat\sigma_y^{[2]})g(\hat\sigma_y^{[3]})g(\hat\sigma_y^{[4]})g(\hat\sigma_y^{[5]})g(\hat\sigma_y^{[6]})g(\hat\sigma_y^{[7]})$$
$$= -g(\hat\Sigma_y)^* = g(\hat\Sigma_y^*)^* = g^*(\hat\Sigma_y)$$

$$\hat\rho_{\text{physical}} = \frac{1}{2}(\hat 1 + \boldsymbol{P}\cdot\hat{\boldsymbol{\Sigma}}) \mapsto g^*(\hat\rho_{\text{physical}})$$

Clifford group

Pauli group: $V \equiv \{\pm 1, \pm i\} \times \{\hat{1}, \hat{\sigma}_z, \hat{\sigma}_x, \hat{\sigma}_z\hat{\sigma}_x\}$

$\boxed{\hat{G}\hat{v}\hat{G}^{\dagger} \in V \quad \text{for all} \quad \hat{v} \in V}$ (Elements of Clifford group)

Elements of the Pauli group belongs to the Clifford group

Hadamard gate

$$H \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \begin{cases} \hat{\sigma}_z \mapsto \hat{\sigma}_x \\ \hat{\sigma}_x \mapsto \hat{\sigma}_z \\ \hat{\sigma}_y \mapsto -\hat{\sigma}_y \end{cases}$$

Phase gate

$$S \equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad \begin{cases} \hat{\sigma}_z \mapsto \hat{\sigma}_z \\ \hat{\sigma}_x \mapsto \hat{\sigma}_y \\ \hat{\sigma}_y \mapsto -\hat{\sigma}_x \end{cases}$$

Two-qubit gates

$\boxed{\hat{G}(\hat{v} \otimes \hat{v}')\hat{G}^{\dagger} \in V \otimes V \quad \text{for all} \quad \hat{v}, \hat{v}' \in V}$

Controlled-NOT gate: $|0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes \sigma_x$

$$\begin{cases} \hat{\sigma}_x \otimes \hat{1} \mapsto \hat{\sigma}_x \otimes \hat{\sigma}_x \\ \hat{1} \otimes \hat{\sigma}_x \mapsto \hat{1} \otimes \hat{\sigma}_x \\ \hat{\sigma}_z \otimes \hat{1} \mapsto \hat{\sigma}_z \otimes \hat{1} \\ \hat{1} \otimes \hat{\sigma}_z \mapsto \hat{\sigma}_z \otimes \hat{\sigma}_z \end{cases}$$