4-2. Distinguishability

Trace distance

- Trace norm and polar decomposition
- Minimum-error discrimination

Fidelity

- Fidelity and distinguishability
- No-cloning theorem

Relation between fidelity and trace distance

Distinguishability

Examples

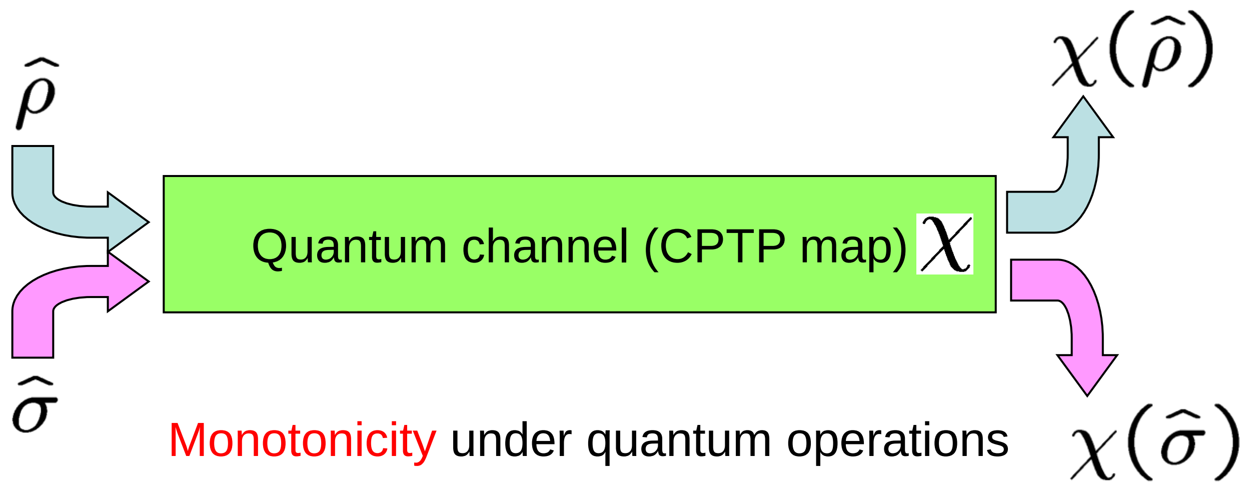
$$\frac{1}{2}\|\hat{\rho}-\hat{\sigma}\|$$

$$1-F(\hat{\rho},\hat{\sigma})$$

Measure of distinguishability between two states $D(\hat{\rho},\hat{\sigma})$

A quantity describing how we can distinguish between the two states in principle.

The distinguishability should never be improved by a quantum operation.



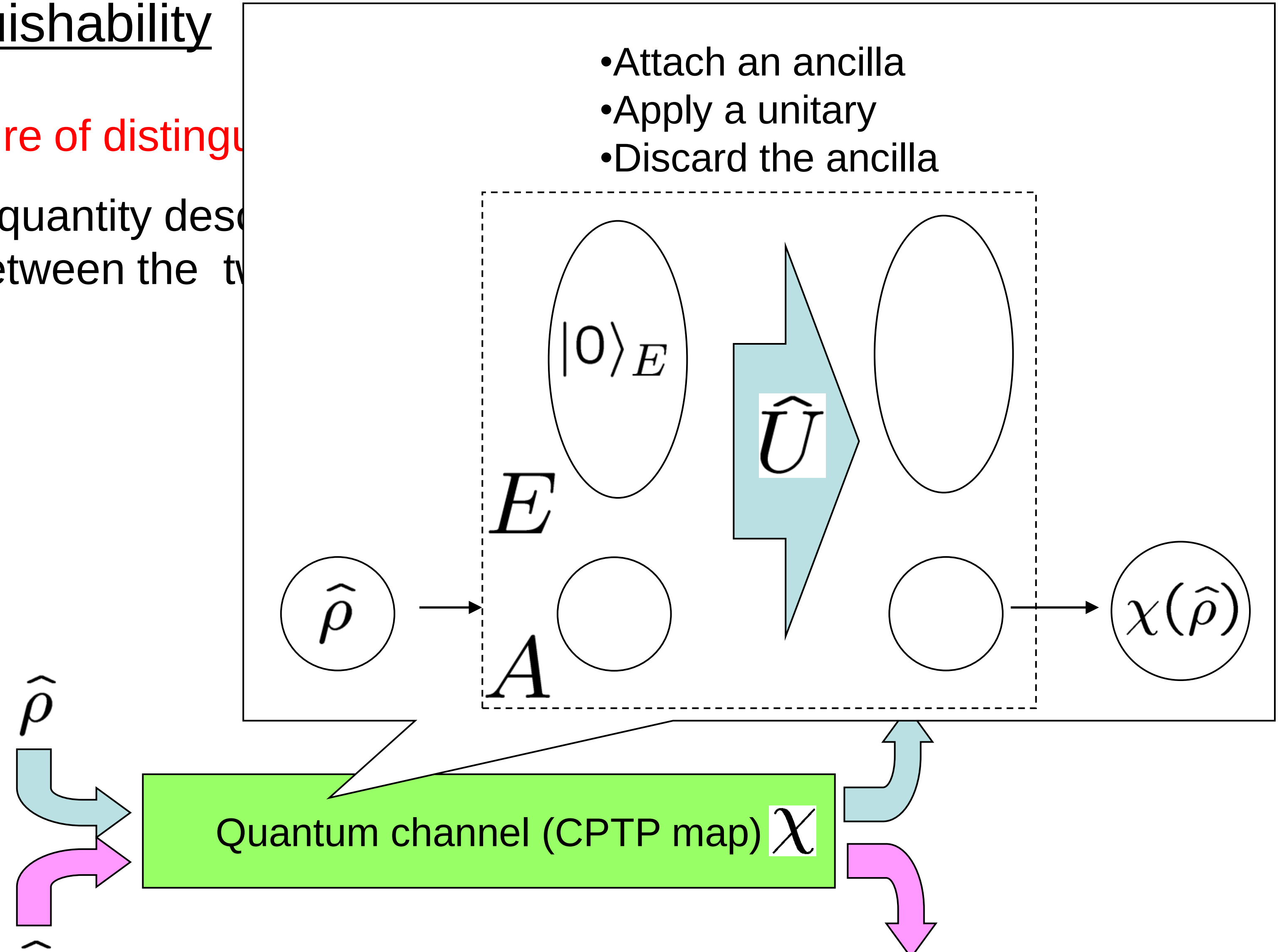
Monotonicity under quantum operations

$$D(\hat{\rho},\hat{\sigma}) \geq D(\chi(\hat{\rho}),\chi(\hat{\sigma}))$$

Distinguishability

Measure of distingu

A quantity desc
between the tw



Monotonicity under quantum operations

$$D(\hat{\rho},\hat{\sigma}) \geq D(\chi(\hat{\rho}),\chi(\hat{\sigma}))$$

Trace norm

$$\boxed{\|\hat{A}\| = \|\hat{A}\|_1 \equiv \mathrm{Tr}|\hat{A}| = \mathrm{Tr}\sqrt{\hat{A}^\dagger \hat{A}}}$$

In particular, when $\hat{A}$ is normal (diagonalizable),

$$\mathrm{Tr}(|\hat{A}|) = \sum_j |\lambda_j| \qquad \lambda_j\text{: Eigenvalues of } \hat{A}$$

$$\boxed{\|\hat{A}\| = \max_{\hat{U}} |\mathrm{Tr}(\hat{A}\hat{U})|}$$

$$|\hat{A}| = \sum_j \nu_j |j\rangle\langle j| \qquad \|\hat{A}\| = \sum_j \nu_j$$

$$\mathrm{Tr}(\hat{A}\hat{U}) = \mathrm{Tr}(\hat{V}|\hat{A}|\hat{U}) = \sum_j \nu_j \langle j|\hat{U}\hat{V}|j\rangle$$

$|\langle j|\hat{U}\hat{V}|j\rangle| \leq 1$ for any $\hat{U}$

$$\hat{U} = \hat{V}^\dagger \to |\langle j|\hat{U}\hat{V}|j\rangle| = 1$$

$$\|\hat{A} \otimes \hat{B}\| = \|\hat{A}\| \times \|\hat{B}\|$$

proof: $|\hat{A} \otimes \hat{B}| = |\hat{A}| \otimes |\hat{B}|$

$$\|\hat{U}\hat{A}\hat{U}^\dagger\| = \|\hat{A}\|$$

proof: $|\hat{U}\hat{A}\hat{U}^\dagger| = \hat{U}|\hat{A}|\hat{U}^\dagger$

Polar decomposition

number $\qquad \alpha = e^{i\theta}|\alpha|$

linear operator $\qquad \hat{A} = \hat{V}|\hat{A}|$

unitary $\qquad$ positive

$$|\hat{A}| \equiv \sqrt{\hat{A}^\dagger \hat{A}}$$

$$\hat{V} \equiv \hat{A}|\hat{A}|^{-1}$$

(when $\hat{A}$ is invertible)

$$\hat{V}^\dagger \hat{V} = |\hat{A}|^{-1}\hat{A}^\dagger \hat{A}|\hat{A}|^{-1} = \hat{1}$$

Trace distance

$\| \cdot \|$: trace norm

$$\frac{1}{2}\|\hat{\rho} - \hat{\sigma}\|$$

Zero when $\hat{\rho} = \hat{\sigma}$ (the same state)

Unity when $\hat{\rho}\hat{\sigma} = 0$ (perfectly distinguishable)

Monotonicity $\boxed{\|\hat{\rho} - \hat{\sigma}\| \geq \|\chi(\hat{\rho}) - \chi(\hat{\sigma})\|}$

- Attach an ancilla $\hat{\rho} \to \hat{\rho} \otimes \hat{\tau}$ $\quad \hat{\sigma} \to \hat{\sigma} \otimes \hat{\tau}$

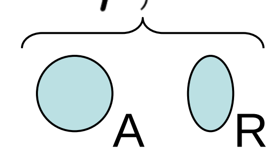
$$\|\hat{\rho}\otimes\hat{\tau} - \hat{\sigma}\otimes\hat{\tau}\| = \|(\hat{\rho}-\hat{\sigma})\otimes\hat{\tau}\| = \|\hat{\rho}-\hat{\sigma}\|\times\|\hat{\tau}\| = \|\hat{\rho}-\hat{\sigma}\|$$

- Apply a unitary $\hat{\rho} \to \hat{U}\hat{\rho}\hat{U}^\dagger$ $\quad \hat{\sigma} \to \hat{U}\hat{\sigma}\hat{U}^\dagger$

$$\|\hat{U}\hat{\rho}\hat{U}^\dagger - \hat{U}\hat{\sigma}\hat{U}^\dagger\| = \|\hat{U}(\hat{\rho} - \hat{\sigma})\hat{U}^\dagger\| = \|\hat{\rho} - \hat{\sigma}\|$$

- Discard the ancilla $\hat{\rho} \to \mathrm{Tr}_R(\hat{\rho})$ $\quad \hat{\sigma} \to \mathrm{Tr}_R(\hat{\sigma})$

$\hat{\rho}, \hat{\sigma}$

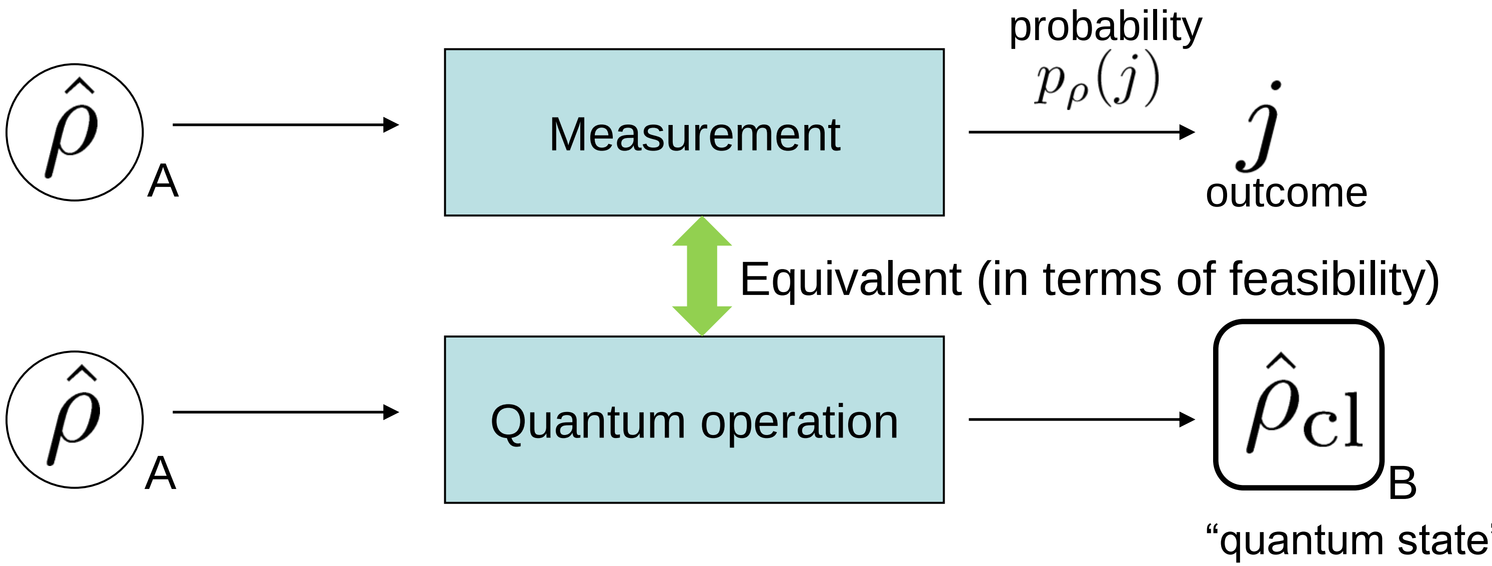


$$\|\mathrm{Tr}_R(\hat{\rho}) - \mathrm{Tr}_R(\hat{\sigma})\| = \max_{\hat{V}_A} |\mathrm{Tr}\,[\mathrm{Tr}_R(\hat{\rho} - \hat{\sigma})\hat{V}_A]|$$

$$= \max_{\hat{V}_A} |\mathrm{Tr}\,[(\hat{\rho} - \hat{\sigma})(\hat{V}_A \otimes \hat{1}_R)]| \leq \max_{\hat{U}_{AR}} |\mathrm{Tr}\,[(\hat{\rho} - \hat{\sigma})\hat{U}_{AR}]|$$

$$= \|\hat{\rho} - \hat{\sigma}\|$$

Measurements and quantum operations

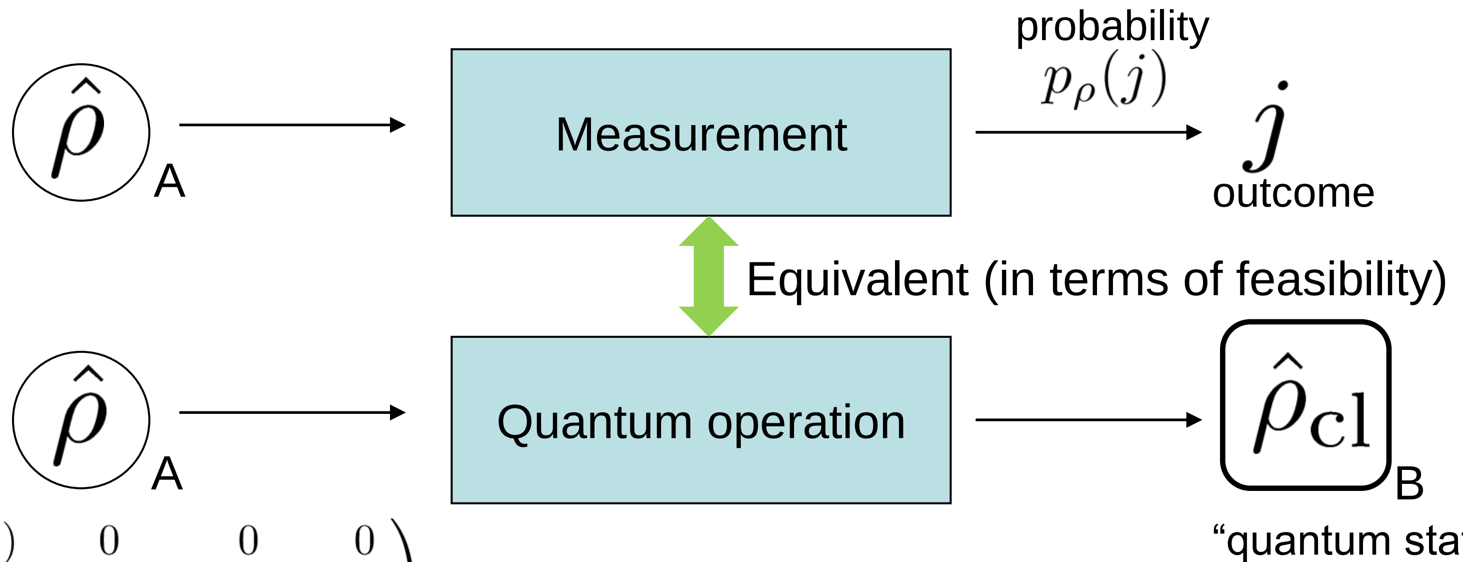


$$\hat{\rho}_{\mathrm{cl}} = \begin{pmatrix} p_\rho(1) & 0 & 0 & 0 \\ 0 & p_\rho(2) & 0 & 0 \\ 0 & 0 & p_\rho(3) & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

Any measurement has a description in terms of a quantum operation.

We may apply rules and bounds for quantum operations to measurements

Monotonicity of trace distance and measurements



$$\hat{\rho}_{\mathrm{cl}} = \begin{pmatrix} p_\rho(1) & 0 & 0 & 0 \\ 0 & p_\rho(2) & 0 & 0 \\ 0 & 0 & p_\rho(3) & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

$$\frac{1}{2}\|\hat{\rho}-\hat{\sigma}\| \geq \frac{1}{2}\|\hat{\rho}_{\mathrm{cl}}-\hat{\sigma}_{\mathrm{cl}}\| = \frac{1}{2}\sum_j |p_\rho(j)-p_\sigma(j)|$$

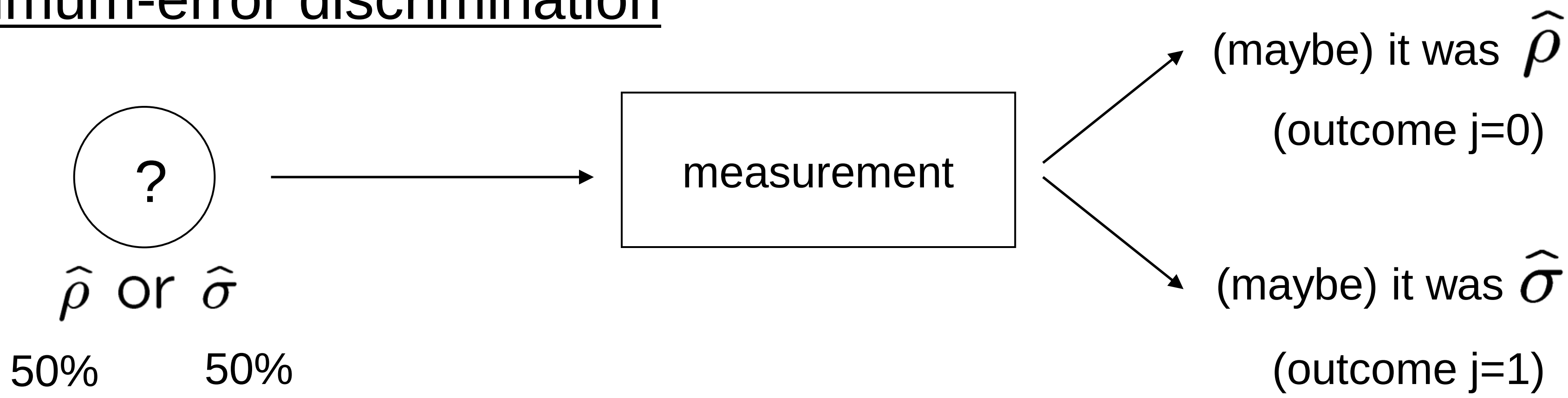
(total variation distance)

Total variation distance between the probabilities of the outcomes never exceeds the trace distance.

The equality is always achieved by the orthogonal measurement on a basis diagonalizing $\hat{\rho}-\hat{\sigma}$.

$$\hat{\rho}-\hat{\sigma} = \sum_j \lambda_j |j\rangle\langle j| \longrightarrow \sum_j |p_\rho(j)-p_\sigma(j)| = \sum_j |\lambda_j|$$

Minimum-error discrimination



probability of error: $p_{\mathrm{err}} = \frac{1}{2} p_\rho(1) + \frac{1}{2} p_\sigma(0)$

total variation distance:

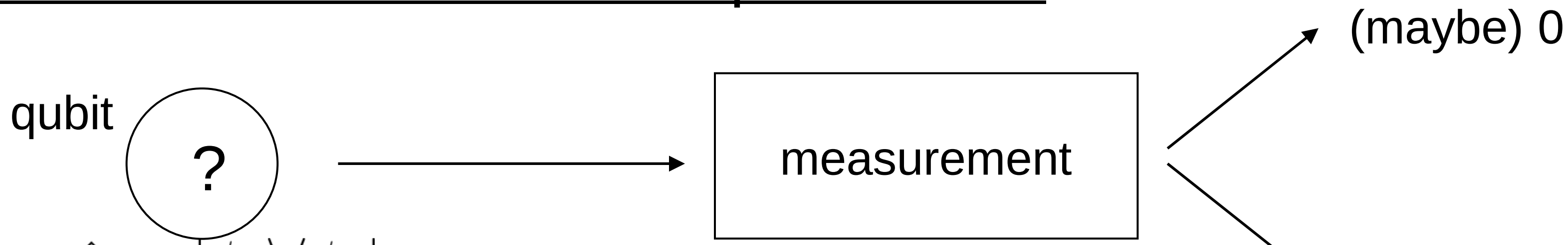
$$\frac{1}{2} \sum_{j=0,1} |p_\rho(j) - p_\sigma(j)| = \frac{1}{2} |1 - p_\rho(1) - p_\sigma(0)| + \frac{1}{2} |p_\rho(1) - [1 - p_\sigma(0)]|$$
$$= 1 - 2p_{\mathrm{err}}$$

The minimum error probability:

$$\boxed{p_{\mathrm{err}}^{(\mathrm{min})} = \frac{1}{2}\left(1 - \frac{1}{2} \|\hat{\rho} - \hat{\sigma}\|\right)}$$

An operational meaning of the trace distance

Discrimination between two pure states



50% $\hat{\rho}_0 \equiv |\phi_0\rangle\langle\phi_0|$

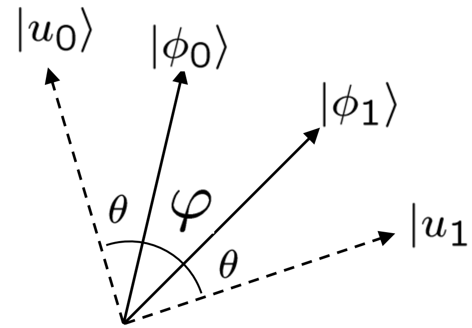
or

50% $\hat{\rho}_1 \equiv |\phi_1\rangle\langle\phi_1|$

$$\langle\phi_0|\phi_1\rangle = s = \cos\varphi = \sin 2\theta > 0$$

$$|\phi_0\rangle = \cos\theta|u_0\rangle + \sin\theta|u_1\rangle$$

$$|\phi_1\rangle = \sin\theta|u_0\rangle + \cos\theta|u_1\rangle$$



$$\hat{\rho}_0 - \hat{\rho}_1 = \cos 2\theta\,(|u_0\rangle\langle u_0| - |u_1\rangle\langle u_1|)$$

$$\frac{1}{2}\|\hat{\rho}_0 - \hat{\rho}_1\| = \cos 2\theta = \sqrt{1-s^2}$$

$$\boxed{p_{\mathrm{err}}^{(\mathrm{min})} = \frac{1-\sqrt{1-s^2}}{2}}$$

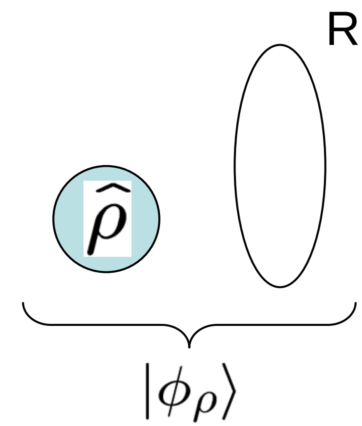
The inner product determines the distinguishability of two pure states.

Fidelity

$$\boxed{F(\hat{\rho}, \hat{\sigma}) \equiv \max |\langle \phi_\rho | \phi_\sigma \rangle|^2}$$

$$\mathrm{Tr}_R[|\phi_\rho\rangle\langle\phi_\rho|] = \hat{\rho}$$
$$\mathrm{Tr}_R[|\phi_\sigma\rangle\langle\phi_\sigma|] = \hat{\sigma}$$

(purifications)



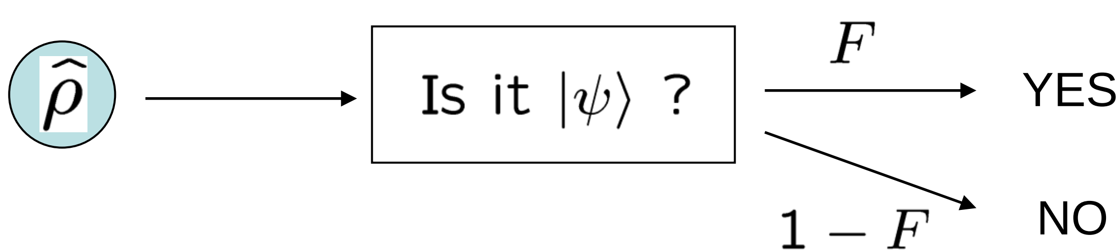
$$F(\hat{\rho}, \hat{\sigma}) = 1 \Leftrightarrow \hat{\rho} = \hat{\sigma} \qquad F(\hat{\rho}, \hat{\sigma}) = 0 \Leftrightarrow \hat{\rho}\hat{\sigma} = 0$$

$$F(|\varphi\rangle\langle\varphi|, |\psi\rangle\langle\psi|) = |\langle\varphi|\psi\rangle|^2$$

$$F(\hat{\rho}, |\psi\rangle\langle\psi|) = \langle\psi|\hat{\rho}|\psi\rangle$$

proof:

$$F = \max_{|u\rangle} \left| {}_R\langle u|_A\langle\psi||\phi_\rho\rangle_{AR} \right|^2$$
$$= \| {}_A\langle\psi||\phi_\rho\rangle_{AR} \|^2$$

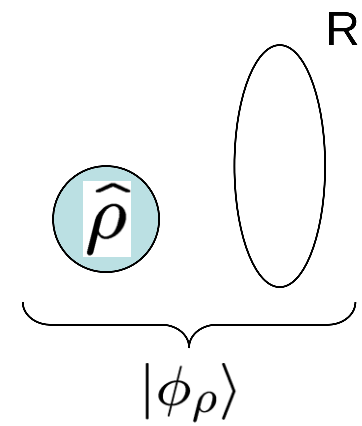


Operational meaning of the fidelity
(to a pure state)

Fidelity

$$F(\hat{\rho},\hat{\sigma}) \equiv \max|\langle\phi_\rho|\phi_\sigma\rangle|^2$$

$$F(\hat{\rho},\hat{\sigma}) = \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|^2$$



Any purification can be written as $|\phi_\rho\rangle = \sum_k \sqrt{\hat{\rho}}|k\rangle \otimes \hat{U}_R|k\rangle_R$

$$= \sum_k \sqrt{\hat{\rho}}\hat{U}'|k\rangle \otimes |k\rangle_R$$

$$F(\hat{\rho},\hat{\sigma}) = \max_{\hat{U},\hat{V}}\left|\sum_{kl}\langle k|\hat{U}^\dagger\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\hat{V}|l\rangle \times {}_R\langle k|l\rangle_R\right|^2$$

$$= \max_{\hat{U},\hat{V}}\left|\mathrm{Tr}(\hat{U}^\dagger\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\hat{V})\right|^2 = \max_{\hat{V}}\left|\mathrm{Tr}(\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\hat{V})\right|^2$$

Monotonicity of fidelity

$$F(\hat\rho,\hat\sigma) \equiv \max|\langle\phi_\rho|\phi_\sigma\rangle|^2 = \|\sqrt{\hat\rho}\sqrt{\hat\sigma}\|^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\hat\sigma}\hat\rho\sqrt{\hat\sigma}}\right)^2$$

$1 - F(\hat\rho,\hat\sigma)$ is a measure of distinguishability. (not a distance)

Monotonicity $\boxed{F(\hat\rho,\hat\sigma) \le F(\chi(\hat\rho),\chi(\hat\sigma))}$

•Attach an ancilla $\hat\rho \to \hat\rho \otimes \hat\tau \qquad \hat\sigma \to \hat\sigma \otimes \hat\tau$

$$F(\hat\rho \otimes \hat\tau, \hat\sigma \otimes \hat\tau) = \|\sqrt{\hat\rho}\sqrt{\hat\sigma}\|^2\|\tau\|^2 = F(\hat\rho,\hat\sigma)$$

•Apply a unitary $\hat\rho \to \hat U\hat\rho\hat U^\dagger \qquad \hat\sigma \to \hat U\hat\sigma\hat U^\dagger$

$$F(\hat U\hat\rho\hat U^\dagger, \hat U\hat\sigma\hat U^\dagger) = \|\hat U\sqrt{\hat\rho}\sqrt{\hat\sigma}\hat U^\dagger\|^2 = \|\sqrt{\hat\rho}\sqrt{\hat\sigma}\|^2 = F(\hat\rho,\hat\sigma)$$

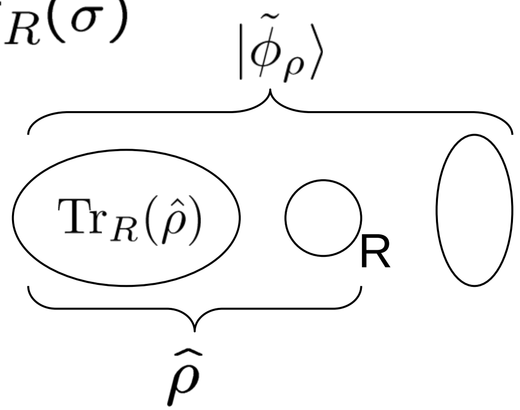
•Discard the ancilla $\hat\rho \to \mathrm{Tr}_R(\hat\rho) \qquad \hat\sigma \to \mathrm{Tr}_R(\hat\sigma)$

Choose purifications achieving the maximum

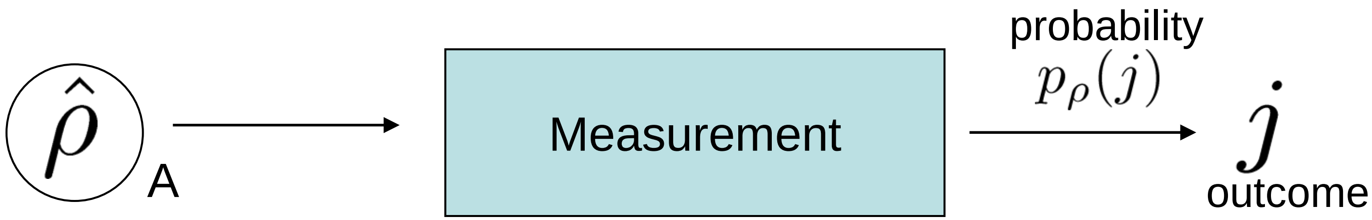
$$F(\hat\rho,\hat\sigma) = |\langle\tilde\phi_\rho|\tilde\phi_\sigma\rangle|^2$$

They are also purifications of $\mathrm{Tr}_R(\hat\rho), \mathrm{Tr}_R(\hat\sigma)$

$$F(\mathrm{Tr}_R(\hat\rho), \mathrm{Tr}_R(\hat\sigma)) \ge |\langle\tilde\phi_\rho|\tilde\phi_\sigma\rangle|^2 = F(\hat\rho,\hat\sigma)$$



Operational meaning of the fidelity?



$$F(\hat{\rho},\hat{\sigma}) \leq F(\hat{\rho}_{\mathrm{cl}},\hat{\sigma}_{\mathrm{cl}}) = \left(\sum_j \sqrt{p_\rho(j)}\sqrt{p_\sigma(j)}\right)^2$$

No clear operational meaning ...

The equality is always achieved by the orthogonal measurement on a basis "diagonalizing $\sqrt{\hat{\rho}}/\sqrt{\hat{\sigma}}$."

$$\mathcal{H} = \underline{(\mathrm{Ran}\ \hat{\sigma})} \oplus (\mathrm{Ker}\ \hat{\sigma})$$

$\downarrow$

Diagonalize the positive operator $\hat{\sigma}^{-1/2}\,|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}|\,\hat{\sigma}^{-1/2}$

$$= \sum_j \sqrt{\mu_j}|j\rangle\langle j|$$

Proof:

$$\sqrt{\mu_j}\hat{\sigma}^{1/2}|j\rangle = |\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}|\,\hat{\sigma}^{-1/2}|j\rangle$$

$$\Rightarrow \mu_j\langle j|\hat{\sigma}|j\rangle = \langle j|\hat{\sigma}^{-1/2}|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}|^2\,\hat{\sigma}^{-1/2}|j\rangle = \langle j|\hat{\rho}|j\rangle$$

$$\Rightarrow \sum_j \sqrt{\mu_j}\langle j|\hat{\sigma}|j\rangle = \sum_j \langle j|\hat{\sigma}^{1/2}|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}|\,\hat{\sigma}^{-1/2}|j\rangle = \mathrm{Tr}\,|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}| = \sqrt{F(\hat{\rho},\hat{\sigma})}$$

Fidelity

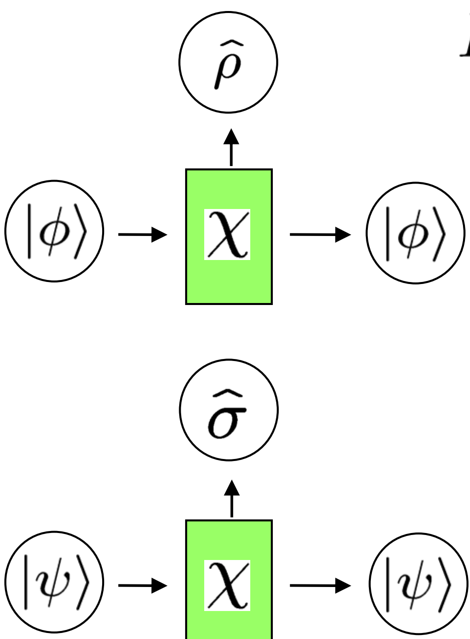
Multiplicativity

$$F(\hat{\rho}_1 \otimes \hat{\rho}_2, \hat{\sigma}_1 \otimes \hat{\sigma}_2) = F(\hat{\rho}_1, \hat{\sigma}_1)F(\hat{\rho}_2, \hat{\sigma}_2)$$

Proof: $$\|\sqrt{\hat{\rho}_1 \otimes \hat{\rho}_2}\sqrt{\hat{\sigma}_1 \otimes \hat{\sigma}_2}\|^2 = \|\sqrt{\hat{\rho}_1}\sqrt{\hat{\sigma}_1} \otimes \sqrt{\hat{\rho}_2}\sqrt{\hat{\sigma}_2}\|^2$$
$$= \|\sqrt{\hat{\rho}_1}\sqrt{\hat{\sigma}_1}\|^2 \times \|\sqrt{\hat{\rho}_2}\sqrt{\hat{\sigma}_2}\|^2$$

This property is not shared by the trace distance.

Applications



$$F(|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|) \leq F(\ \chi(|\phi\rangle\langle\phi|),\ \chi(|\psi\rangle\langle\psi|)\ )$$
$$= F(|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|)F(\hat{\rho}, \hat{\sigma})$$

$$|\langle\phi|\psi\rangle|^2\,[1 - F(\hat{\rho}, \hat{\sigma})] \leq 0$$

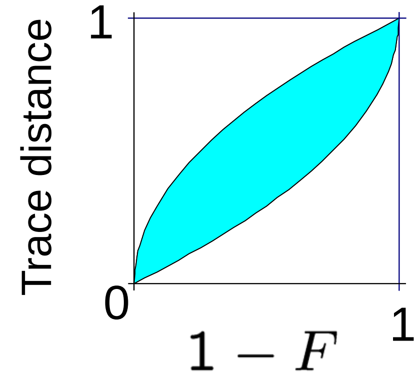
$$|\langle\phi|\psi\rangle| > 0 \longrightarrow \hat{\rho} = \hat{\sigma}$$

Basic principle for a quantum cryptography scheme called B92 protocol.

$$\begin{cases} \rho = |\phi\rangle\langle\phi| \\ \sigma = |\psi\rangle\langle\psi| \end{cases} \longrightarrow \langle\phi|\psi\rangle = 0, 1$$

No-cloning theorem

Fidelity and trace distance



$$1-\sqrt{F(\hat{\rho},\hat{\sigma})} \leq \frac{1}{2}\|\hat{\rho}-\hat{\sigma}\| \leq \sqrt{1-F(\hat{\rho},\hat{\sigma})}$$

Proof:

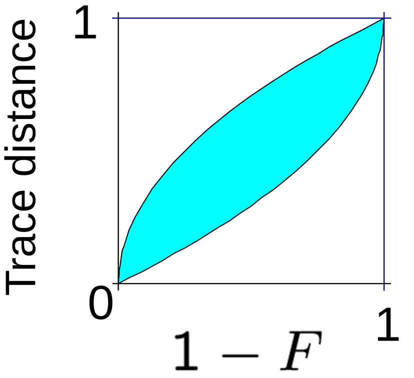
There exists a measurement that preserves the fidelity:

$$\sqrt{F(\hat{\rho},\hat{\sigma})} = \sum_j \sqrt{p_\rho(j)}\sqrt{p_\sigma(j)}$$

$$\begin{aligned}
\frac{1}{2}\|\hat{\rho}-\hat{\sigma}\| &\geq \frac{1}{2}\sum_j |p_\rho(j)-p_\sigma(j)| \\
&= \frac{1}{2}\sum_j \left|\sqrt{p_\rho(j)}-\sqrt{p_\sigma(j)}\right|\left(\sqrt{p_\rho(j)}+\sqrt{p_\sigma(j)}\right) \\
&\geq \frac{1}{2}\sum_j \left|\sqrt{p_\rho(j)}-\sqrt{p_\sigma(j)}\right|^2 = 1-\sum_j \sqrt{p_\rho(j)}\sqrt{p_\sigma(j)} \\
&= 1-\sqrt{F(\hat{\rho},\hat{\sigma})}
\end{aligned}$$

Fidelity and trace distance

$$1-\sqrt{F(\hat{\rho},\hat{\sigma})}\le\frac{1}{2}\|\hat{\rho}-\hat{\sigma}\|\le\sqrt{1-F(\hat{\rho},\hat{\sigma})}$$



Proof:

There exists a pair of purifications satisfying

$$F(\hat{\rho},\hat{\sigma})=|\langle\tilde{\phi}_\rho|\tilde{\phi}_\sigma\rangle|^2\equiv s^2$$

$$\frac{1}{2}\left\||\tilde{\phi}_\rho\rangle\langle\tilde{\phi}_\rho|-|\tilde{\phi}_\sigma\rangle\langle\tilde{\phi}_\sigma|\right\|=\sqrt{1-s^2}=\sqrt{1-F(\hat{\rho},\hat{\sigma})}$$

Consider the quantum operation of discarding the subsystem used for purifying.

$$|\tilde{\phi}_\rho\rangle\to\hat{\rho}$$
$$|\tilde{\phi}_\sigma\rangle\to\hat{\sigma}$$

$$\frac{1}{2}\left\||\tilde{\phi}_\rho\rangle\langle\tilde{\phi}_\rho|-|\tilde{\phi}_\sigma\rangle\langle\tilde{\phi}_\sigma|\right\|\ge\frac{1}{2}\|\hat{\rho}_0-\hat{\rho}_1\|$$

5. Communication resources

- Classical channel
- Quantum channel
- Entanglement
  - How does the state evolve under LOCC?
  - Properties of maximally entangled states
  - Bell basis
- Quantum dense coding
- Quantum teleportation
  - Entanglement swapping
- Resource conversion protocols and bounds

Classical channel

$$\begin{matrix}0\\1\end{matrix} \dashrightarrow \begin{matrix}0\\1\end{matrix}$$

Ideal classical channel: faithful transfer of any signal chosen from d symbols

Parallel use of channels

$$\left.\begin{matrix}\begin{matrix}0\\1\end{matrix} \dashrightarrow \begin{matrix}0\\1\end{matrix}\\ \\ \begin{matrix}a\\b\\c\end{matrix} \dashrightarrow \begin{matrix}a\\b\\c\end{matrix}\end{matrix}\right\} \quad \begin{matrix}0a\\0b\\0c\\1a\\1b\\1c\end{matrix} \dashrightarrow \begin{matrix}0a\\0b\\0c\\1a\\1b\\1c\end{matrix}$$

$$\left.\begin{matrix}\text{d-symbol ideal classical channel}\\ \text{d'-symbol ideal classical channel}\end{matrix}\right\} \quad \text{(dd')-symbol ideal classical channel}$$

Measure of usefulness

d-symbol ideal classical channel $\longrightarrow$ (log d) bits

Additive for ideal channels

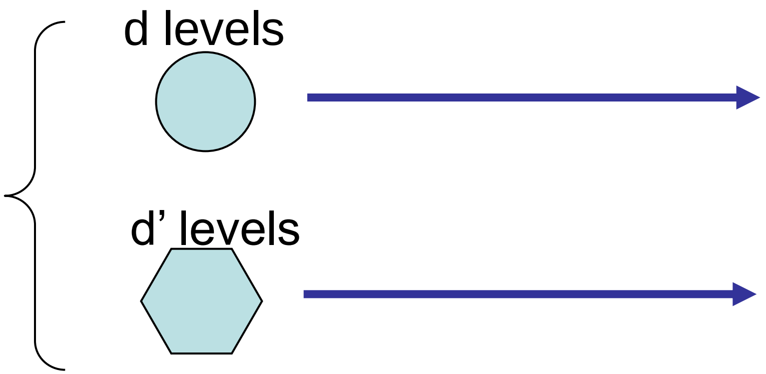
Quantum channel

$\alpha|0\rangle + \beta|1\rangle$ $\alpha|0\rangle + \beta|1\rangle$

Ideal quantum channel: faithful transfer of any state of an d-level system
(Hilbert space of dimension d)

Parallel use of channels

(dd')-level system



Measure of usefulness

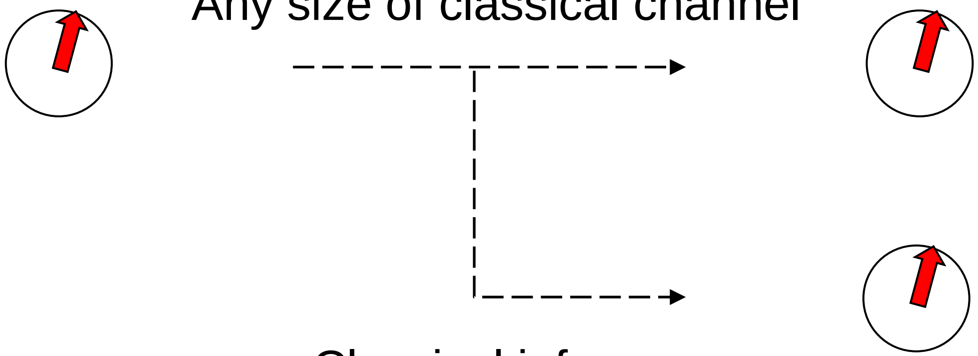
d-level ideal quantum channel $\longrightarrow$ (log d) qubits
Additive for ideal channels

Can classical channels substitute a quantum channel?

NO (with no other resources)

Suppose that it was possible …

Any size of classical channel



Classical info
can be copied

The same procedure should
result in the same state.

This amounts to the cloning of unknown quantum states,
which is forbidden.

Can a quantum channel substitute a classical channel?

Of course yes.

But not so bizarre (with no other resources).

> n-qubit ideal quantum channel can only substitute a n-bit classical channel.

(Holevo bound)

Suppose that transfer of an d-level system can convey any signal from s symbols faithfully.

Measurement

$$j = 1, 2, \ldots, s \qquad \hat{\rho}_j \longrightarrow \hat{\rho}_j \longrightarrow j'$$

$$\dim \mathcal{H} = d$$

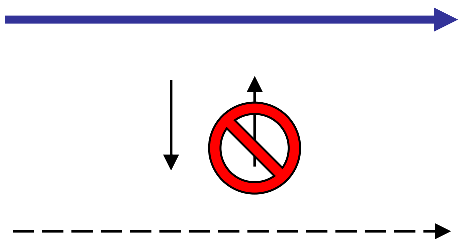
Always $j' = j$

Recall that any measurement must be described by a POVM.

$$\mathrm{Tr}\,(\hat{F}_j \hat{\rho}_j) = 1 \qquad \sum_{j'=1}^{s} \hat{F}_{j'} \leq \hat{1}$$

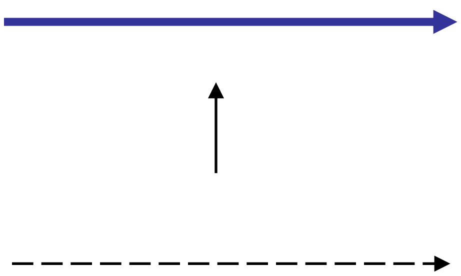
$$s = \sum_{j=1}^{s} \mathrm{Tr}\,(\hat{F}_j \hat{\rho}_j) \leq \sum_{j=1}^{s} \mathrm{Tr}\,(\hat{F}_j \hat{1}) = \mathrm{Tr}\left(\sum_{j=1}^{s} \hat{F}_j\right) \leq \mathrm{Tr}\,\hat{1} = d$$

Difference between quantum and classical channels



We have seen that a quantum channel is more powerful than a classical channel.

Can we pin down what is missing in a classical channel?



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Other resource

Entanglement

I’ve already bought a classical channel, but now I want to use a quantum channel. Do I have to buy the quantum channel?

Oh, you can buy this optional package for a cheaper price, and upgrade the classical channel to a quantum channel!

Operational definition of entanglement

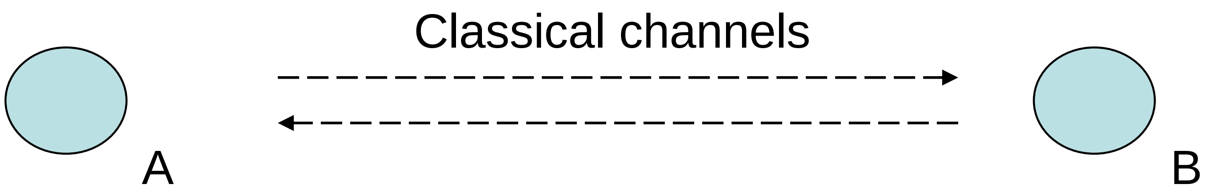
"Correlations that cannot be created over classical channels"

LOCC: Local operations and classical communication

Alice has a subsystem A, and Bob has a subsystem B.

Operations (including measurements) on a local subsystem are allowed.

Communication between Alice and Bob only uses classical channels.



Separable states: The states that can be created under LOCC from scratch.

Entangled states: The states that cannot be created under LOCC from scratch.

Entangled states and separable states

$|\phi\rangle_A \otimes |\psi\rangle_B$ $\qquad$ $\sum_k \alpha_k |\phi_k\rangle_A \otimes |\psi_k\rangle_B$

Separable states $\qquad$ Entangled states

Are there any procedure to distinguish between the two classes?

$\longrightarrow$ Schmidt decomposition $|\Phi\rangle_{AB} = \sum_{i=1}^{s} \sqrt{p_i} |a_i\rangle_A |b_i\rangle_B$

$p_1 \geq p_2 \geq \cdots \geq p_s > 0$

Schmidt number

Number of nonzero coefficients in Schmidt decomposition

= The rank of the marginal density operators

$\{p_j\}$: The eigenvalues of the marginal density operators (the same for A and B)

'Symmetry' between A and B

$\hat{\rho}_A, \hat{\rho}_B$ The same set of eigenvalues

$s = \mathrm{Rank}(\hat{\rho}_A) = \mathrm{Rank}(\hat{\rho}_B)$

Separable states $\qquad$ Schmidt number = 1

$p_1 = 1$

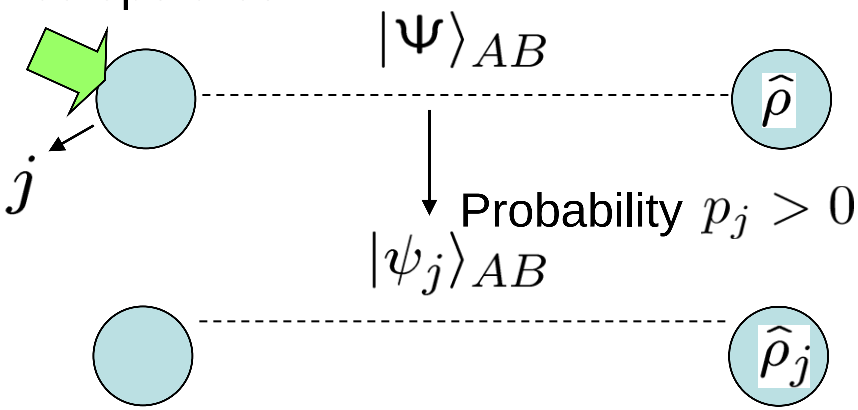
Entangled states $\qquad$ Schmidt number > 1

$p_1 \geq p_2 > 0$

How does the state evolve under LOCC?

Any LOCC procedure can be made a sequential one:
- Alice appies local operations
- Alice communicates to Bob
- Bob applies local operations
- Bob communicates to Alice
- Alice …..

When Alice operates



$$\boxed{\sum_j p_j \hat{\rho}_j = \hat{\rho}}$$

$$\hat{\rho} \geq p_j \hat{\rho}_j$$

$$\mathrm{Ran}(\hat{\rho}) \supset \mathrm{Ran}(\hat{\rho}_j)$$

$$\mathrm{Rank}(\hat{\rho}) \geq \mathrm{Rank}(\hat{\rho}_j)$$

Schmidt number never increases under LOCC (even probabilistically)

Schmidt number >1 $\longrightarrow$ Impossible to create under LOCC

If a concave functional S only depends on the eigenvalues,

$$S(\hat{\rho}) \geq \sum_j p_j S(\hat{\rho}_j)$$

Any such functional of the marginal density operator (e.g., von Neumann entropy) is monotone decreasing under LOCC on average.

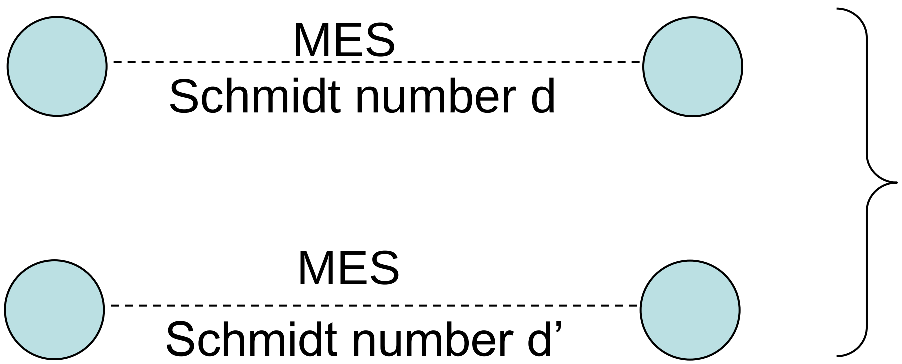
Maximally entangled states (MES)

"ideal" entangled states

$$\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

An MES with Schmidt number d

Putting two MESs together



MES with

Schmidt number dd'

$$\left(\sum_{j=1}^{d} \frac{1}{\sqrt{d}} |j\rangle_A \otimes |j\rangle_B \right) \otimes \left(\sum_{k=1}^{d'} \frac{1}{\sqrt{d'}} |k\rangle_{A'} \otimes |k\rangle_{B'} \right) = \sum_{j,k} \frac{1}{\sqrt{dd'}} |jk\rangle_{AA'} \otimes |jk\rangle_{BB'}$$

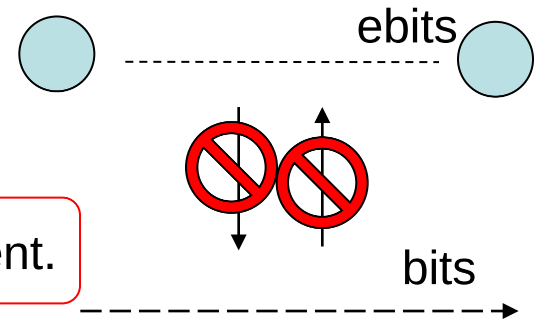
Measure of entanglement

MES with Schmidt number d $\longrightarrow$ (log d) ebits

$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$ 1ebit

Additive for MESs

Ebits and bits are mutually exclusive



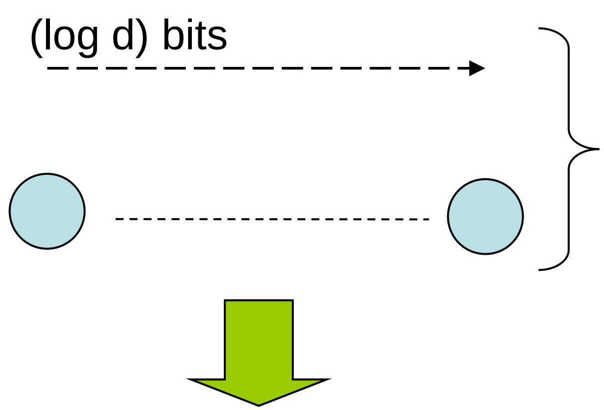
Schmidt number never increases under LOCC.

Classical channels cannot increase (ideal) entanglement.

d-symbol ideal classical channel

The outcome can be correctly predicted with probability at least 1/d.

Transfer of s symbols



Success probability 1

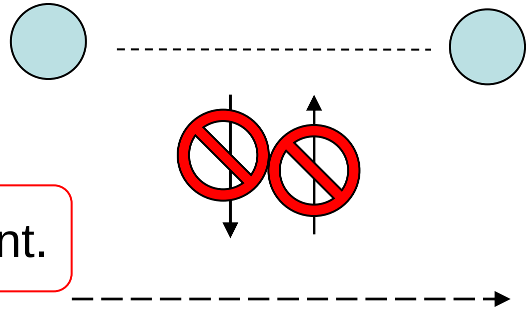
Transfer of s symbols

Success probability 1/d

$\longrightarrow \quad s \leq d$

Entanglement cannot assist (ideal) classical channels

Ebits and bits are mutually exclusive



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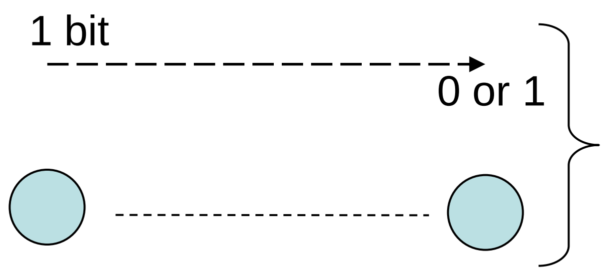
The outcome can be correctly predicted with probability at least 1/d.



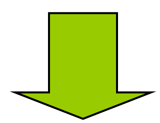
Transfer of 3 symbols

♥ ♦ ♠

Which is on my mind?

Success probability 1

That must be ♥!!



Transfer of 3 symbols

♥ ♦ ♠

Which is on my mind?

Success probability 1/2

That must be ♥!!

Entanglement cannot assist (ideal) classical channels

Resource conversion protocols

Conversion to ebits

Entanglement sharing

1 qubit ⟶ 1 ebit

Conversion to bits

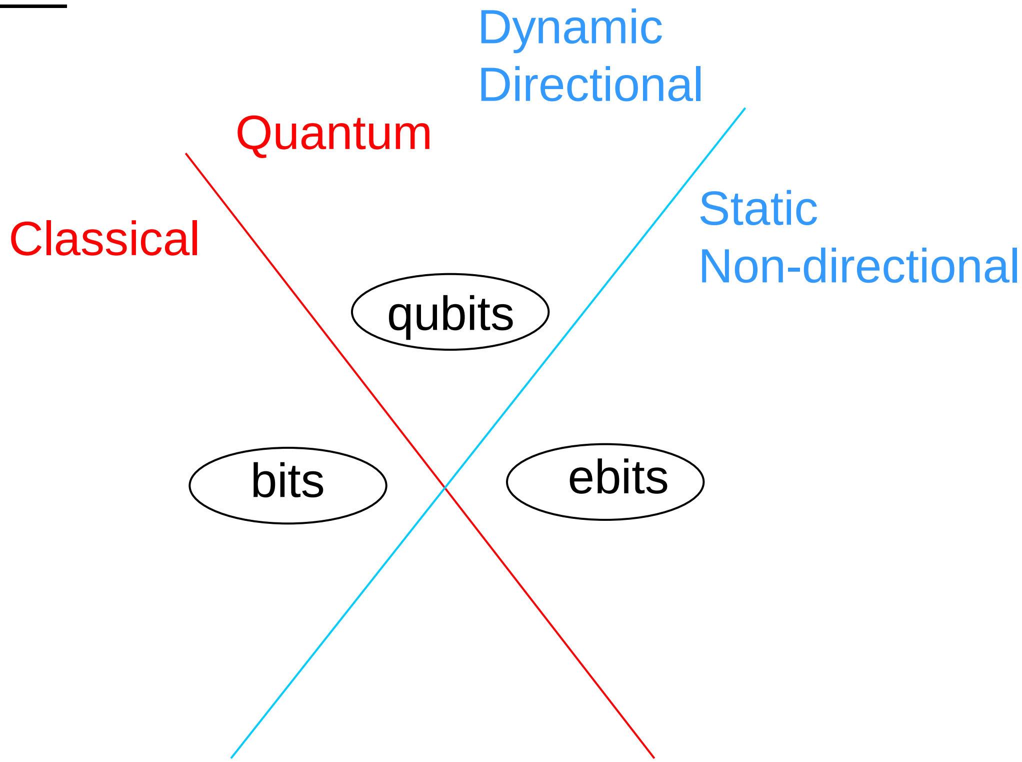
Quantum dense coding

1 qubit + 1 ebit ⟶ 2 bits

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit ⟶ 1 qubit



Restrictions

bits alone ⟶ no ebits

ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit

Properties of maximally entangled states $|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$

(I) Convertibility via local unitary

$$\boxed{|\Phi'\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Phi\rangle_{AB}}$$

(II) Pair of local states (relative states)

$$\boxed{\frac{1}{\sqrt{d}}|\phi\rangle_A = {}_B\langle\phi^*||\Phi\rangle_{AB}}$$

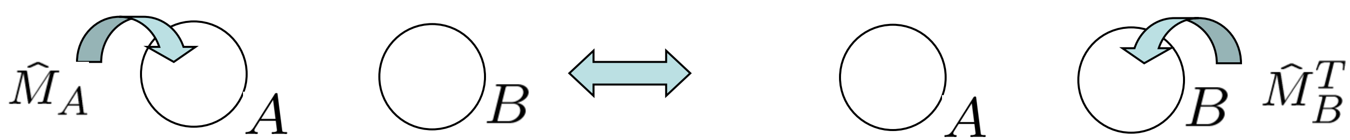
$|\phi\rangle_A = \sum_k \alpha_k |k\rangle_A$ ← A

B measurement → $|\phi^*\rangle_B = \sum_k \overline{\alpha_k}|k\rangle_B$

$p = 1/d$

(III) Pair of local operations

$$\boxed{(\hat{M}_A \otimes \hat{1}_B)|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{M}_B^T)|\Phi\rangle_{AB}}$$



(IV) Orthonormal basis (Bell basis)

$$\boxed{\langle\Phi_j|\Phi_k\rangle = \delta_{jk}\ (j,k = 1,\ldots d^2)}$$

There exists an orthonormal basis composed of MESs.

Bell basis for a pair of qubits

$(d = 2)$

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B) = \hat{Z}_B|\Phi_+\rangle$$

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B) = \hat{X}_A|\Phi_+\rangle$$

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B) = (\hat{X}_A \otimes \hat{Z}_B)|\Phi_+\rangle$$

$$\hat{X} \equiv \hat{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$\hat{Z} \equiv \hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Bell basis

$$\beta \equiv \exp(2\pi i/d) \qquad (\beta^d = 1, \beta^{-1} = \overline{\beta})$$

Basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\} \quad (|d\rangle \equiv |0\rangle)$

$$\hat{X} \equiv \sum_{j=0}^{d-1} |j+1\rangle\langle j| \qquad \hat{Z} \equiv \sum_{j=0}^{d-1} \beta^j |j\rangle\langle j| \qquad \text{(Unitary)}$$

$$\hat{X}^T = \hat{X}^{-1} \qquad \hat{Z}^T = \hat{Z}$$

$\hat{Z}^d = \hat{X}^d = \hat{1}$ Eigenvalues: $1, \beta, \beta^2, \ldots, \beta^{d-1}$

$$\hat{Z}\hat{X} = \beta\hat{X}\hat{Z} \qquad \hat{Z}^m\hat{X}^l = \beta^{lm}\hat{X}^l\hat{Z}^m$$

$$|\Phi_{0,0}\rangle \equiv \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle_A \otimes |k\rangle_B$$

$$(\hat{X}_A \otimes \hat{X}_B)|\Phi_{0,0}\rangle = |\Phi_{0,0}\rangle$$
$$(\hat{Z}_A \otimes \hat{Z}_B^{-1})|\Phi_{0,0}\rangle = |\Phi_{0,0}\rangle$$

Bell basis: $\{|\Phi_{l,m}\rangle\}$ $(l = 0, 1, \ldots, d-1;\ m = 0, 1, \ldots, d-1)$

$$|\Phi_{l,m}\rangle \equiv (\hat{X}_A^l \otimes \hat{Z}_B^m)|\Phi_{0,0}\rangle$$

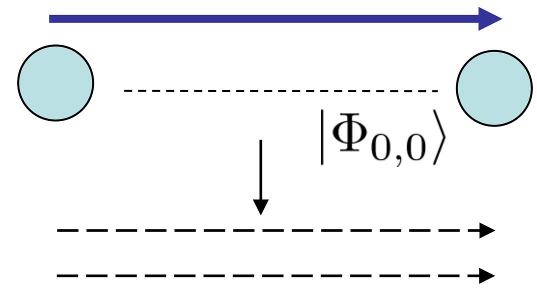
$$\left.\begin{array}{l} (\hat{X}_A \otimes \hat{X}_B)|\Phi_{l,m}\rangle = \beta^{-m}|\Phi_{l,m}\rangle \\ (\hat{Z}_A \otimes \hat{Z}_B^{-1})|\Phi_{l,m}\rangle = \beta^{l}|\Phi_{l,m}\rangle \end{array}\right\} \longrightarrow \text{All states are orthogonal.}$$

Quantum dense coding

1 qubit + 1 ebit $\longrightarrow$ 2 bits

n qubits + n ebits $\longrightarrow$ 2n bits

(Dimension d) + (Schmidt number d)
$\rightarrow$ (d^2 symbols)

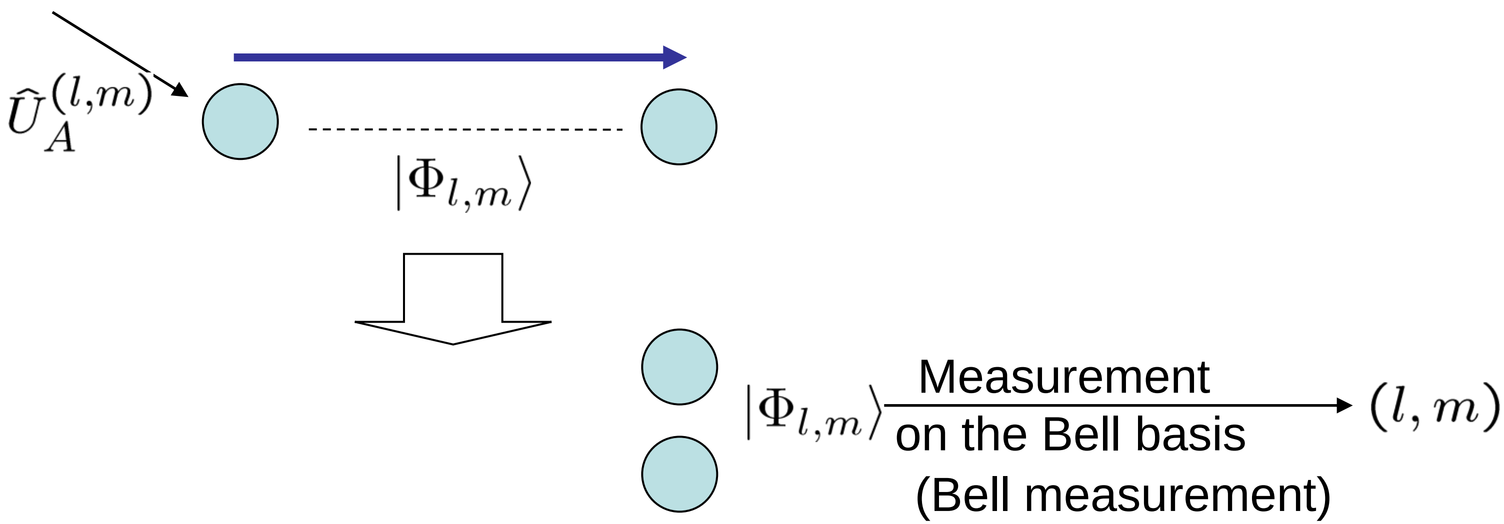


MES

Convertibility via local unitary

Orthonormal basis (Bell basis)

d^2 symbols (l, m)

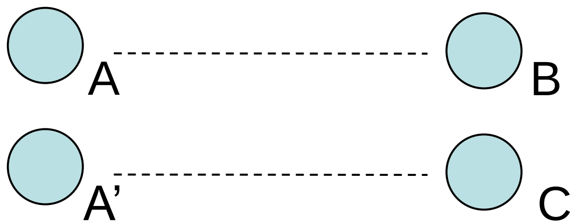


Creating entanglement by nonlocal measurement

$$|\Phi_{0,0}\rangle_{AB} \otimes |\Phi_{0,0}\rangle_{A'C} = \sum_{j,k} \frac{1}{\sqrt{d^2}} |jk\rangle_{AA'} \otimes |jk\rangle_{BC}$$

measurement

$|\Psi\rangle_{AA'} \longleftarrow$



Relative state of $|\Psi\rangle_{AA'}$

$\dashrightarrow |\Psi^*\rangle_{BC}$

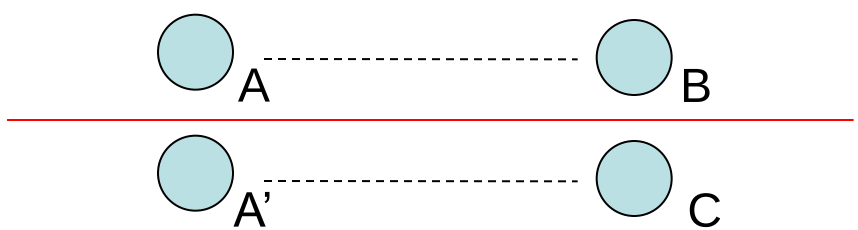
(More precisely, obtaining an outcome corresponding to a POVM element $\mu|\Psi\rangle\langle\Psi|$)

$\left.\begin{matrix} |\Psi\rangle_{AA'} \\ |\Psi^*\rangle_{BC} \end{matrix}\right\}$ The same entanglement

$$\hat{\rho}_A \equiv \mathrm{Tr}_{A'}\, |\Psi\rangle\langle\Psi|$$
$$\hat{\rho}_B \equiv \mathrm{Tr}_{C}\, |\Psi^*\rangle\langle\Psi^*|$$
$$_A\langle j|\hat{\rho}_A|j'\rangle_A = \overline{_B\langle j|\hat{\rho}_B|j'\rangle_B}$$

$\hat{\rho}_A,\ \hat{\rho}_B$:the same set of eigenvalues

When $|\Psi\rangle_{AA'}$ is an entangled state, (e.g., Bell measurement)



Initially no entanglement $\longrightarrow$ entangled $|\Psi^*\rangle_{BC}$

Such a measurement cannot be implemented over LOCC.

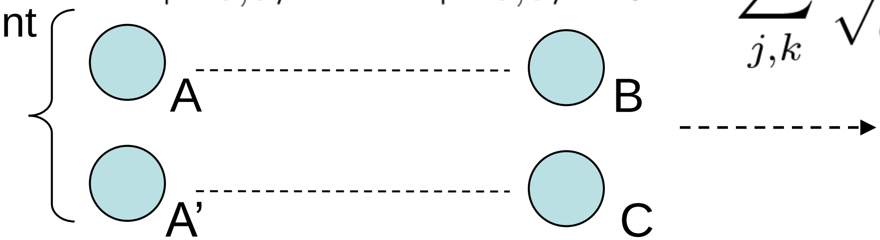
Entanglement swapping

$$|\Phi_{0,0}\rangle \equiv \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle \otimes |k\rangle$$

$$|\Phi_{0,0}\rangle_{AB} \otimes |\Phi_{0,0}\rangle_{A'C} = \sum_{j,k} \frac{1}{\sqrt{d^2}} |jk\rangle_{AA'} \otimes |jk\rangle_{BC}$$

Bell measurement

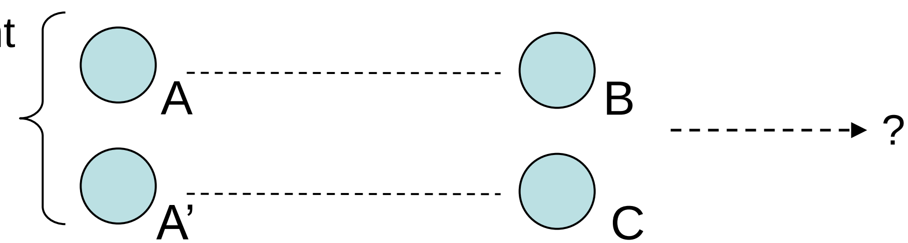
$|\Phi_{0,0}\rangle_{AA'}$



$|\Phi_{0,0}\rangle_{BC}$

Bell measurement

$|\Phi_{l,m}\rangle_{AA'}$

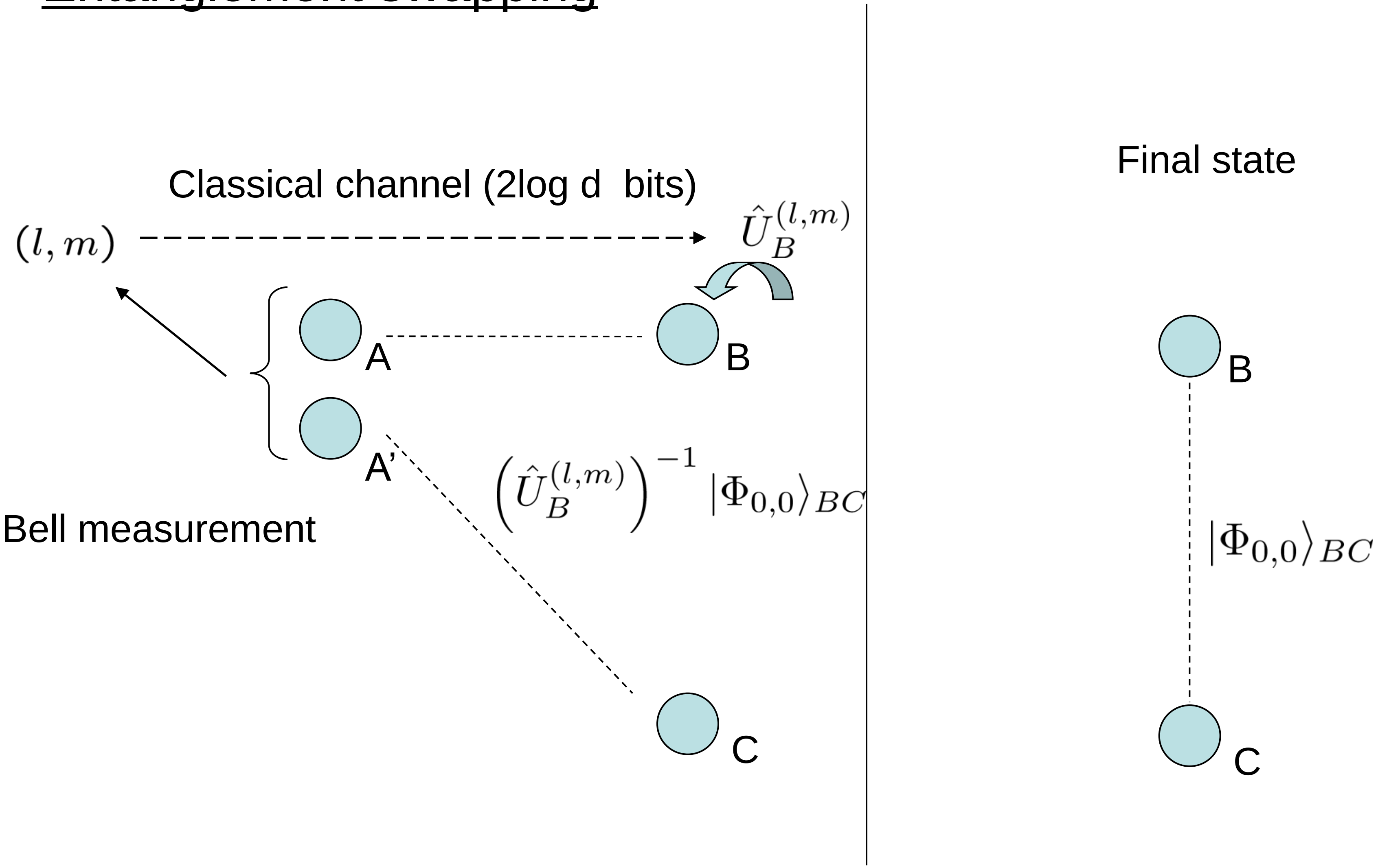


A maximally entangled state

It should be written as

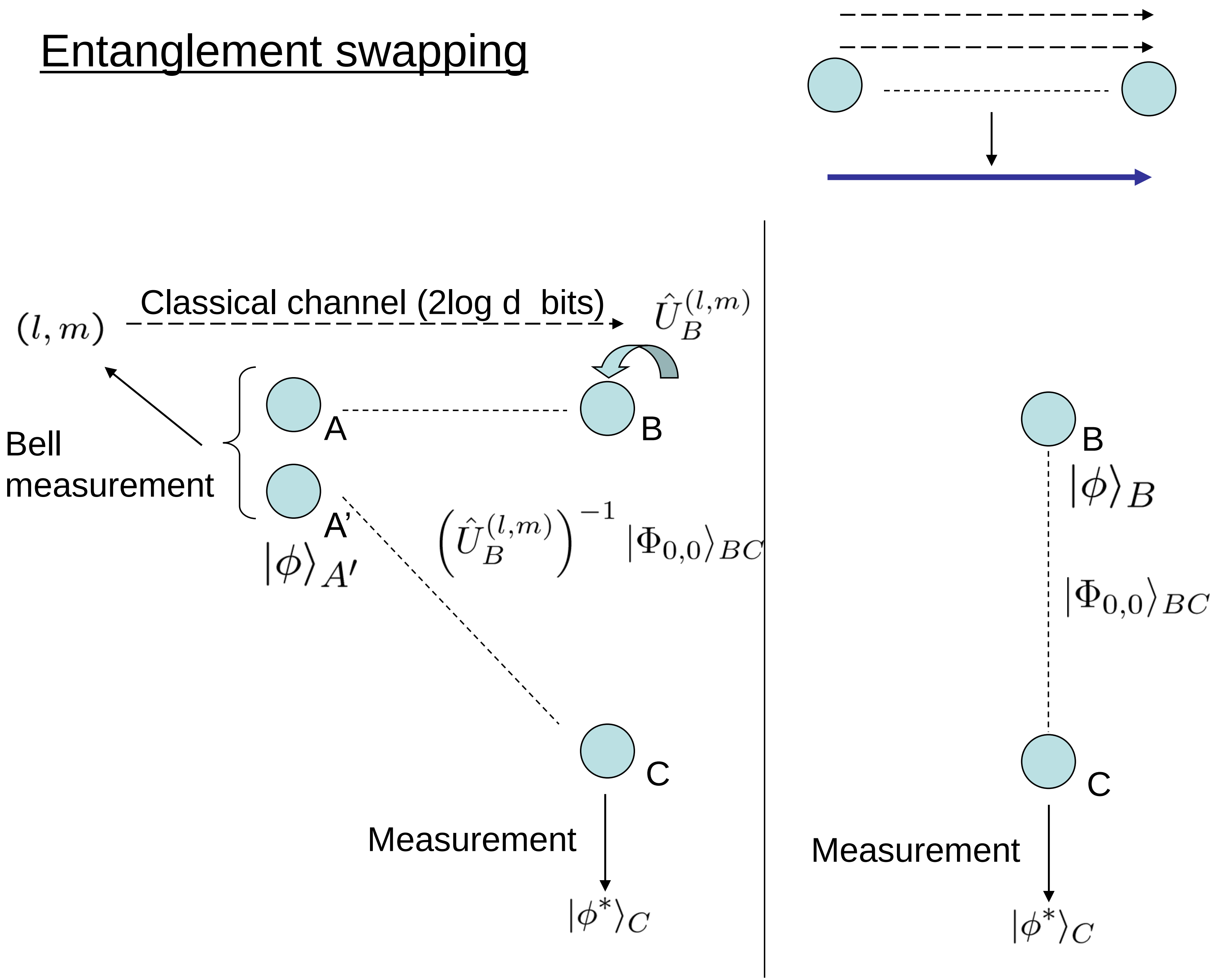
$$\left(\hat{U}_B^{(l,m)}\right)^{-1} |\Phi_{0,0}\rangle_{BC}$$

Entanglement swapping



It is possible to creating entanglement over two subsystems without letting them directly interacted to each other.

Entanglement swapping



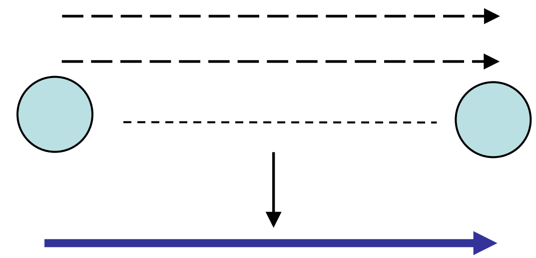
Quantum teleportation

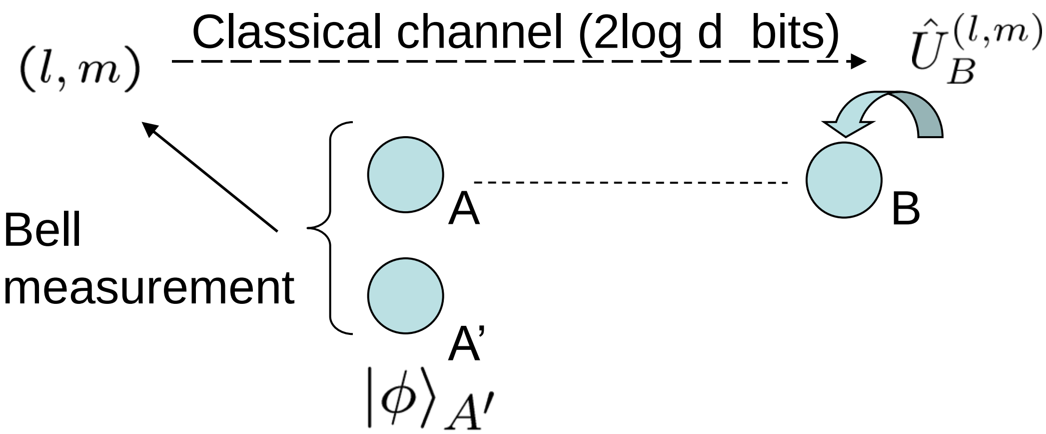
1 ebit + 2 bit $\longrightarrow$ 1 qubit

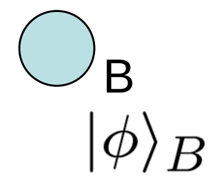
n ebits + 2n bits $\longrightarrow$ n qubits

(d^2 symbols) + (Schmidt number d)

$\rightarrow$ (Dimension d)

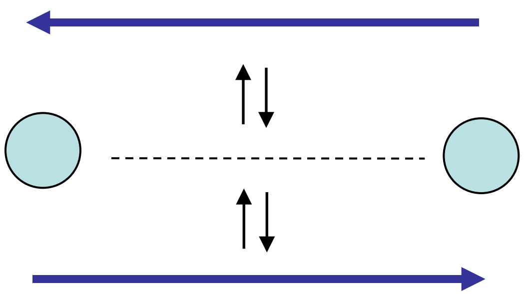






Quantum teleportation

If the cost of classical communication is neglected …



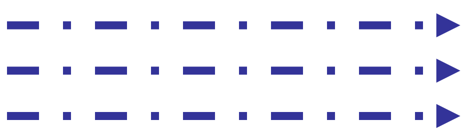
One can reserve the quantum channel by storing a quantum state.

One can use a quantum channel in the opposite direction.

A convenient way of quantum error correction (failure $\longrightarrow$ retry).

Noisy quantum channel

$|\phi\rangle$



$\hat{\rho} \rightarrow |\phi\rangle$

Recovering

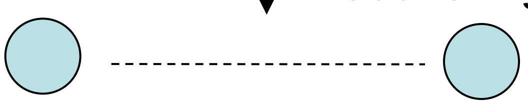
Failure $\longrightarrow$ no recovery.

Noisy quantum channel

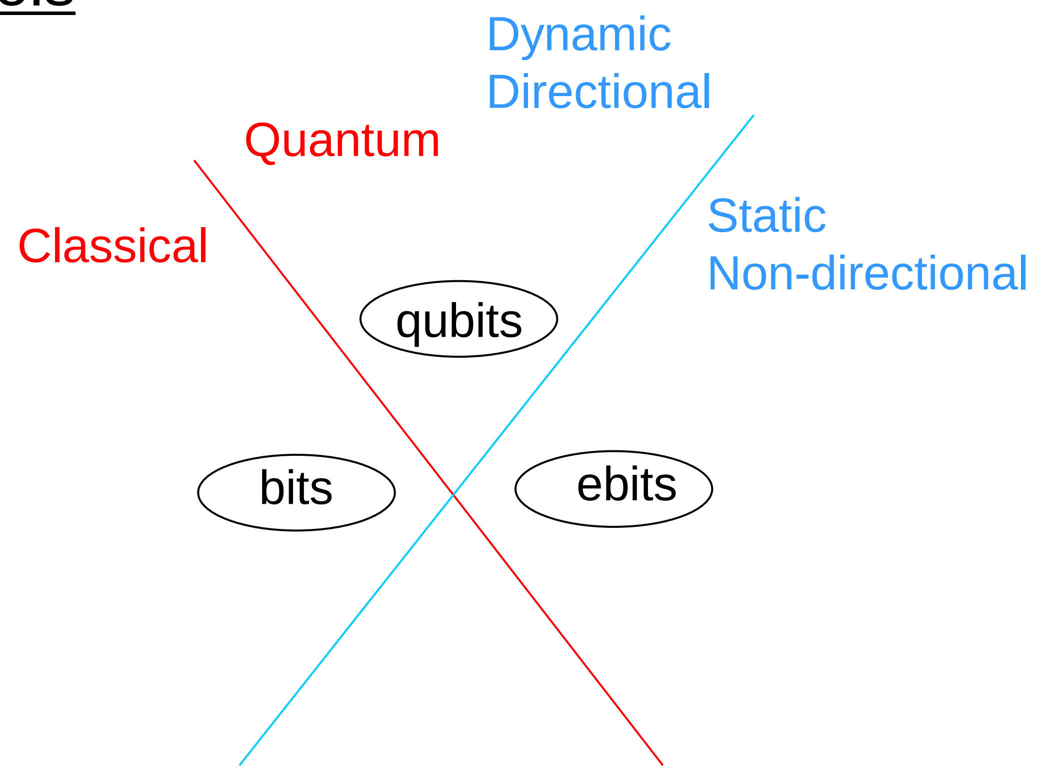
$\downarrow$

Noisy entanglement

$\downarrow$ Recovering



Resource conversion protocols



Conversion to ebits

Entanglement sharing

1 qubit $\longrightarrow$ 1 ebit

Conversion to bits

Quantum dense coding

1 qubit + 1 ebit $\longrightarrow$ 2 bits

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit $\longrightarrow$ 1 qubit

Restrictions

bits alone $\longrightarrow$ no ebits

ebits alone $\longrightarrow$ no bits

1 qubit alone $\longrightarrow$ no more than 1 bit

Resource conversion protocols and bounds

We can do the following…

Conversion to ebits

Entanglement sharing

1 qubit $\longrightarrow$ 1 ebit

$$(\Delta q, \Delta e, \Delta c) = (-1, 1, 0)$$

Conversion to bits

Quantum dense coding

1 qubit + 1 ebit $\longrightarrow$ 2 bits

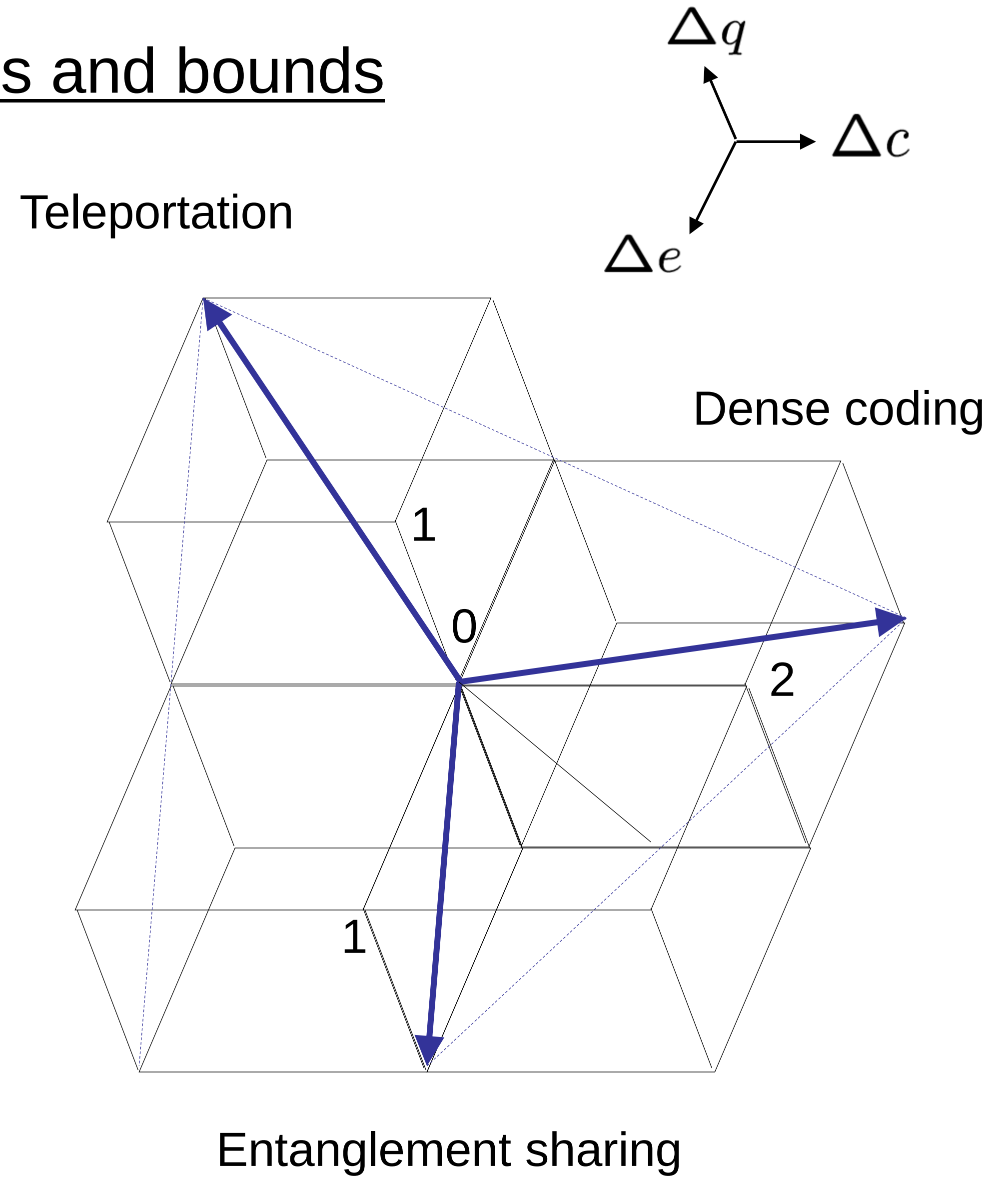
$$(\Delta q, \Delta e, \Delta c) = (-1, -1, 2)$$

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit $\longrightarrow$ 1 qubit

$$(\Delta q, \Delta e, \Delta c) = (1, -1, -2)$$



Resource conversion protocols and bounds

We can do the following…

Restrictions

bits alone ⟶ no ebits

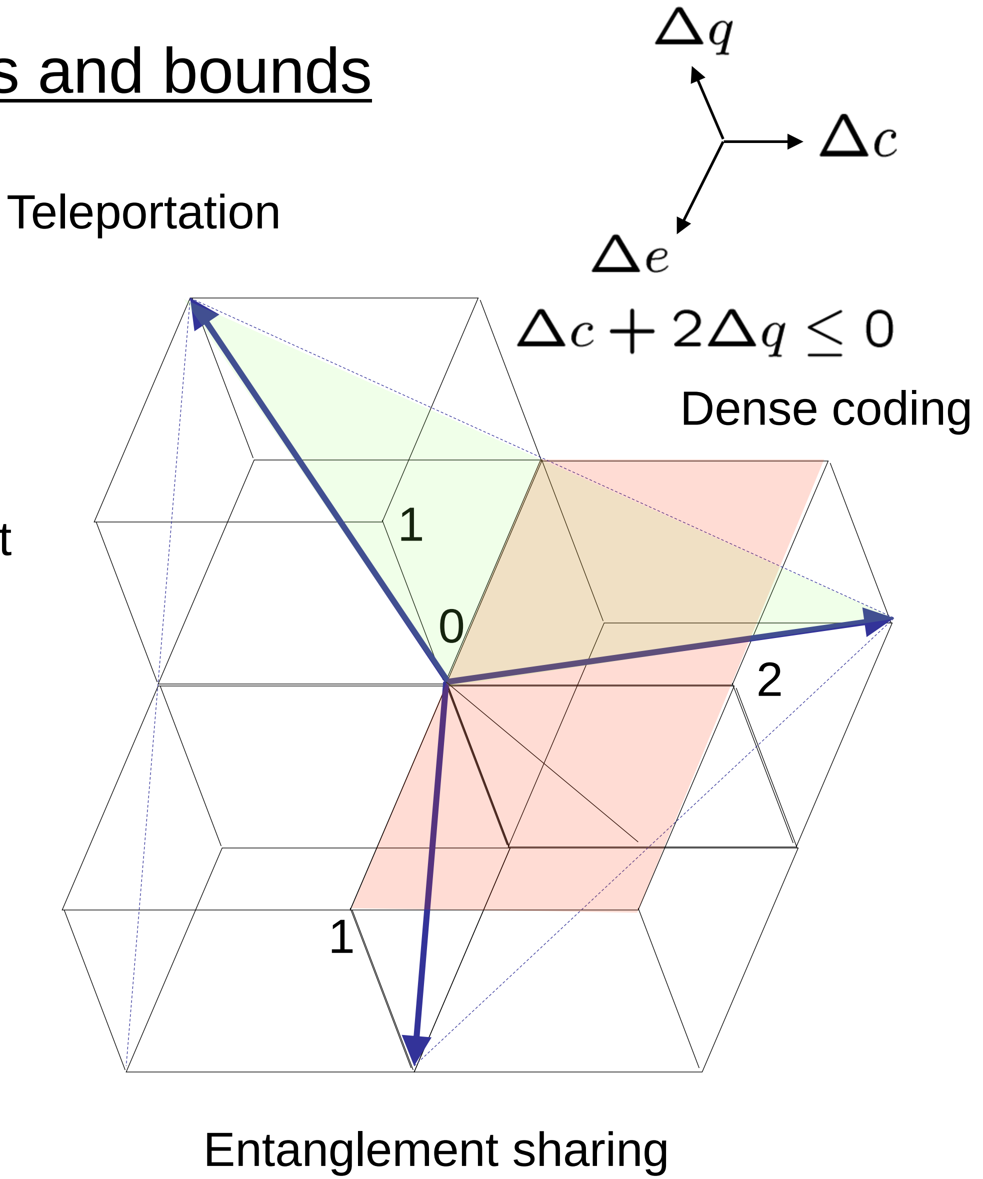
ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit

•The red region should be unreachable.

•From a point above the blue plane, the red region is accessible through 'Teleportation' and 'Dense coding.'

The region above the blue plane should be unreachable.



Resource conversion protocols and bounds

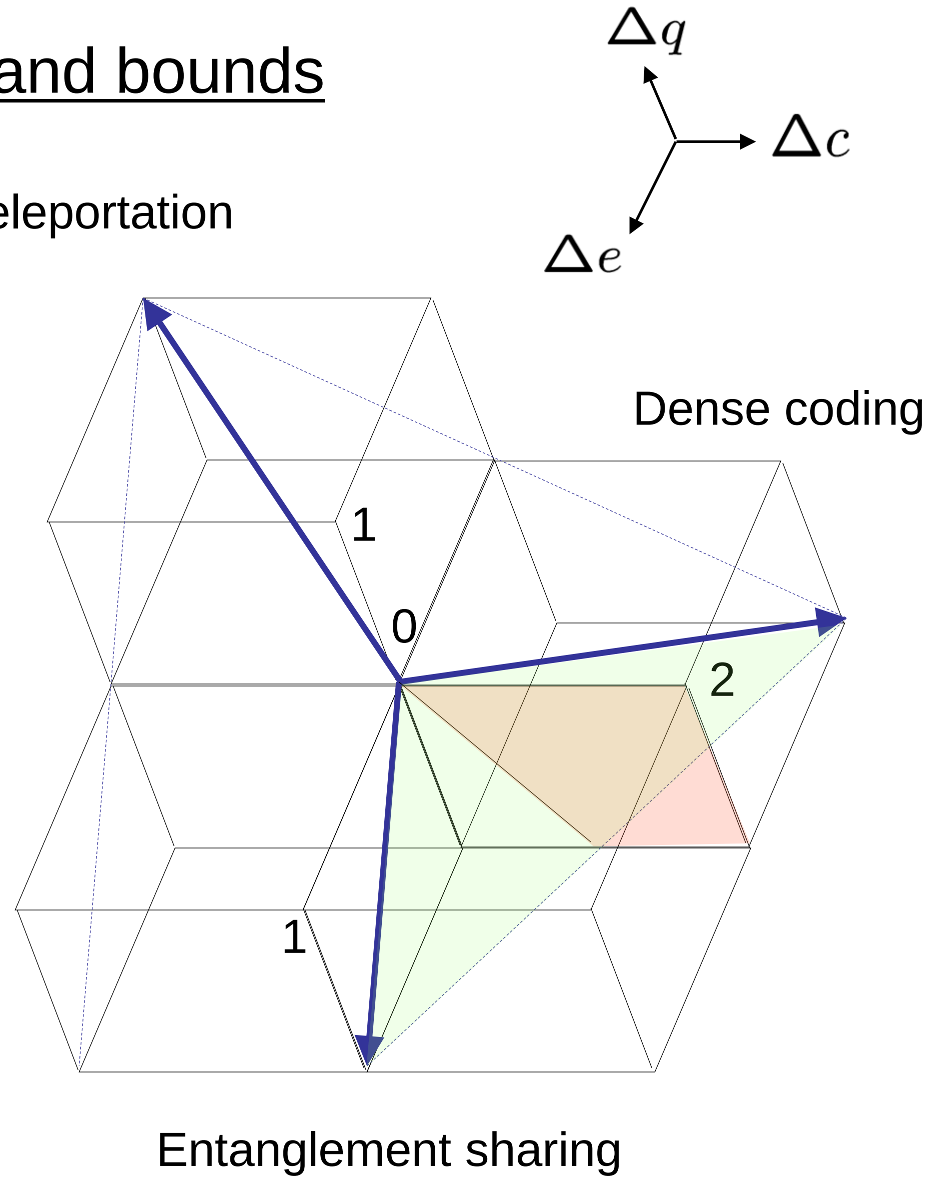
We can do the following…

Restrictions

bits alone $\longrightarrow$ no ebits

ebits alone $\longrightarrow$ no bits

1 qubit alone $\longrightarrow$ no more than 1 bit



$$\Delta c + \Delta q + \Delta e \leq 0$$

•The red region should be unreachable.

•From a point above the blue plane, the red region is accessible through 'Dense coding' and 'Entanglement sharing.'

⇨ The region above the blue plane should be unreachable.

Resource conversion protocols and bounds

We can do the following…

Conversion to ebits

Entanglement sharing (ES)

1 qubit $\longrightarrow$ 1 ebit

$$(\Delta q, \Delta e, \Delta c) = (-1, 1, 0)$$

Conversion to bits

Quantum dense coding (QD)

1 qubit + 1 ebit $\longrightarrow$ 2 bits

$$(\Delta q, \Delta e, \Delta c) = (-1, -1, 2)$$

Conversion to qubits

Quantum teleportation (QT)

2 bits + 1 ebit $\longrightarrow$ 1 qubit

$$(\Delta q, \Delta e, \Delta c) = (1, -1, -2)$$

We cannot violate the following …

Entanglement alone never assists classical channels

+ QD,QT

$$\Delta c + 2\Delta q \leq 0$$

Classical channels alone cannot increase entanglement

+ QT,ES

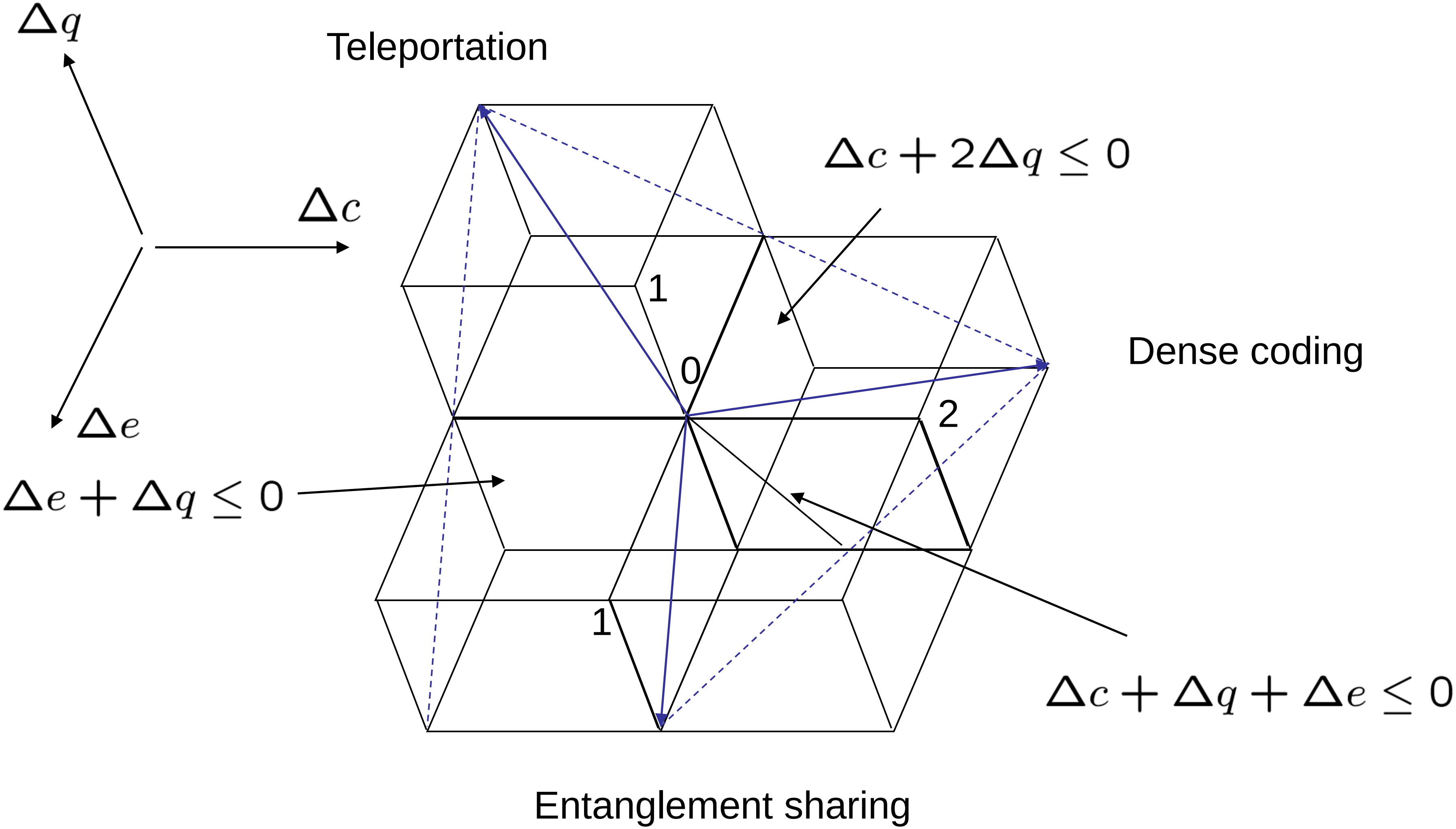
$$\Delta e + \Delta q \leq 0$$

1-qubit channel alone can convey no more than 1 classical bit

+ ES,QD

$$\Delta q + \Delta e + \Delta c \leq 0$$

Resource conversion protocols and bounds



Resource conversion protocols

Conversion to ebits

Entanglement sharing

1 qubit ⟶ 1 ebit

Conversion to bits

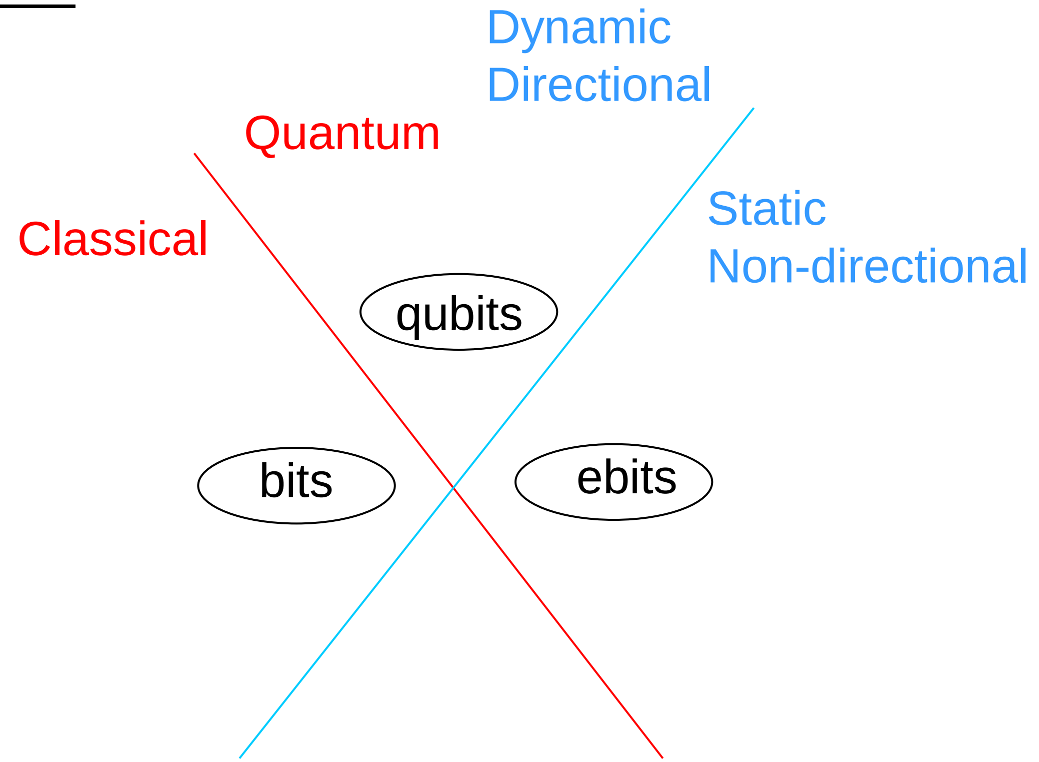
Quantum dense coding

1 qubit + 1 ebit ⟶ 2 bits

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit ⟶ 1 qubit



Restrictions

bits alone ⟶ no ebits

ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit