3. Qubits

Pauli operators (Pauli matrices)

Bloch representation (Bloch sphere)

Orthogonal measurement

Unitary operation
Qubit

\( \dim \mathcal{H} = 2 \)

Take a standard basis \( \{ |0\rangle, |1\rangle \} \)

Linear operator \( \hat{A} \)

Matrix representation (for \( \{ |0\rangle, |1\rangle \} \) )

\[
\hat{A} = \begin{pmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{pmatrix}
\]

\( A_{ij} = \langle i | \hat{A} | j \rangle \)

\( \hat{A} = \sum_{ij} A_{ij} |i\rangle \langle j| \)

4 complex parameters

\[
\hat{A} = \alpha_0 \hat{\sigma}_0 + \alpha_1 \hat{\sigma}_1 + \alpha_2 \hat{\sigma}_2 + \alpha_3 \hat{\sigma}_3
\]
Pauli operators (Pauli matrices)

Take a standard basis \( \{ |0\rangle, |1\rangle \} \)

\[
\hat{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_x = \hat{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
\hat{\sigma}_y = \hat{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \hat{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Unitary and self-adjoint

\[
[\hat{\sigma}_i, \hat{\sigma}_j] = 2i \epsilon_{ijk} \hat{\sigma}_k
\]

\[
\hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i = 2 \delta_{i,j} \hat{1}
\]

\[
\text{Tr}(\hat{\sigma}_i) = 0, \quad \text{Tr}(\hat{\sigma}_i \hat{\sigma}_j) = 2 \delta_{i,j}.
\]

\(i,j = 1,2,3\)

\[
[\hat{\sigma}_x, \hat{\sigma}_y] = 2i \hat{\sigma}_z
\]

\[
\hat{\sigma}_x^2 = \hat{1}
\]

\[
\{ \hat{\sigma}_x, \hat{\sigma}_z \} = \hat{\sigma}_x \hat{\sigma}_z + \hat{\sigma}_z \hat{\sigma}_x = 0
\]

\[
\text{Tr}(\hat{\sigma}_\mu \hat{\sigma}_\nu) = 2 \delta_{\mu,\nu}
\]

\(\mu, \nu = 0, 1, 2, 3; \quad \sigma_0 \equiv \hat{1} \)

'Orthogonality' with respect to

\( (\hat{A}, \hat{B}) \equiv \text{Tr}(\hat{A}^\dagger \hat{B}) \)

Levi-Civita symbol

\[
\begin{cases} 
\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\
\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \\
\text{Otherwise} \ \epsilon_{ijk} = 0
\end{cases}
\]

Einstein notation

\( \sum_k \) is omitted.
Pauli operators (Pauli matrices)

\[
[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k
\]
\[
\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{i,j}\hat{1}
\]
\[
\text{Tr}(\hat{\sigma}_i) = 0, \quad \text{Tr}(\hat{\sigma}_i\hat{\sigma}_j) = 2\delta_{i,j}.
\]

Linear operator \( \hat{A} \) 4 complex parameters \((P_0, P_x, P_y, P_z)\)

\[
\hat{A} = \frac{1}{2} \left( P_0\hat{1} + \mathbf{P} \cdot \hat{\mathbf{\sigma}} \right) = \frac{1}{2} \left( \begin{array}{cc} P_0 + P_z & P_x - iP_y \\ P_x + iP_y & P_0 - P_z \end{array} \right)
\]

\[
\mathbf{P} = (P_x, P_y, P_z)
\]
\[
\hat{\mathbf{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)
\]

\[
P_0 = \text{Tr}(\hat{A}) \quad \mathbf{P} = \text{Tr}(\hat{\mathbf{\sigma}}\hat{A})
\]
Pauli operators (Pauli matrices)

\[
\hat{A} = \frac{1}{2} \left( P_0 \hat{1} + \mathbf{P} \cdot \hat{\sigma} \right) = \frac{1}{2} \begin{pmatrix}
P_0 + P_z & P_x - iP_y \\
P_x + iP_y & P_0 - P_z
\end{pmatrix}
\]

\(\hat{A}\) is self-adjoint. \(\iff\) \(P_0\) and \(\mathbf{P}\) are real.

Eigenvalues \(\lambda_+,\lambda_-\)

\[
\det(\hat{A}) = \lambda_+\lambda_- = \frac{1}{4}(P_0^2 - |\mathbf{P}|^2)
\]

\[
\text{Tr}(\hat{A}) = \lambda_+ + \lambda_- = P_0
\]

\[
\downarrow
\]

\[
\lambda_\pm = \frac{(P_0 \pm |\mathbf{P}|)}{2}
\]

\(\hat{A}\) is positive. \(\iff\) \(P_0\) and \(\mathbf{P}\) are real, \(P_0 \geq |\mathbf{P}|\)
Bloch representation (Bloch sphere)

Density operator: \( \hat{\rho} = \frac{1}{2} \left( \hat{1} + \mathbf{P} \cdot \hat{\sigma} \right) \)

Positive & Unit trace: \( P_0 \geq |\mathbf{P}| \quad P_0 = 1 \)

Density operator for a qubit system:

A 3D real vector of length no greater than 1

A point inside or on the sphere of radius 1

\( \mathbf{P} = (P_x, P_y, P_z) \)

Bloch vector

\( \lambda_{\pm} = (P_0 \pm |\mathbf{P}|)/2 = (1 \pm |\mathbf{P}|)/2 \)

Pure states: \( \lambda_+ = 1, \lambda_- = 0 \)

On the sphere: \( |\mathbf{P}| = 1 \)
**Pure states**

\[ \hat{\rho}_j = |\phi_j\rangle\langle\phi_j| \]

\[ \hat{\rho}_j = \frac{1}{2} (\hat{1} + P_j \cdot \hat{\sigma}) \]

\[ |\langle \phi_1 | \phi_2 \rangle|^2 = \text{Tr}[\hat{\rho}_1 \hat{\rho}_2] \]

\[ = \frac{1 + P_1 \cdot P_2}{2} = \cos^2 \frac{\theta}{2} \]

**Orthogonal states**

\[ \theta = \pi \]

**Orthonormal basis**

A line through the origin

Vectors in the Hilbert space

\[ P_1 \cdot P_2 = \cos \theta \]
Examples

Spin ½ particle

- \( |0\rangle \)
- \( |0\rangle - |1\rangle \)
- \( |0\rangle - i|1\rangle \)
- \( |0\rangle + i|1\rangle \)
- \( |0\rangle + |1\rangle \)

Polarization of a single photon

Bloch vector = Spin vector

Bloch sphere = Poincaré sphere
Orthogonal measurement

Orthonormal basis \[ \{ |\phi_1\rangle, |\phi_2\rangle \} \]  →  A line through the origin

\[
P(1) = \langle \phi_1 | \hat{\rho} | \phi_1 \rangle = \text{Tr}(\hat{\rho}_1 \hat{\rho}) = \frac{1 + P_1 \cdot P}{2}
\]

\[
P(2) = \frac{1 - P_1 \cdot P}{2}
\]

Example

Measurement of observable \( \hat{\sigma}_z \)

Z axis
Unitary operation

|ψ⟩, e^{iθ}|ψ⟩  The same physical state

\( \hat{U}, \ e^{iθ}\hat{U} \)  The same physical operation

\[
\det(e^{iθ}\hat{U}) = e^{2iθ} \det\hat{U}
\]

group  \( SU(2) \) : Set of \( \hat{U} \) with \( \det \hat{U} = 1 \)

\( \hat{U} \in SU(2) \leftrightarrow -\hat{U} \in SU(2) \)

(2 to 1 correspondence to the physical unitary operations)

\[
\hat{U} = \exp[i\hat{S}]
\]

Self-adjoint, traceless

\[
\hat{S} = \frac{1}{2}(P \cdot \hat{σ})
\]

We can parameterize the elements of \( SU(2) \) as

\[
\hat{U}(n, \varphi) \equiv \exp[-i(\varphi/2)n \cdot \hat{σ}]
\]

Unit vector
Unitary operation

\[ \hat{\rho} = \frac{1}{2} (\hat{1} + P \cdot \hat{\sigma}) \quad \xrightarrow{\hat{U}(n, \varphi)} \quad \hat{\rho}' = \frac{1}{2} (\hat{1} + P' \cdot \hat{\sigma}) \]

How does the Bloch vector change?

Infinitesimal change \( \hat{U}(n, \delta\varphi) \sim \hat{1} - i(\delta\varphi/2)n \cdot \hat{\sigma} \)

\[ \delta P \equiv P' - P = \text{Tr}[\hat{\sigma} \hat{\rho}'] - \text{Tr}[\hat{\sigma} \hat{\rho}] \]

\[ = \text{Tr}[\hat{\sigma} \hat{U}(n, \delta\varphi) \hat{\rho} \hat{U}^\dagger(n, \delta\varphi)] - \text{Tr}[\hat{\sigma} \hat{\rho}] \]

\[ = \text{Tr}[\hat{U}^\dagger(n, \delta\varphi) \hat{\sigma} \hat{U}(n, \delta\varphi) \hat{\rho}] - \text{Tr}[\hat{\sigma} \hat{\rho}] \]

\[ \sim \text{Tr}\{(i\delta\varphi/2)[(n \cdot \hat{\sigma}), \hat{\sigma}] \hat{\rho}\} = -\delta\varphi \text{Tr}[n_i \varepsilon_{ijk} \hat{\sigma}_k \hat{\rho}] \]

\[ = \delta\varphi \text{Tr}[(n \times \hat{\sigma}) \hat{\rho}] = \delta\varphi n \times P. \]

Rotation around axis \( n \) by angle \( \delta\varphi \)
Unitary operation

\[ \hat{U} \in SU(2) \]

\[ \hat{U} = \exp[-i(\varphi/2)n \cdot \hat{\sigma}] \]

Rotation around axis \( n \) by angle \( \varphi \)

Examples

\( \hat{\sigma}_z : \pi \) rotation around \( z \) axis
\( \hat{\sigma}_x : \pi \) rotation around \( x \) axis

\[ \hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

Hadamard transform

\( \pi \) rotation (interchanges \( z \) and \( x \) axes)
4. Power of an ancillary system

Kraus representation (Operator-sum rep.)

Generalized measurement
  Unambiguous state discrimination

Quantum operation (Quantum channel, CPTP map)

Relation between quantum operations and bipartite states
  A maximally entangled state and relative states

What can we do in principle?
Power of an ancilla system

Basic operations
Unitary operations
Orthogonal measurements

An auxiliary system (ancilla)

\[ \hat{\rho} \xrightarrow{\hat{U}} \rho_{out} \]

\[ |0\rangle_E \xrightarrow{\hat{U}} \]

Measurement \{ |j\rangle_E \}

Probability \( p_j \)
Power of an ancilla system

Basic operations
Unitary operations
Orthogonal measurements

An auxiliary system (ancilla)

$\hat{\rho}$ → $A$ → $|0\rangle_E$ → $U$ → $E'$ → $j$ → $\hat{\rho}_{\text{out}}$

Measurement: $\{|j\rangle_{E'}\}$
Probability: $p_j$
Power of an ancilla system

\[ \hat{\rho} \otimes |0\rangle_E \langle 0| \]

\[ \hat{U} (\hat{\rho} \otimes |0\rangle_E \langle 0|) \hat{U}^\dagger \]

\[ p_j \hat{\rho}_{\text{out}}^{(j)} = E \langle j | \hat{U} (\hat{\rho} \otimes |0\rangle_E \langle 0|) \hat{U}^\dagger | j \rangle_E \]

\[ = \hat{M}(j) \hat{\rho} \hat{M}(j)^\dagger \]

\[ \hat{M}(j) \equiv E \langle j | \hat{U} | 0 \rangle_E \]

\[ E \langle j | \hat{U} | 0 \rangle_E \]

\[ \hat{M}(j) : \mathcal{H}_A \rightarrow \mathcal{H}_A \]
Kraus representation (Operator-sum rep.)

\[ p_j \hat{\rho}^{(j)}_{\text{out}} = E \langle j | \hat{\mathcal{U}} (\hat{\rho} \otimes |0\rangle_E) \hat{\mathcal{U}}^\dagger | j \rangle_E \]

\[ \downarrow \hat{M}^{(j)} \equiv E \langle j | \hat{\mathcal{U}} | 0 \rangle_E \quad \text{Kraus operators} \]

\[ p_j \hat{\rho}^{(j)}_{\text{out}} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)} \dagger \quad \text{with} \quad \sum_j \hat{M}^{(j)} \dagger \hat{M}^{(j)} = \hat{1} \]

Representation with no reference to the ancilla system

\[ \sum_j \hat{M}^{(j)} \dagger \hat{M}^{(j)} = \sum_j E \langle 0 | \hat{\mathcal{U}}^\dagger | j \rangle_E E \langle j | \hat{\mathcal{U}} | 0 \rangle_E \]

\[ = E \langle 0 | \hat{\mathcal{U}}^\dagger \hat{\mathcal{U}} | 0 \rangle_E \]

\[ = E \langle 0 | \hat{1}_A \otimes \hat{1}_E | 0 \rangle_E \]

\[ = \hat{1}_A \]
Kraus operators $\rightarrow$ Physical realization

$$p_j \hat{\rho}^{(j)}_{out} = E \langle j | \hat{U} (\hat{\rho} \otimes |0\rangle_E E \langle 0| ) \hat{U}^\dagger | j \rangle_E$$

$$\uparrow \quad \downarrow \quad \hat{M}^{(j)} \equiv E \langle j | \hat{U} | 0 \rangle_E \quad \text{Kraus operators}$$

$$p_j \hat{\rho}^{(j)}_{out} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)} \dagger \quad \text{with} \quad \sum_j \hat{M}^{(j)} \dagger \hat{M}^{(j)} = \hat{1}$$

Arbitrary set $\{ \hat{M}^{(j)} \}$ satisfying $\sum_j \hat{M}^{(j)} \dagger \hat{M}^{(j)} = \hat{1}$

$$|\phi\rangle_A \otimes |0\rangle_E \mapsto \sum_j \hat{M}^{(j)} |\phi\rangle_A \otimes |j\rangle_E \quad \text{is linear.}$$

preserves inner products.

For any two states $|\phi\rangle_A$ and $|\psi\rangle_A$,

$$\left( \sum_{j'} \hat{M}^{(j')} |\psi\rangle_A \otimes |j'\rangle_E \right) \left( \sum_j \hat{M}^{(j)} |\phi\rangle_A \otimes |j\rangle_E \right) = A \langle \psi | \phi \rangle_A = (|\psi\rangle_A \otimes |0\rangle_E) \dagger (|\phi\rangle_A \otimes |0\rangle_E) .$$

There exists a unitary satisfying

$$\hat{U} (|\phi\rangle_A \otimes |0\rangle_E) = \sum_j \hat{M}^{(j)} |\phi\rangle_A \otimes |j\rangle_E$$
Generalized measurement

\[ p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \quad \text{with} \quad \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1} \]

\[ p_j = \text{Tr}[\hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger}] = \text{Tr}[\hat{F}^{(j)} \hat{\rho}] \]

\[ \hat{F}^{(j)} \equiv \hat{M}^{(j)\dagger} \hat{M}^{(j)} \geq 0 \quad \text{positive} \]

\[ p_j = \text{Tr}[\hat{F}^{(j)} \hat{\rho}] \quad \text{with} \quad \sum_j \hat{F}^{(j)} = \hat{1} \]

\[ \{ \hat{F}^{(j)} \} \quad \text{POVM} \]

Positive operator valued measure
Generalized measurement

\[ p_j = \text{Tr}[\hat{F}(j) \hat{\rho}] \quad \text{with} \quad \sum_j \hat{F}(j) = \hat{1} \]

Examples

**Orthogonal measurement on basis \( \{|a_j\rangle\} \)**

\[ \hat{F}(j) = |a_j\rangle\langle a_j| \]

**Trine measurement on a qubit**

\[ \hat{F}(j) = \frac{2}{3} |b_j\rangle\langle b_j| \]

\[ |b_j\rangle\langle b_j| = \frac{1}{2} \left( \hat{1} + P_j \cdot \hat{\sigma} \right) \]

\[ \sum_j P_j = 0 \quad \rightarrow \quad \sum_j \hat{F}(j) = \hat{1} \]
Distinguishing two nonorthogonal states

Minimum-error discrimination

\[ \langle \phi_0 | \phi_1 \rangle = s > 0 \]

- qubit
- \(|\phi_0\rangle \) or \(|\phi_1\rangle\)
- 50% 50%

Unambiguous state discrimination

- qubit
- \(|\phi_0\rangle \) or \(|\phi_1\rangle\)
- 50% 50%

- \(p_{\text{fail}}\)
- (surely) 0
- (surely) 1
- 2 (I don’t know)
Unambiguous state discrimination

qubit \( \begin{array}{c}
|\phi_0\rangle \\
|\phi_1\rangle
\end{array} \)

50\% 50\%

\( \langle \phi_0 | \phi_1 \rangle = s > 0 \)

Orthogonal measurement

\{ |\phi_0\rangle, |\phi_0^\perp\rangle \}

2 (I don’t know) (surely) 1

\{ |\phi_1\rangle, |\phi_1^\perp\rangle \}

If the initial state is \( |\phi_0\rangle \)

it always fails.

If the initial state is \( |\phi_1\rangle \)

it fails with prob. \( |\langle \phi_0 | \phi_1 \rangle|^2 = s^2 \)

\( p_{\text{fail}} = \frac{1 + s^2}{2} \)
Unambiguous state discrimination

qubit \( |\phi_0 \rangle \text{ or } |\phi_1 \rangle \)  
50\%  50\%  
\[ \langle \phi_0 | \phi_1 \rangle = s > 0 \]

Generalized measurement

\[ \hat{F}_0 := \mu |\phi_1 \rangle\langle \phi_1 | \]
\[ \hat{F}_1 := \mu |\phi_0 \rangle\langle \phi_0 | \]
\[ \hat{F}_2 := \hat{1} - \hat{F}_0 - \hat{F}_1 \]

The only constraint on \( \mu \) comes from \( \hat{F}_2 \geq 0 \)

\[ \langle \phi_0 \rangle| \phi_1 \rangle \rangle = s \]
\[ (\hat{F}_0 + \hat{F}_1)(|\phi_0 \rangle \pm |\phi_1 \rangle) \]
\[ = \mu(1 \pm s)(|\phi_0 \rangle \pm |\phi_1 \rangle) \]

The optimum: \( \mu = (1 + s)^{-1} \)

\[ p_{\text{fail}} = 1 - \frac{\mu}{2}|\langle \phi_0 | \phi_1 \rangle|^2 - \frac{\mu}{2}|\langle \phi_1 | \phi_0 \rangle|^2 \]
\[ = 1 - \mu(1 - s^2) \]

\[ p_{\text{fail}} = s \]
Quantum operation (Quantum channel, CPTP map)

\[ p_j \hat{\rho}^{(j)}_{\text{out}} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1} \]

\[ \hat{\rho}_{\text{out}} = \sum_j p_j \hat{\rho}^{(j)}_{\text{out}} = \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \]
\[ = \sum_j E \langle j | \hat{U} (\hat{\rho} \otimes |0\rangle_{EE} \langle 0|) \hat{U}^\dagger |j\rangle_E \]
\[ = \text{Tr}_E [\hat{U} (\hat{\rho} \otimes |0\rangle_{EE} \langle 0|) \hat{U}^\dagger] \]

\[ \hat{\rho}_{\text{out}} = \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \]
\[ = \text{Tr}_E [\hat{U} (\hat{\rho} \otimes |0\rangle_{EE} \langle 0|) \hat{U}^\dagger] \]

\[ \hat{\rho}_{\text{out}} = \mathcal{C}(\hat{\rho}) \] completely-positive trace-preserving map
CPTP map
Quantum operation (Quantum channel, CPTP map)

\[\hat{\rho} \rightarrow \hat{\rho}_{\text{out}}\]

\[\hat{\rho}_{\text{out}} = \sum_j \hat{M}(j) \hat{\rho} \hat{M}(j)^\dagger \quad \text{with} \quad \sum_j \hat{M}(j)^\dagger \hat{M}(j) = \hat{1}\]

\[= \text{Tr}_E[\hat{U}(\hat{\rho} \otimes |0\rangle_E \langle 0|) \hat{U}^\dagger]\]

\[\hat{\rho}_{\text{out}} = C(\hat{\rho}) \quad \text{completely-positive trace-preserving map}
\]

CPTP map
Positive maps and completely-positive maps

Linear map
\[ \hat{\rho}_A \mapsto C_A(\hat{\rho}_A) \]

“positive”: \( C_A(\hat{\rho}_A) \) is positive whenever \( \hat{\rho}_A \) is positive

\[ \hat{\rho}_A \xrightarrow{C_A} C_A(\hat{\rho}_A) \]

“completely-positive”: \( (C_A \otimes I_B)(\hat{\rho}_{AB}) \) is positive whenever \( \hat{\rho}_{AB} \) is positive

\[ \hat{\rho}_{AB} \xrightarrow{C_A \otimes I_B} (C_A \otimes I_B)(\hat{\rho}_{AB}) \]

\( (C_A \otimes I_B)(\hat{\rho}_{AB}) = \sum_j \hat{M}_A^{(j)} \hat{\rho}_{AB} \hat{M}_A^{(j)\dagger} \)
Power of an ancilla system

Basic operations
Unitary operations
Orthogonal measurements

An auxiliary system (ancilla)

\[ \hat{\rho} \rightarrow E \quad \hat{U} \quad \text{measurement} \quad \{ |j\rangle_E \} \rightarrow j \quad \rho_{\text{out}}(j) \]

\[ p_j \quad \text{probability} \]
What can we do in principle?

We have seen what we can (at least) do by using an ancilla system.

\[ p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}(j) \hat{\rho} \hat{M}(j)^\dagger \text{ with } \sum_j \hat{M}(j)^\dagger \hat{M}(j) = \hat{1} \]

We also want to know what we cannot do.

Black box with classical and quantum outputs
This is what we can do in principle

Any physical process should be represented in the following form:

\[ p_m \rho_{\text{out}} = \sum_k \hat{M}(k,m) \hat{\rho} \hat{M}^\dagger(k,m) + \sum_{m,k} \hat{M}^\dagger(k,m) \hat{M}(k,m) = 1_A \]

Orthogonal measurement
What can we do in principle?

- Attach an ancilla
- Apply a unitary
- Discard the ancilla

Black box with quantum output (Quantum channel)
Maximally entangled states (MES)

\[ \dim \mathcal{H}_A = \dim \mathcal{H}_B = d \]

Orthonormal bases

\[ \{ |k\rangle_A \}_{k=1,2,...,d} \quad \{ |k\rangle_B \}_{k=1,2,...,d} \]

\[ \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B \]

Maximally entangled state
(with Schmidt number \( d \))
Properties of MES (II): Relative states

Fix a maximally entangled state

\[ |\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A |k\rangle_B \]

\[ \dim \mathcal{H}_A = \dim \mathcal{H}_B = d \]

Relative states

\[ |\phi\rangle_A = \sum_k \alpha_k |k\rangle_A \]

\[ |\phi^*\rangle_B = \sum_k \overline{\alpha_k} |k\rangle_B \]

\[ = \sqrt{d} \times_B \langle \phi^* | \Phi \rangle_{AB} \]

\[ = \sqrt{d} \times_A \langle \phi | \Phi \rangle_{AB} \]

Orthogonal measurement

\[ \{ |v_j\rangle_B \}_{j=1,2,...,d} \]

\[ |v_1\rangle_B = |\phi^*\rangle_B \]

\[ p_1 = \frac{1}{d} \]

outcome \( j = 1 \)

\[ \frac{1}{\sqrt{d}} |\phi\rangle_A = B \langle \phi^* | \Phi \rangle_{AB} \]
Properties of MES (III): Pair of equivalent local operations

$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A |k\rangle_B$$

$$(\hat{T}_A \otimes \hat{1}_B)|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{T}'_B)|\Phi\rangle_{AB}$$

$$_A\langle l| \otimes _B\langle k|$$

$$_A\langle l| \hat{T}_A |k\rangle_A = _B\langle k| \hat{T}'_B |l\rangle_B$$

transpose

Equivalent
Quantum operation and bipartite state

We can remotely prepare system A in any state with a nonzero success probability.

At any time

\[ \langle \Phi \rangle_{AR} \]

\[ \hat{\rho}_{out} \]

\[ \hat{\sigma}_{AR} \]

\[ d \times B \langle \phi^* | \hat{\sigma}_{AR} | \phi^* \rangle_B \]

\[ \phi^* \]

\[ \frac{1}{d} \]

\[ \hat{\sigma}_{AR} : \text{The state obtained when a half of an MES is fed to the channel.} \]

If this state is known,

\[ \hat{\rho}_{out} = B \langle \phi^* | \hat{\sigma}_{AR} | \phi^* \rangle_B d \]

Output for every input state is known!

Characterization of a process = Characterization of a state
Quantum operation and bipartite state

\[ \hat{\rho}_{\text{out}} = \sqrt{d_R} \langle \phi^* | \hat{\sigma}_{AR} | \phi^* \rangle_R \sqrt{d} \]

\[ R \langle \phi^* | = \sqrt{d} \ AR \langle \Phi | | \phi \rangle \]

\[ \hat{\sigma}_{AR} = \sum_j |\Psi_j\rangle_{AR} AR \langle \Psi_j | \]

unnormalized

\[ \sqrt{d_R} \langle \phi^* | | \Psi_j \rangle_{AR} = \hat{M}^{(j)} | \phi \rangle_A \]

(A linear map)

\[ \hat{\rho}_{\text{out}} = \sum_j \hat{M}^{(j)} | \phi \rangle_{AA} \langle \phi | \hat{M}^{(j)\dagger} \]

\[ AR \langle \Phi | | \Psi_j \rangle \]

(A linear map)
What we can do in principle

Black box with quantum output
(Quantum channel)
\[ \hat{\rho}_{\text{out}} = \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \]
Size of the ancilla system

\[ \dim \mathcal{H}_E = (\dim \mathcal{H}_A)^2 \]
is enough.

\[ |\Phi\rangle_{AR} \]
\[ \dim \mathcal{H}_A = \dim \mathcal{H}_R = d \]

\[ |\xi\rangle_{ARE} \equiv \hat{U}(|\Psi\rangle_{AR} \otimes |0\rangle_E) \]

\[ \dim(\text{Ran } \hat{\rho}_E) = \dim(\text{Ran } \hat{\rho}_{AR}) \leq \dim \mathcal{H}_{AR} = d^2 \]
Universal NOT? Spin reversal?

Bloch vector

\[ \mathbf{P} \rightarrow -\mathbf{P} \]

linear map \( \hat{\rho} \rightarrow \mathcal{C}(\hat{\rho}) \)

\[ \mathcal{C}(\hat{1}) = \hat{1} \quad \mathcal{C}(\hat{\sigma}_x) = -\hat{\sigma}_x \]
\[ \mathcal{C}(\hat{\sigma}_y) = -\hat{\sigma}_y \quad \mathcal{C}(\hat{\sigma}_z) = -\hat{\sigma}_z \]

\[ \mathcal{C}(\lvert 0 \rangle \langle 0 \rvert) = \lvert 1 \rangle \langle 1 \rvert \]
\[ \mathcal{C}(\lvert 1 \rangle \langle 1 \rvert) = \lvert 0 \rangle \langle 0 \rvert \]
\[ \mathcal{C}(\lvert 0 \rangle \langle 1 \rvert) = -\lvert 0 \rangle \langle 1 \rvert \]
\[ \mathcal{C}(\lvert 1 \rangle \langle 0 \rvert) = -\lvert 1 \rangle \langle 0 \rvert \]

\[ \hat{\sigma}_x = \lvert 1 \rangle \langle 0 \rvert + \lvert 0 \rangle \langle 1 \rvert \]
\[ \hat{\sigma}_y = i \lvert 1 \rangle \langle 0 \rvert - i \lvert 0 \rangle \langle 1 \rvert \]
\[ \hat{\sigma}_z = \lvert 0 \rangle \langle 0 \rvert - \lvert 1 \rangle \langle 1 \rvert \]
\[ \hat{1} = \lvert 0 \rangle \langle 0 \rvert + \lvert 1 \rangle \langle 1 \rvert \]

This map is positive, but...
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\[ C(|0\rangle \langle 0|) = |1\rangle \langle 1| \]
\[ C(|1\rangle \langle 1|) = |0\rangle \langle 0| \]
\[ C(|0\rangle \langle 1|) = -|0\rangle \langle 1| \]
\[ C(|1\rangle \langle 0|) = -|1\rangle \langle 0| \]

\[ 2|\Phi\rangle \langle \Phi| = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \]
\[ = |00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11| \]

\[ 2\hat{\rho}_{AR} \equiv 2(C \otimes I)|\Phi\rangle \langle \Phi| = \]
\[ = |10\rangle \langle 10| - |00\rangle \langle 11| - |11\rangle \langle 00| + |01\rangle \langle 01| \]

\[ 2\hat{\rho}_{AR}(|00\rangle + |11\rangle) = -|11\rangle - |00\rangle = -(|00\rangle + |11\rangle) \]

\[ \hat{\rho}_{AR} \] has a negative eigenvalue! (The map is not completely positive.)

Universal NOT is impossible.