3. Qubits

Pauli operators (Pauli matrices)

Bloch representation (Bloch sphere)

Orthogonal measurement

Unitary operation

Qubit

$$\dim \mathcal{H} = 2$$

Take a standard basis $\{|0\rangle, |1\rangle\}$

Linear operator $\hat{A}$

Matrix representation (for $\{|0\rangle, |1\rangle\}$)

$$\hat{A} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

$$A_{ij} = \langle i|\hat{A}|j\rangle$$

$$\hat{A} = \sum_{ij} A_{ij}|i\rangle\langle j|$$

4 complex parameters

$$\hat{A} = \alpha_0\hat{\sigma}_0 + \alpha_1\hat{\sigma}_1 + \alpha_2\hat{\sigma}_2 + \alpha_3\hat{\sigma}_3$$

Pauli operators (Pauli matrices)

Take a standard basis $\{|0\rangle, |1\rangle\}$

$$\hat{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_x = \hat{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{\sigma}_y = \hat{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \hat{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Unitary and self-adjoint

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k$$

$$\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{i,j}\hat{1}$$

$$\mathrm{Tr}(\hat{\sigma}_i) = 0, \ \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j) = 2\delta_{i,j}.$$

$i, j = 1, 2, 3$

Levi-Civita symbol

$$\begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \\ \text{Otherwise } \epsilon_{ijk} = 0 \end{cases}$$

Einstein notation

$\sum_k$ is omitted.

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z$$

$$\hat{\sigma}_x^2 = \hat{1}$$

$$\{\hat{\sigma}_x, \hat{\sigma}_z\} \equiv \hat{\sigma}_x\hat{\sigma}_z + \hat{\sigma}_z\hat{\sigma}_x = 0$$

$$\mathrm{Tr}(\hat{\sigma}_\mu\hat{\sigma}_\nu) = 2\delta_{\mu,\nu}$$

$$(\mu, \nu = 0, 1, 2, 3; \ \sigma_0 \equiv \hat{1})$$

'Orthogonality' with respect to

$$(\hat{A}, \hat{B}) \equiv \mathrm{Tr}(\hat{A}^\dagger\hat{B})$$

Pauli operators (Pauli matrices)

$$
\begin{aligned}
[\hat{\sigma}_i, \hat{\sigma}_j] &= 2i\epsilon_{ijk}\hat{\sigma}_k \\
\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i &= 2\delta_{i,j}\hat{1} \\
\mathrm{Tr}(\hat{\sigma}_i) = 0,\ \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j) &= 2\delta_{i,j}.
\end{aligned}
$$

Linear operator $\hat{A}$ 4 complex parameters (P_0, P_x, P_y, P_z)

$$
\hat{A} = \frac{1}{2}\left(P_0\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) = \frac{1}{2}\begin{pmatrix} P_0 + P_z & P_x - iP_y \\ P_x + iP_y & P_0 - P_z \end{pmatrix}
$$

$$
\begin{aligned}
\boldsymbol{P} &= (P_x, P_y, P_z) \\
\hat{\boldsymbol{\sigma}} &= (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)
\end{aligned}
$$

$$
P_0 = \mathrm{Tr}(\hat{A}) \qquad \boldsymbol{P} = \mathrm{Tr}(\hat{\boldsymbol{\sigma}}\hat{A})
$$

Pauli operators (Pauli matrices)

$$\hat{A} = \frac{1}{2}\left(P_0\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) = \frac{1}{2}\left(\begin{array}{cc} P_0 + P_z & P_x - iP_y \\ P_x + iP_y & P_0 - P_z \end{array}\right)$$

$\hat{A}$ is self-adjoint. $\longleftrightarrow$ P_0 and $\boldsymbol{P}$ are real.

Eigenvalues λ_+, λ_-

$$\det(\hat{A}) = \lambda_+\lambda_- = \tfrac{1}{4}(P_0^2 - |\boldsymbol{P}|^2)$$
$$\mathrm{Tr}(\hat{A}) = \lambda_+ + \lambda_- = P_0$$
$$\downarrow$$
$$\lambda_\pm = (P_0 \pm |\boldsymbol{P}|)/2$$

$\hat{A}$ is positive. $\longleftrightarrow$ P_0 and $\boldsymbol{P}$ are real, $P_0 \geq |\boldsymbol{P}|$

Bloch representation (Bloch sphere)

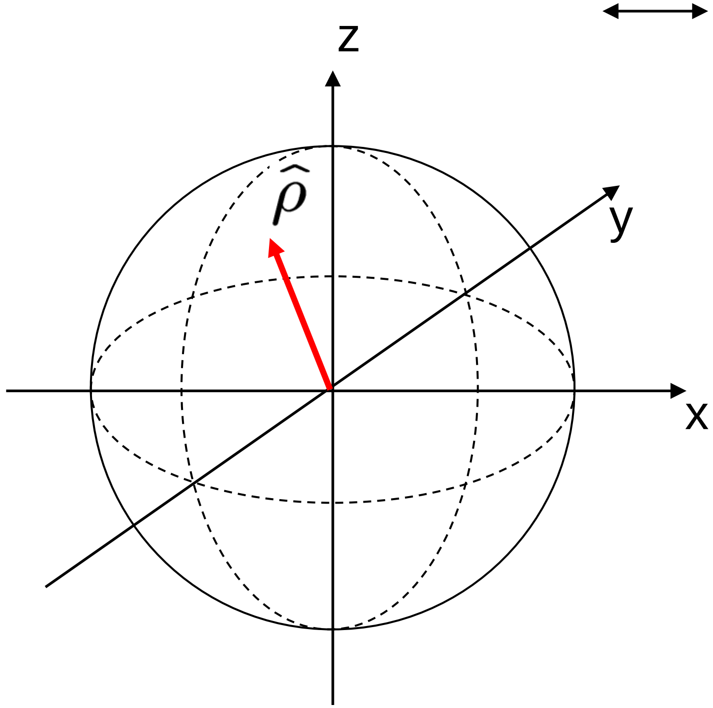
Density operator

Positive & Unit trace

$P_0 \geq |\boldsymbol{P}| \qquad P_0 = 1$

$$\hat{\rho} = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) \qquad |\boldsymbol{P}| \leq 1$$

Density operator for a qubit system



$\longleftrightarrow$ A 3D real vector of length no greater than 1

A point inside or on the sphere of radius 1

$$\boldsymbol{P} = (P_x, P_y, P_z)$$

Bloch vector

$$\lambda_{\pm} = (P_0 \pm |\boldsymbol{P}|)/2 = (1 \pm |\boldsymbol{P}|)/2$$

Pure states $\longleftrightarrow$ $\lambda_+ = 1, \lambda_- = 0$

$\longleftrightarrow$ $|\boldsymbol{P}| = 1$

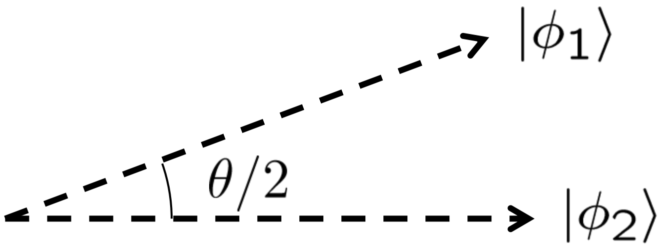
$\longleftrightarrow$ On the sphere

Pure states

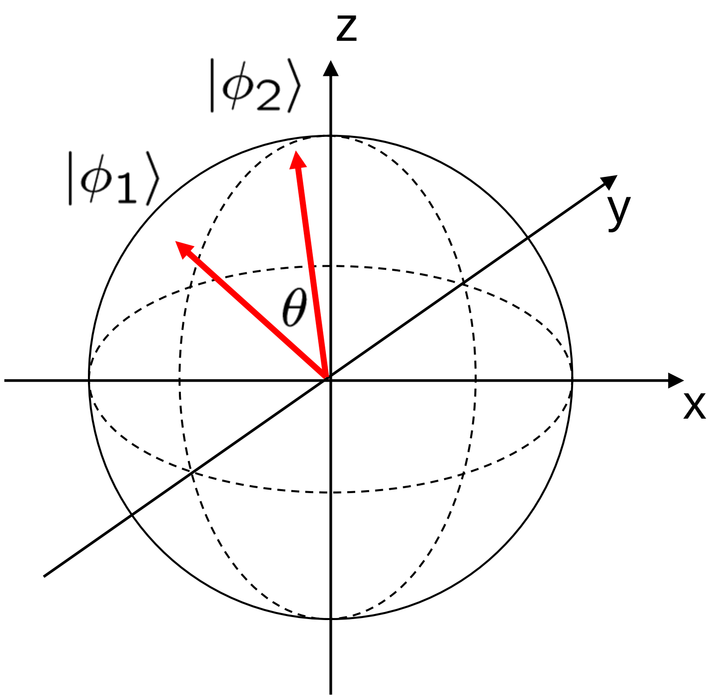
$$\hat{\rho}_j = |\phi_j\rangle\langle\phi_j|$$

$$\hat{\rho}_j = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}_j \cdot \hat{\boldsymbol{\sigma}}\right)$$

$$|\langle\phi_1|\phi_2\rangle|^2 = \mathrm{Tr}[\hat{\rho}_1\hat{\rho}_2] = \frac{1 + \boldsymbol{P}_1 \cdot \boldsymbol{P}_2}{2} = \cos^2\frac{\theta}{2}$$



Vectors in the Hilbert space

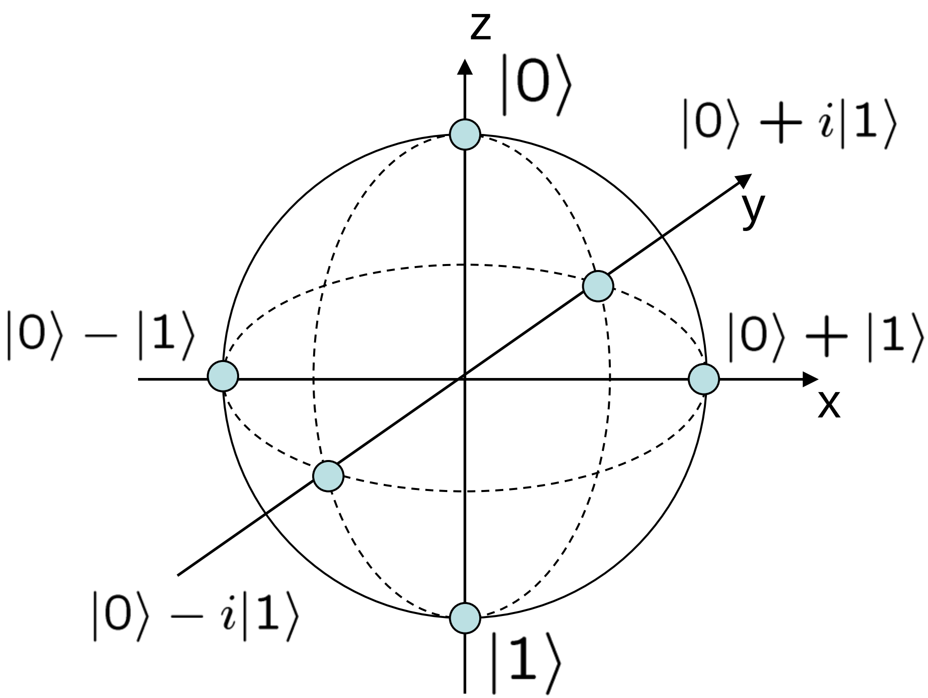


$$\boldsymbol{P}_1 \cdot \boldsymbol{P}_2 = \cos\theta$$

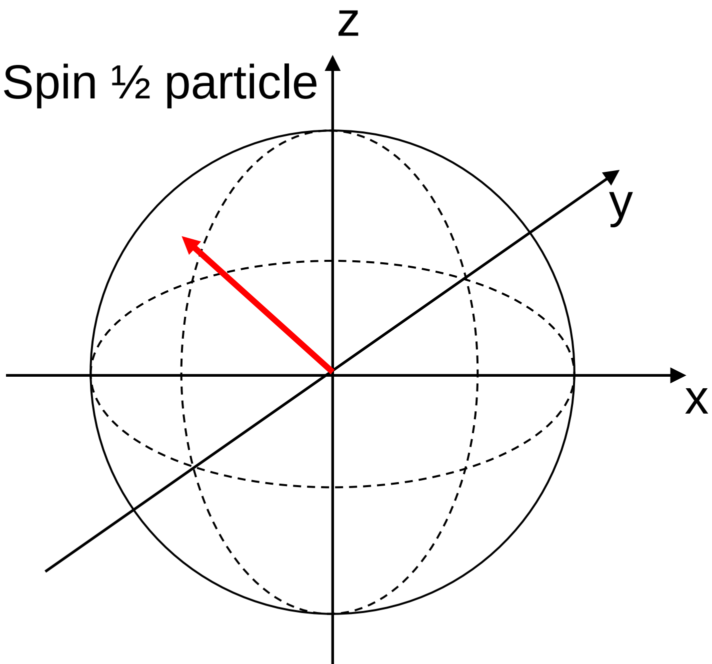
Orthogonal states $\longleftrightarrow$ $\theta = \pi$

Orthonormal basis $\longleftrightarrow$ A line through the origin

Examples

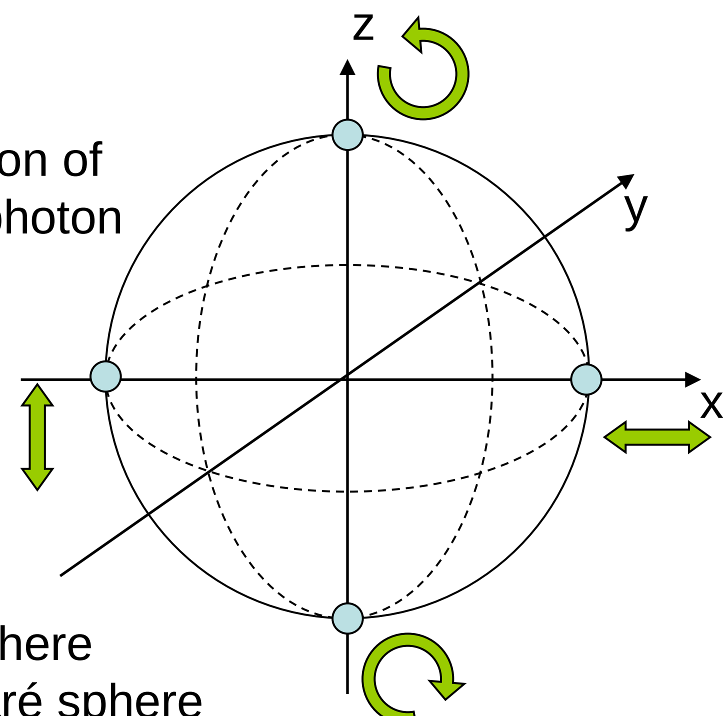


Spin ½ particle



Bloch vector = Spin vector

Polarization of
a single photon



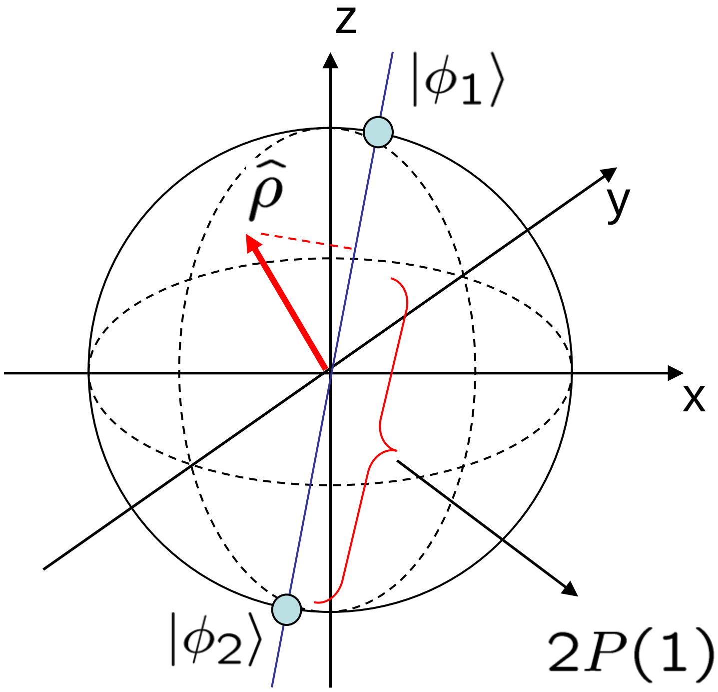
Bloch sphere
= Poincaré sphere

Orthogonal measurement

Orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle\}$ $\longleftrightarrow$ A line through the origin

$$P(1) = \langle\phi_1|\hat{\rho}|\phi_1\rangle = \mathrm{Tr}(\hat{\rho}_1\hat{\rho}) = \frac{1+\boldsymbol{P}_1\cdot\boldsymbol{P}}{2}$$

$$P(2) = \frac{1-\boldsymbol{P}_1\cdot\boldsymbol{P}}{2}$$



Example

Measurement of observable $\hat{\sigma}_z$

↓

Z axis

Unitary operation

$|\psi\rangle, e^{i\theta}|\psi\rangle$ The same physical state

$\widehat{U}, \quad e^{i\theta}\widehat{U}$ The same physical operation

$$\det(e^{i\theta}\widehat{U}) = e^{2i\theta}\det\widehat{U}$$

group $SU(2)$: Set of $\widehat{U}$ with $\det\widehat{U} = 1$ $\widehat{U} \in SU(2) \leftrightarrow -\widehat{U} \in SU(2)$

(2 to 1 correspondence to the physical unitary operations)

$$\widehat{U} = \exp[i\widehat{S}]$$

Self-adjoint, traceless

$$\widehat{U} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

$$\widehat{S} = \frac{1}{2}(\boldsymbol{P}\cdot\widehat{\boldsymbol{\sigma}})$$

$$\widehat{S} = \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}$$

We can parameterize the elements of SU(2) as

$$\widehat{U}(\boldsymbol{n},\varphi) \equiv \exp[-i(\varphi/2)\boldsymbol{n}\cdot\widehat{\boldsymbol{\sigma}}]$$

Unit vector

Unitary operation

$$\hat{\rho} = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) \xrightarrow{\hat{U}(\boldsymbol{n},\varphi)} \hat{\rho}' = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}'\cdot\hat{\boldsymbol{\sigma}}\right)$$

How does the Bloch vector change?

Infinitesimal change $\hat{U}(\boldsymbol{n},\delta\varphi) \sim \hat{1} - i(\delta\varphi/2)\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}$

$$\delta\boldsymbol{P} \equiv \boldsymbol{P}' - \boldsymbol{P} = \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}'] - \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}]$$

$$\begin{aligned}
&= \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{U}(\boldsymbol{n},\delta\varphi)\hat{\rho}\hat{U}^\dagger(\boldsymbol{n},\delta\varphi)] - \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}] \\
&= \mathrm{Tr}[\hat{U}^\dagger(\boldsymbol{n},\delta\varphi)\hat{\boldsymbol{\sigma}}\hat{U}(\boldsymbol{n},\delta\varphi)\hat{\rho}] - \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}] \\
&\sim \mathrm{Tr}\{(i\delta\varphi/2)[(\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}),\hat{\boldsymbol{\sigma}}]\hat{\rho}\} = -\delta\varphi\,\mathrm{Tr}[n_i\epsilon_{ijk}\hat{\sigma}_k\hat{\rho}] \\
&= \delta\varphi\,\mathrm{Tr}[(\boldsymbol{n}\times\hat{\boldsymbol{\sigma}})\hat{\rho}] = \delta\varphi\,\boldsymbol{n}\times\boldsymbol{P}.
\end{aligned}$$

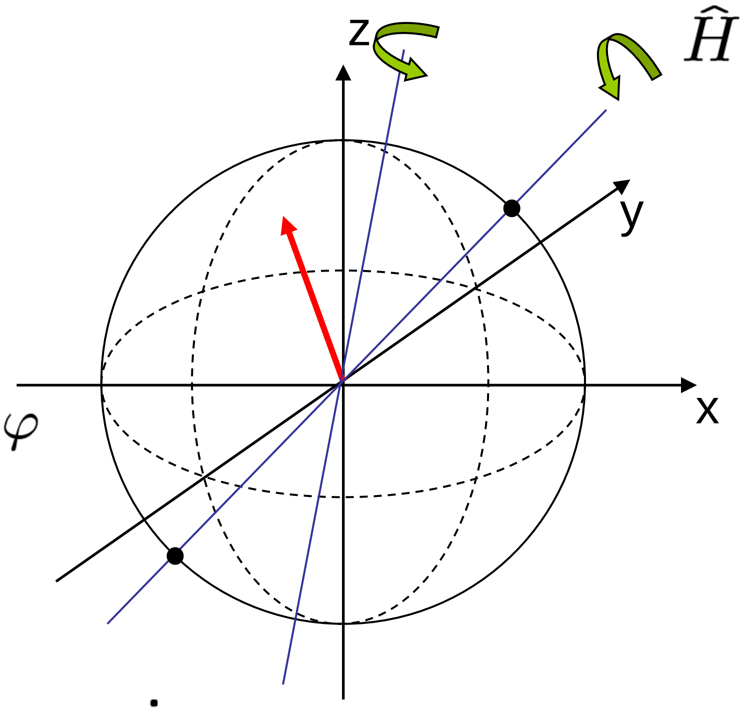
Rotation around axis $\boldsymbol{n}$ by angle $\delta\varphi$

Unitary operation

$$\hat{U} \in SU(2)$$

$$\hat{U} = \exp[-i(\varphi/2)\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}]$$

Rotation around axis $\boldsymbol{n}$ by angle φ



Examples

$\hat{\sigma}_z$: π rotation around z axis

$\hat{\sigma}_x$: π rotation around x axis

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hadamard transform

π rotation (interchanges z and x axes)

4. Power of an ancillary system

Kraus representation (Operator-sum rep.)

Generalized measurement

    Unambiguous state discrimination

Quantum operation (Quantum channel, CPTP map)

Relation between quantum operations and bipartite states

    A maximally entangled state and relative states

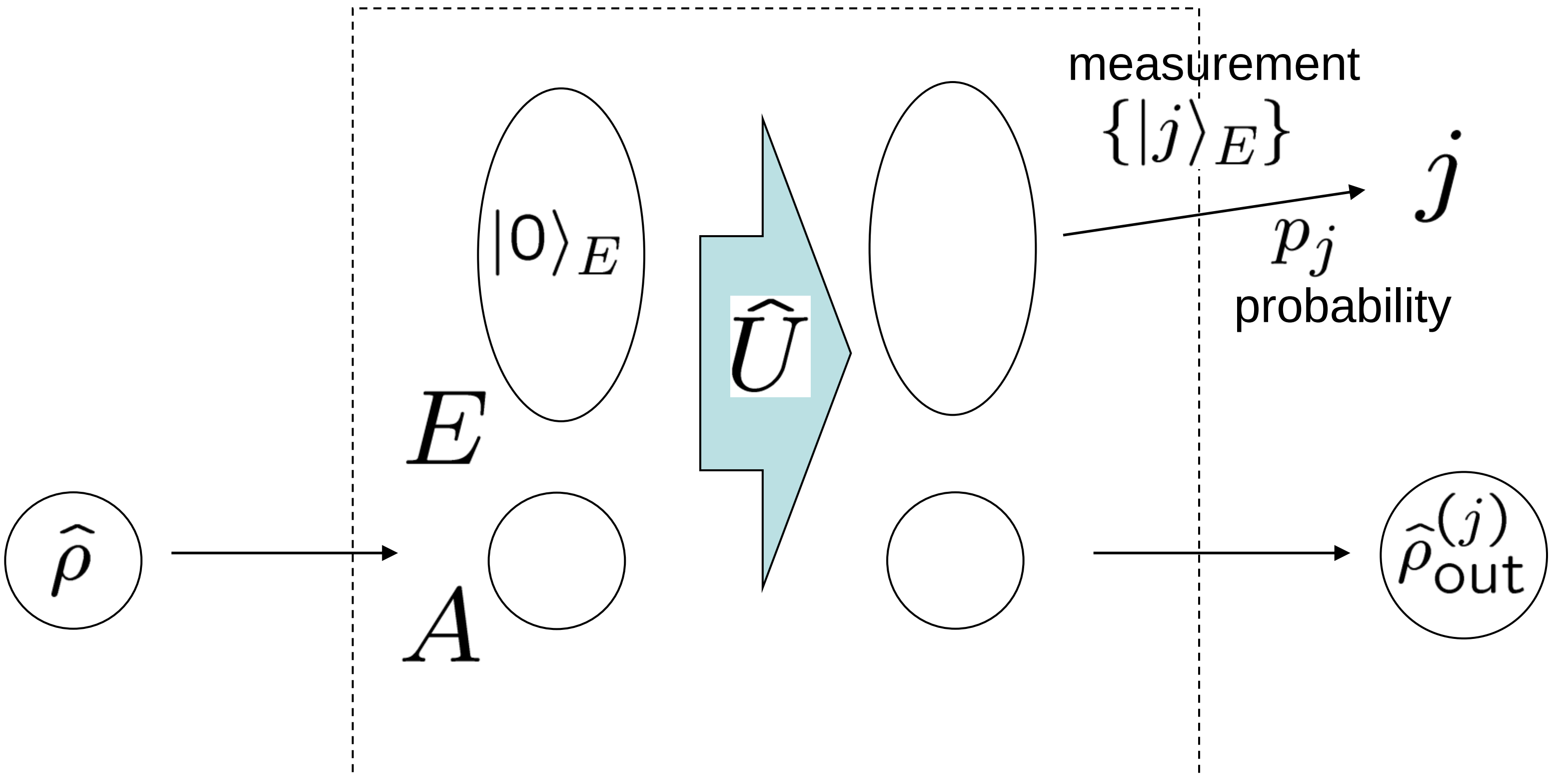
What can we do in principle?

Power of an ancilla system

Basic operations
- Unitary operations
- Orthogonal measurements

+

An auxiliary system (ancilla)

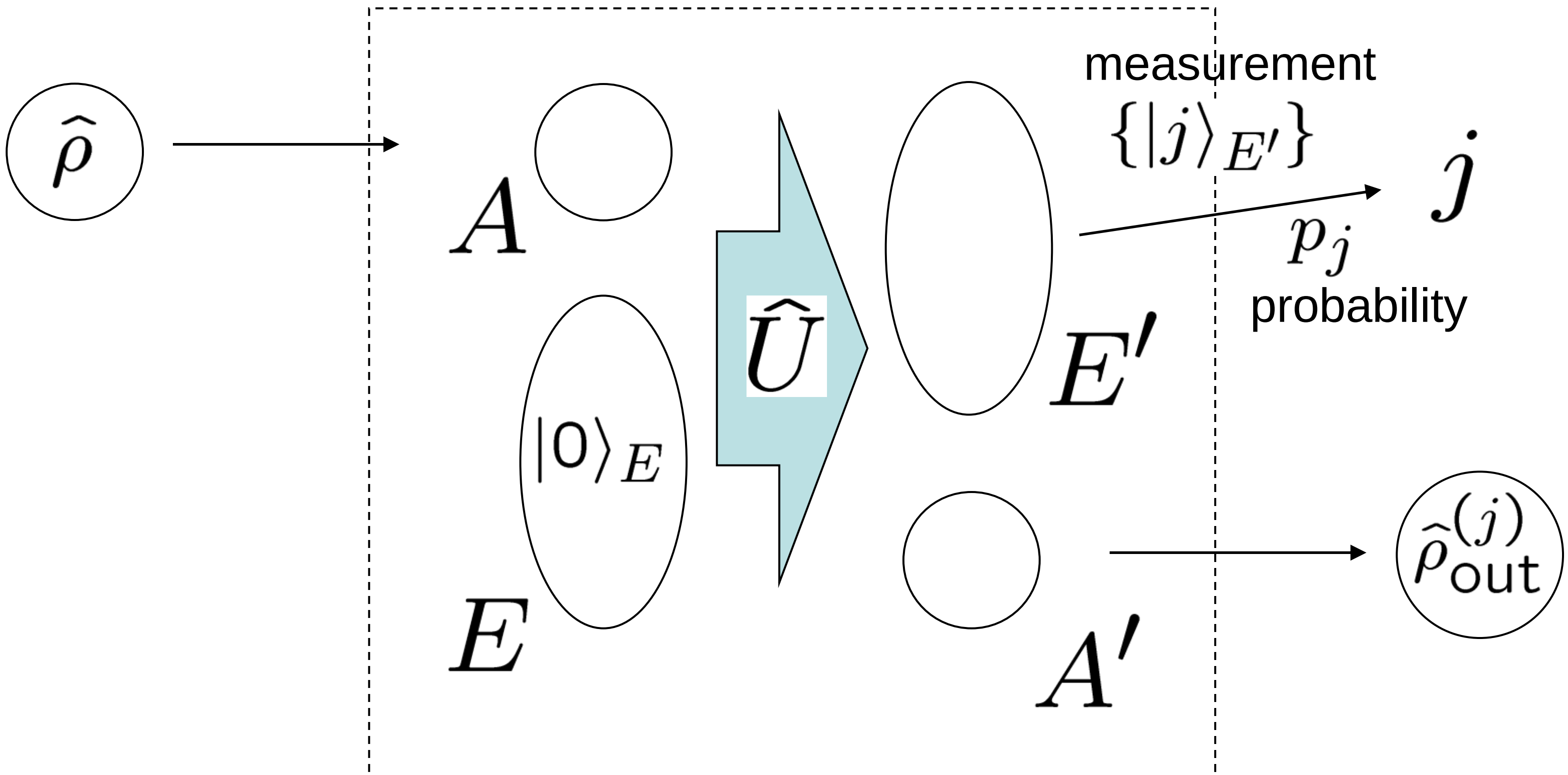


Power of an ancilla system

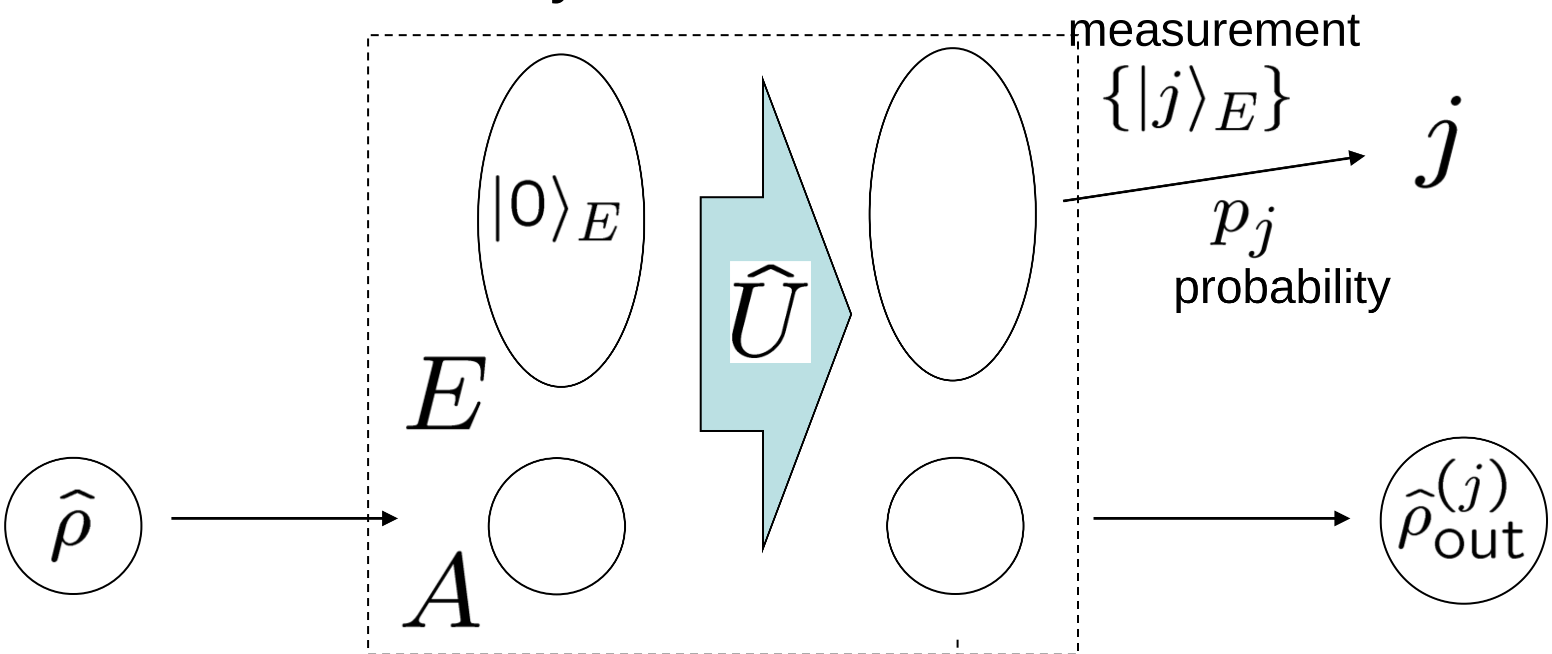
Basic operations
- Unitary operations
- Orthogonal measurements

+

An auxiliary system (ancilla)



Power of an ancilla system



$$\widehat{\rho}\otimes|0\rangle_{EE}\langle 0|$$

$$\widehat{U}(\widehat{\rho}\otimes|0\rangle_{EE}\langle 0|)\widehat{U}^\dagger$$

$$p_j\widehat{\rho}_{\text{out}}^{(j)} = {}_E\langle j|\widehat{U}(\widehat{\rho}\otimes|0\rangle_{EE}\langle 0|)\widehat{U}^\dagger|j\rangle_E$$

$$= \widehat{M}^{(j)}\widehat{\rho}\widehat{M}^{(j)\dagger}$$

$$\widehat{M}^{(j)} \equiv {}_E\langle j|\widehat{U}|0\rangle_E$$

$${}_E\langle j|\ \widehat{U}\ |0\rangle_E$$

$$\widehat{M}^{(j)}:\mathcal{H}_A\to\mathcal{H}_A$$

Kraus representation (Operator-sum rep.)

$$p_j \hat{\rho}^{(j)}_{\text{out}} = {}_E\langle j|\hat{U}(\hat{\rho}\otimes|0\rangle_{EE}\langle 0|)\hat{U}^\dagger|j\rangle_E$$

$$\Big\downarrow \quad \hat{M}^{(j)} \equiv {}_E\langle j|\hat{U}|0\rangle_E$$ Kraus operators

$$p_j \hat{\rho}^{(j)}_{\text{out}} = \hat{M}^{(j)}\hat{\rho}\hat{M}^{(j)\dagger} \text{ with } \textstyle\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$$

Representation with no reference to the ancilla system

$$\begin{aligned}
\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} &= \sum_j {}_E\langle 0|\hat{U}^\dagger|j\rangle_{EE}\langle j|\hat{U}|0\rangle_E \\
&= {}_E\langle 0|\hat{U}^\dagger\hat{U}|0\rangle_E \\
&= {}_E\langle 0|\hat{1}_A\otimes\hat{1}_E|0\rangle_E \\
&= \hat{1}_A
\end{aligned}$$

Kraus operators $\longrightarrow$ Physical realization

$$p_j \hat{\rho}_{\text{out}}^{(j)} = {}_E\langle j|\hat{U}(\hat{\rho}\otimes|0\rangle_{E\,E}\langle 0|)\hat{U}^\dagger|j\rangle_E$$

$$\uparrow \downarrow \quad \hat{M}^{(j)} \equiv {}_E\langle j|\hat{U}|0\rangle_E$$ Kraus operators

$$p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}^{(j)}\hat{\rho}\hat{M}^{(j)\dagger} \text{ with } \textstyle\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$$

Arbitrary set $\{\hat{M}^{(j)}\}$ satisfying $\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$

$|\phi\rangle_A \otimes |0\rangle_E \mapsto \sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E$ is linear.

preserves inner products.

For any two states $|\phi\rangle_A$ and $|\psi\rangle_A$,

$$\left(\sum_{j'} \hat{M}^{(j')}|\psi\rangle_A \otimes |j'\rangle_E\right)^\dagger \left(\sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E\right)$$
$$= {}_A\langle\psi|\phi\rangle_A = (|\psi\rangle_A \otimes |0\rangle_E)^\dagger (|\phi\rangle_A \otimes |0\rangle_E)\,.$$

$\downarrow$

There exists a unitary satisfying
$\hat{U}(|\phi\rangle_A \otimes |0\rangle_E) = \sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E$

Generalized measurement

$$p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

$$p_j = \text{Tr}[\hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger}] = \text{Tr}[\hat{F}^{(j)} \hat{\rho}]$$

$$\hat{F}^{(j)} \equiv \hat{M}^{(j)\dagger} \hat{M}^{(j)} \geq 0$$

positive

$$\boxed{p_j = \text{Tr}[\hat{F}^{(j)} \hat{\rho}] \text{ with } \sum_j \hat{F}^{(j)} = \hat{1}}$$

$\{\hat{F}^{(j)}\}$ POVM

Positive operator valued measure

Generalized measurement

$$p_j = \mathrm{Tr}[\hat{F}^{(j)}\hat{\rho}] \;\text{ with } \textstyle\sum_j \hat{F}^{(j)} = \hat{1}$$

Examples

Orthogonal measurement on basis $\{|a_j\rangle\}$

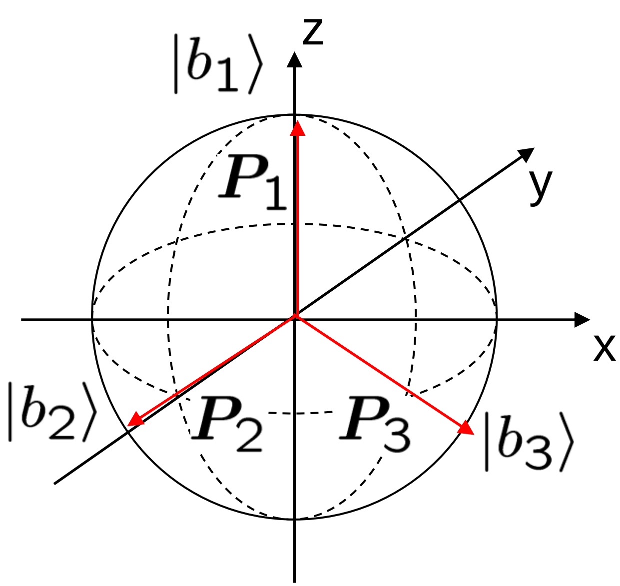
$$\hat{F}^{(j)} = |a_j\rangle\langle a_j|$$

Trine measurement on a qubit

$$\hat{F}^{(j)} = \tfrac{2}{3}|b_j\rangle\langle b_j|$$

$$|b_j\rangle\langle b_j| = \tfrac{1}{2}\left(\hat{1} + \boldsymbol{P}_j \cdot \hat{\boldsymbol{\sigma}}\right)$$

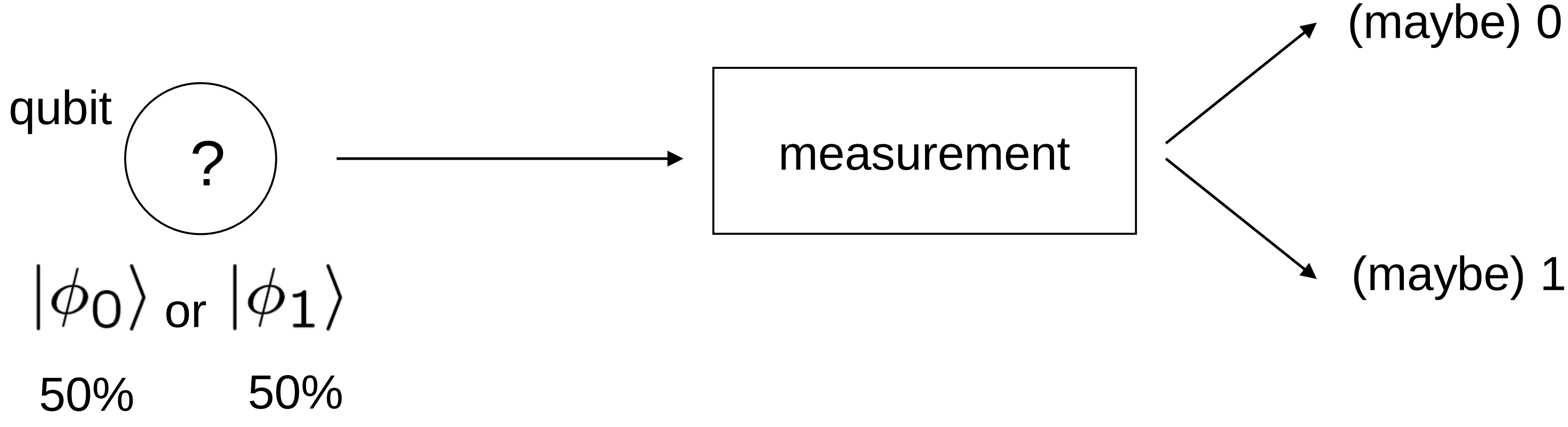
$$\textstyle\sum_j \boldsymbol{P}_j = 0 \longrightarrow \sum_j \hat{F}^{(j)} = \hat{1}$$



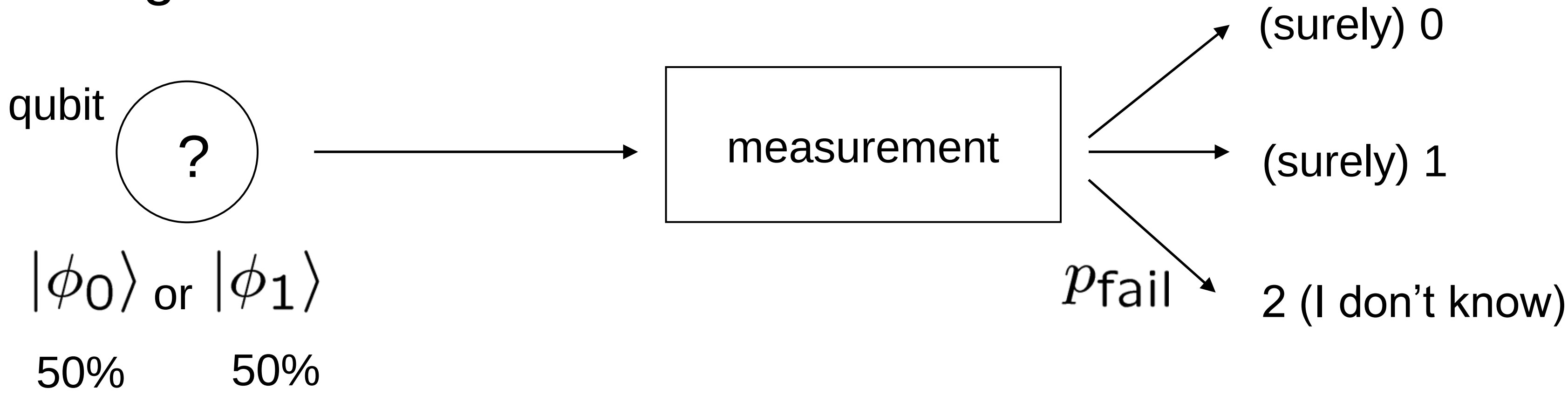
Distinguishing two nonorthogonal states

$$\langle\phi_0|\phi_1\rangle = s > 0$$

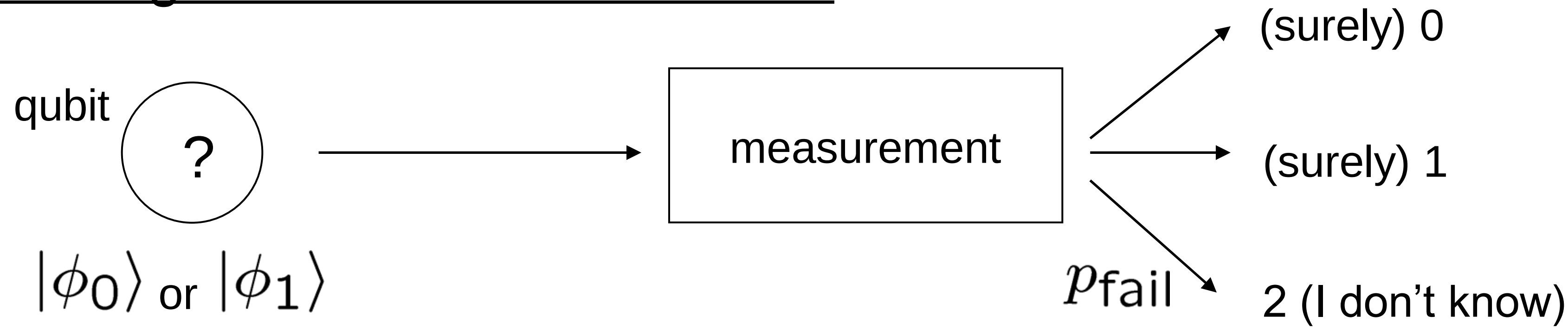
Minimum-error discrimination



Unambiguous state discrimination



Unambiguous state discrimination



$|\phi_0\rangle$ or $|\phi_1\rangle$

50% 50%

$$\langle\phi_0|\phi_1\rangle = s > 0$$

Orthogonal measurement

$$\{|\phi_0\rangle, |\phi_0^{\perp}\rangle\}$$

2 (I don't know) (surely) 1

If the initial state is $|\phi_0\rangle$

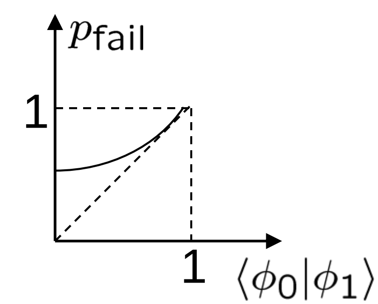
it always fails.

If the initial state is $|\phi_1\rangle$

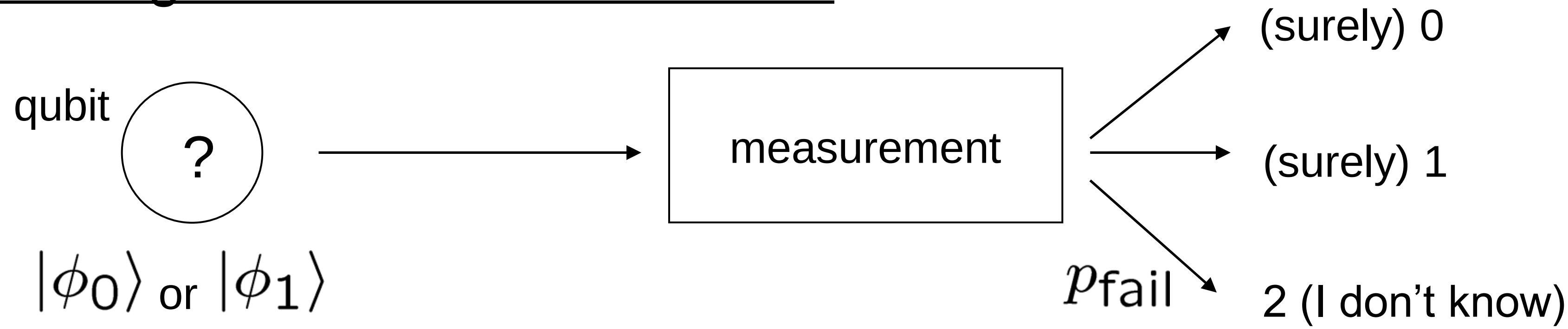
it fails with prob. $|\langle\phi_0|\phi_1\rangle|^2 = s^2$

$$\{|\phi_1\rangle, |\phi_1^{\perp}\rangle\}$$

$$p_{\text{fail}} = \frac{1+s^2}{2}$$



Unambiguous state discrimination



$|\phi_0\rangle$ or $|\phi_1\rangle$

50% 50%

$$\langle \phi_0 | \phi_1 \rangle = s > 0$$

<u>Generalized measurement</u>

$$\hat{F}_0 := \mu |\phi_1^{\perp}\rangle\langle\phi_1^{\perp}|$$

$$\hat{F}_1 := \mu |\phi_0^{\perp}\rangle\langle\phi_0^{\perp}|$$

$$\hat{F}_2 := \hat{1} - \hat{F}_0 - \hat{F}_1$$

The only constraint on μ comes from $\hat{F}_2 \geq 0$ $(\hat{F}_0 + \hat{F}_1 \leq \hat{1})$

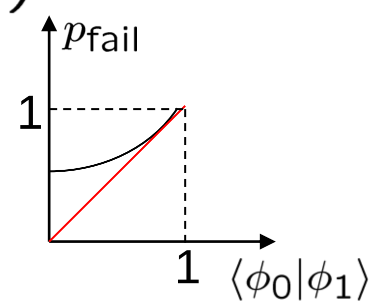
$$\langle \phi_0^{\perp} | \phi_1^{\perp} \rangle = s$$

$$(\hat{F}_0 + \hat{F}_1)(|\phi_0^{\perp}\rangle \pm |\phi_1^{\perp}\rangle) = \mu(1 \pm s)(|\phi_0^{\perp}\rangle \pm |\phi_1^{\perp}\rangle)$$

The optimum: $\mu = (1+s)^{-1}$

$$p_{\mathrm{fail}} = 1 - \frac{\mu}{2}|\langle\phi_0|\phi_1^{\perp}\rangle|^2 - \frac{\mu}{2}|\langle\phi_1|\phi_0^{\perp}\rangle|^2 = 1 - \mu(1 - s^2)$$

$$p_{\mathrm{fail}} = s$$



Quantum operation (Quantum channel, CPTP map)

$$p_j \hat{\rho}_{\rm out}^{(j)} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \ \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

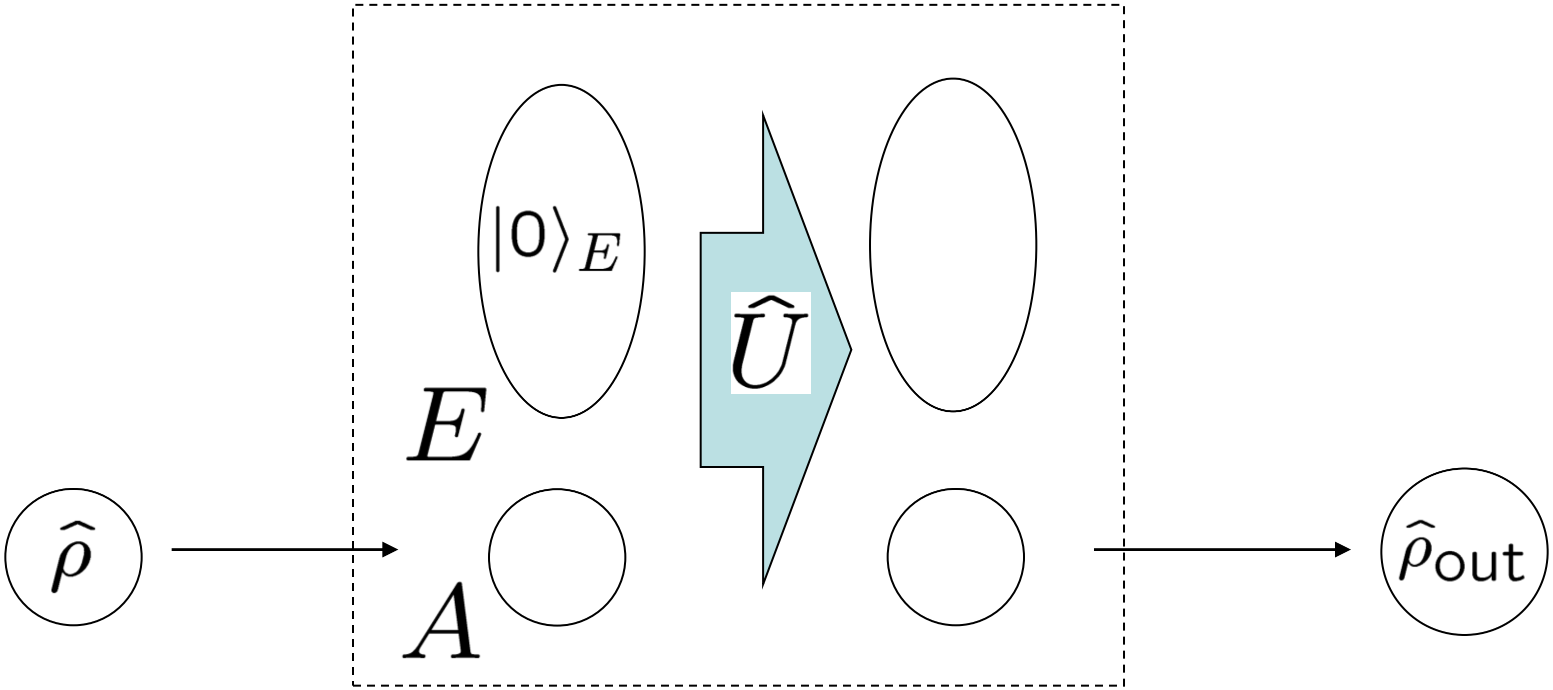
$$\begin{aligned} \hat{\rho}_{\rm out} = \sum_j p_j \hat{\rho}_{\rm out}^{(j)} &= \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \\ &= \sum_j {}_E\langle j| \hat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \hat{U}^\dagger |j\rangle_E \\ &= \mathrm{Tr}_E [\hat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \hat{U}^\dagger] \end{aligned}$$

$$\boxed{\begin{aligned} \hat{\rho}_{\rm out} &= \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \\ &= \mathrm{Tr}_E [\hat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \hat{U}^\dagger] \end{aligned}}$$

$\hat{\rho}_{\rm out} = \mathcal{C}(\hat{\rho})$ completely-positive trace-preserving map

CPTP map

Quantum operation (Quantum channel, CPTP map)



$$\hat{\rho}_{\mathrm{out}} = \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \ \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$
$$= \mathrm{Tr}_E[\hat{U}(\hat{\rho} \otimes |0\rangle_{EE}\langle 0|)\hat{U}^\dagger]$$

$\hat{\rho}_{\mathrm{out}} = \mathcal{C}(\hat{\rho})$ completely-positive trace-preserving map

CPTP map

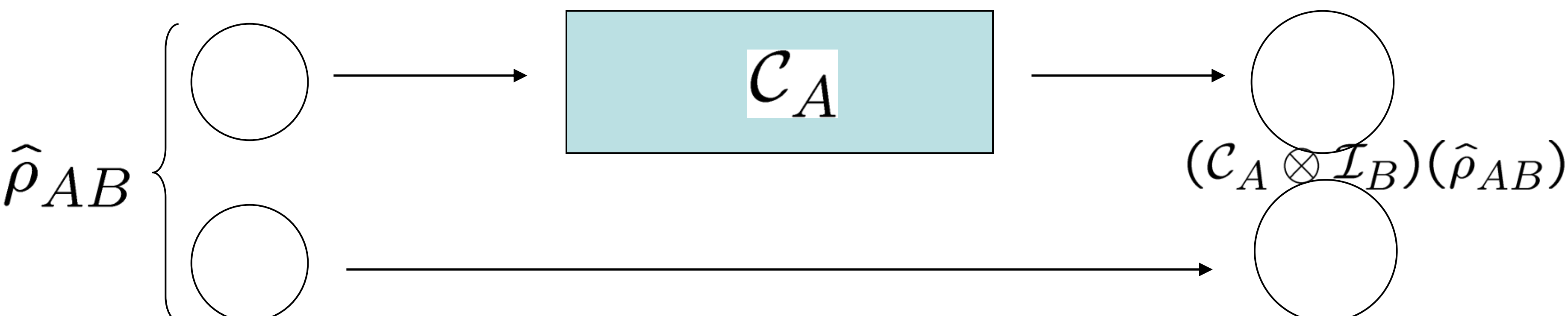
Positive maps and completely-positive maps

Linear map

$$\hat{\rho}_A \mapsto \mathcal{C}_A(\hat{\rho}_A)$$

"positive": $\mathcal{C}_A(\hat{\rho}_A)$ is positive whenever $\hat{\rho}_A$ is positive

"completely-positive": $(\mathcal{C}_A \otimes \mathcal{I}_B)(\hat{\rho}_{AB})$ is positive whenever $\hat{\rho}_{AB}$ is positive



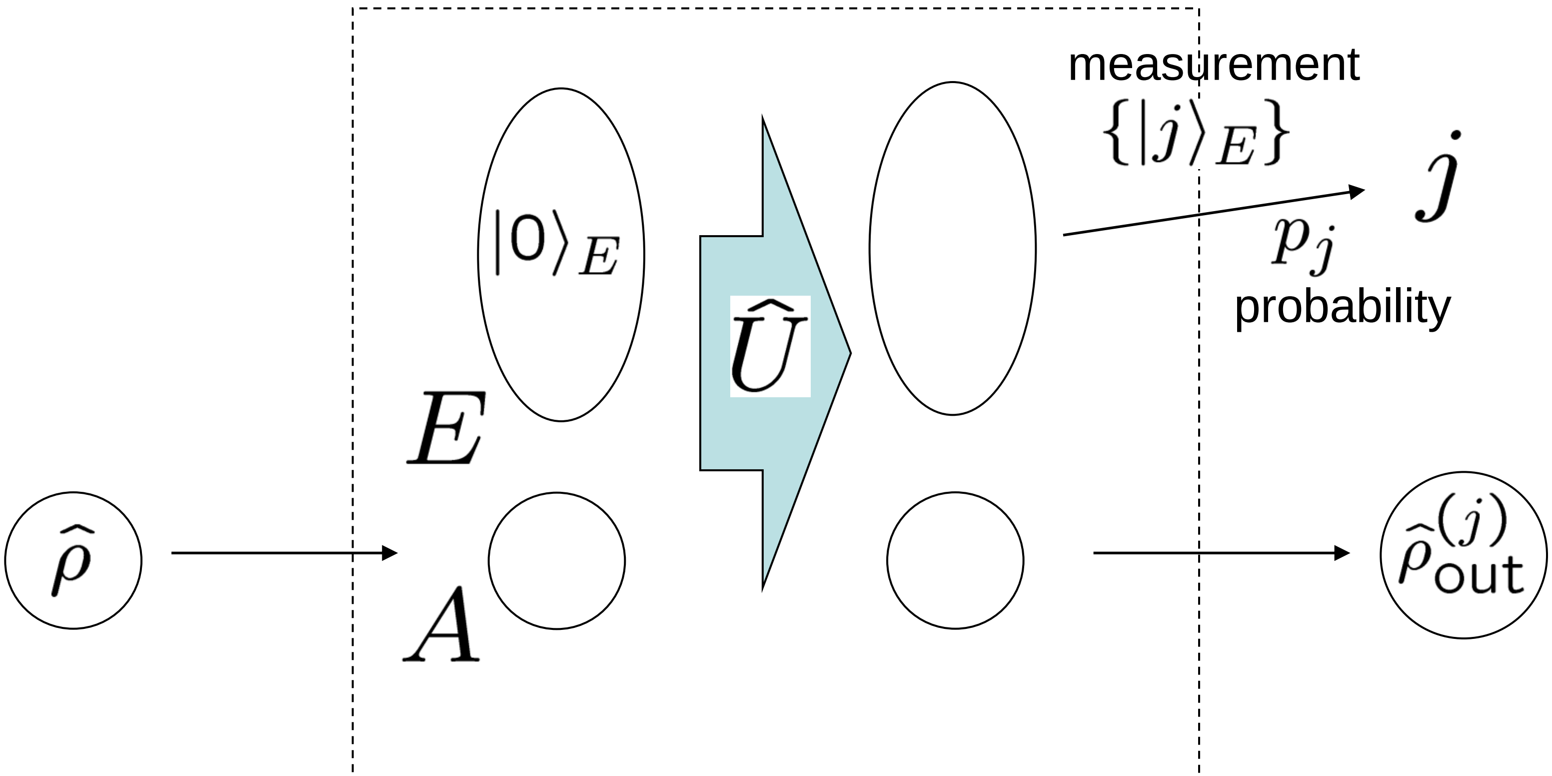
$$(\mathcal{C}_A \otimes \mathcal{I}_B)(\hat{\rho}_{AB}) = \sum_j \hat{M}_A^{(j)} \hat{\rho}_{AB} \hat{M}_A^{(j)\dagger}$$

Power of an ancilla system

Basic operations
Unitary operations
Orthogonal measurements

+

An auxiliary system
(ancilla)

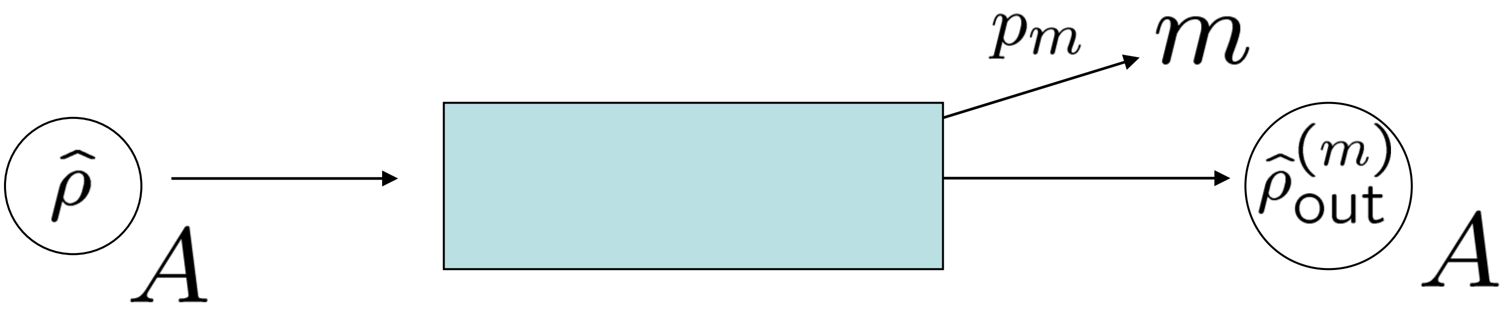


What can we do in principle?

We have seen what we can (at least) do by using an ancilla system.

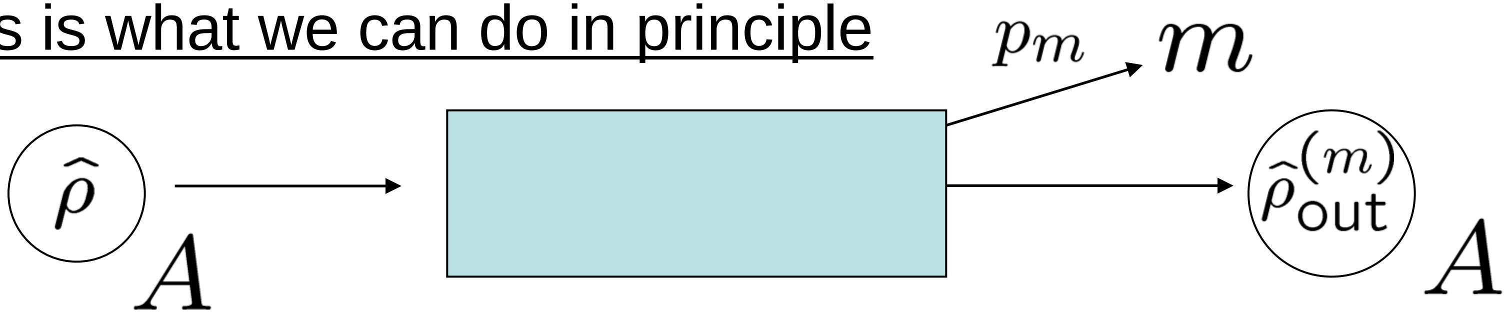
$$p_j \hat{\rho}^{(j)}_{\mathrm{out}} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

We also want to know what we cannot do.



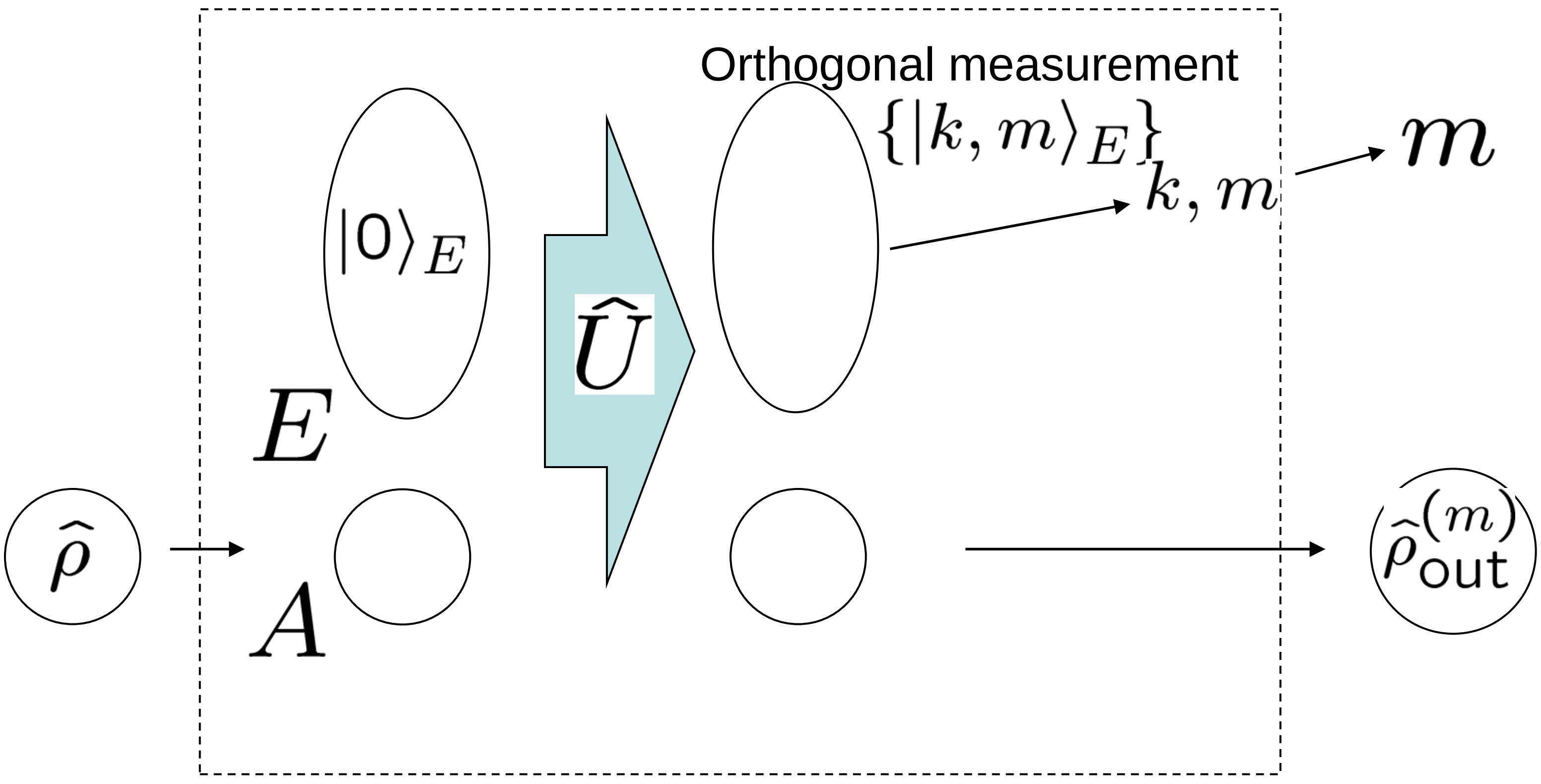
Black box with classical and quantum outputs

This is what we can do in principle

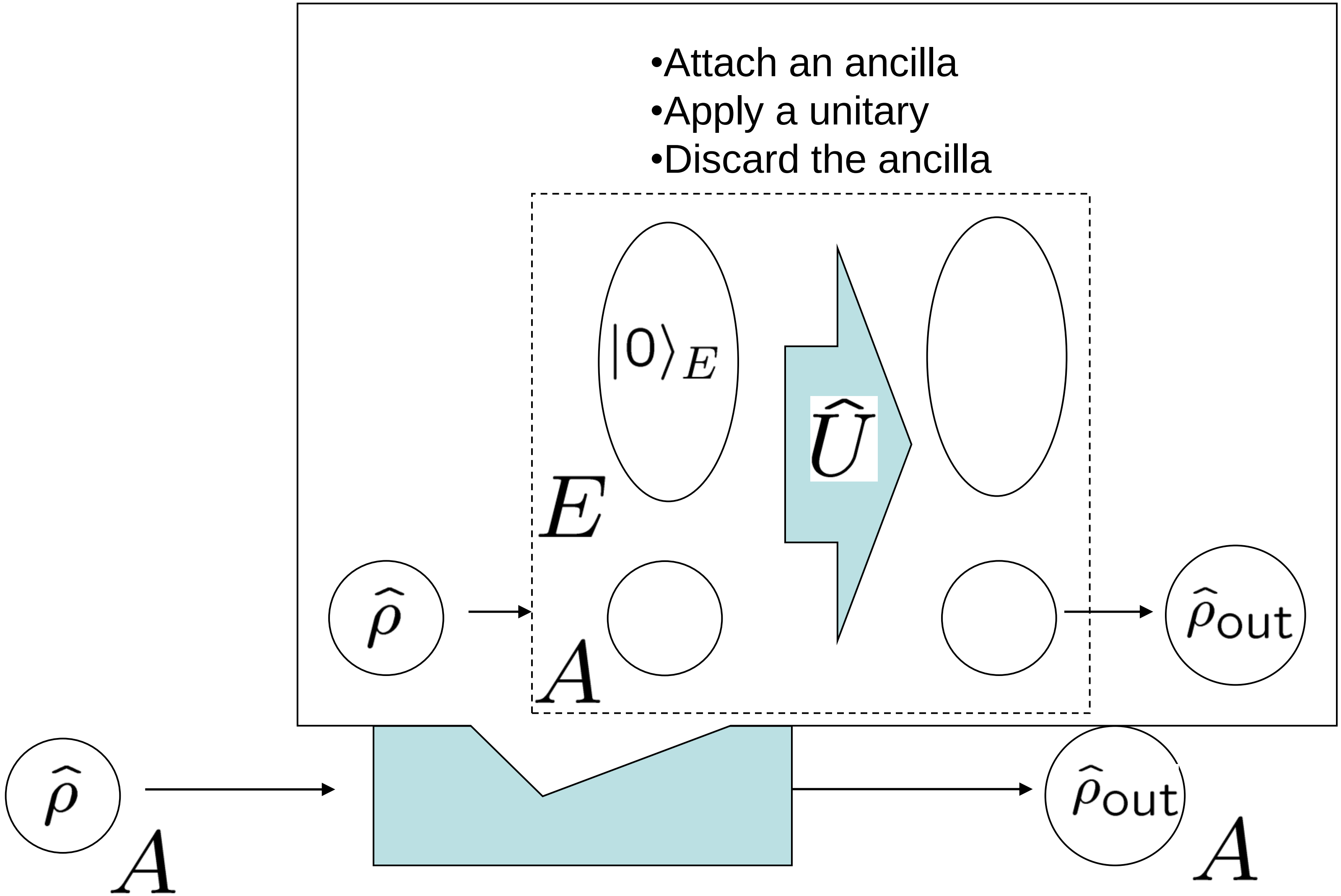


Any physical process should be represented in the following form:

$$p_m \hat{\rho}_{\text{out}}^{(m)} = \sum_k \hat{M}^{(k,m)} \hat{\rho} \hat{M}^{(k,m)\dagger} \qquad \sum_{m,k} \hat{M}^{(k,m)\dagger} \hat{M}^{(k,m)} = \hat{1}_A$$



What can we do in principle?



Black box with quantum output
(Quantum channel)

Maximally entangled states (MES)

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$$

Orthonormal bases $\{|k\rangle_A\}_{k=1,2,\ldots,d}$ $\{|k\rangle_B\}_{k=1,2,\ldots,d}$

$$\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

Maximally entangled state

(with Schmidt number d)

Properties of MES (II): Relative states

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$$

Fix a maximally entangled state

$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A |k\rangle_B$$

A B

Relative states

$$|\phi\rangle_A = \sum_k \alpha_k |k\rangle_A \longleftrightarrow |\phi^*\rangle_B = \sum_k \overline{\alpha_k} |k\rangle_B$$

$$= \sqrt{d} \times {}_B\langle \phi^* | |\Phi\rangle_{AB} \qquad = \sqrt{d} \times {}_A\langle \phi | |\Phi\rangle_{AB}$$

$|\Phi\rangle_{AB}$

$A \longrightarrow |\phi\rangle_A$

B $\xrightarrow{\text{Orthogonal measurement}}$ outcome $j = 1$

$\{|v_j\rangle_B\}_{j=1,2,\ldots,d}$

$p_1 = \frac{1}{d}$

$|v_1\rangle_B = |\phi^*\rangle_B$

$$\boxed{\frac{1}{\sqrt{d}} |\phi\rangle_A = {}_B\langle \phi^* | |\Phi\rangle_{AB}}$$

Properties of MES (III): Pair of equivalent local operations

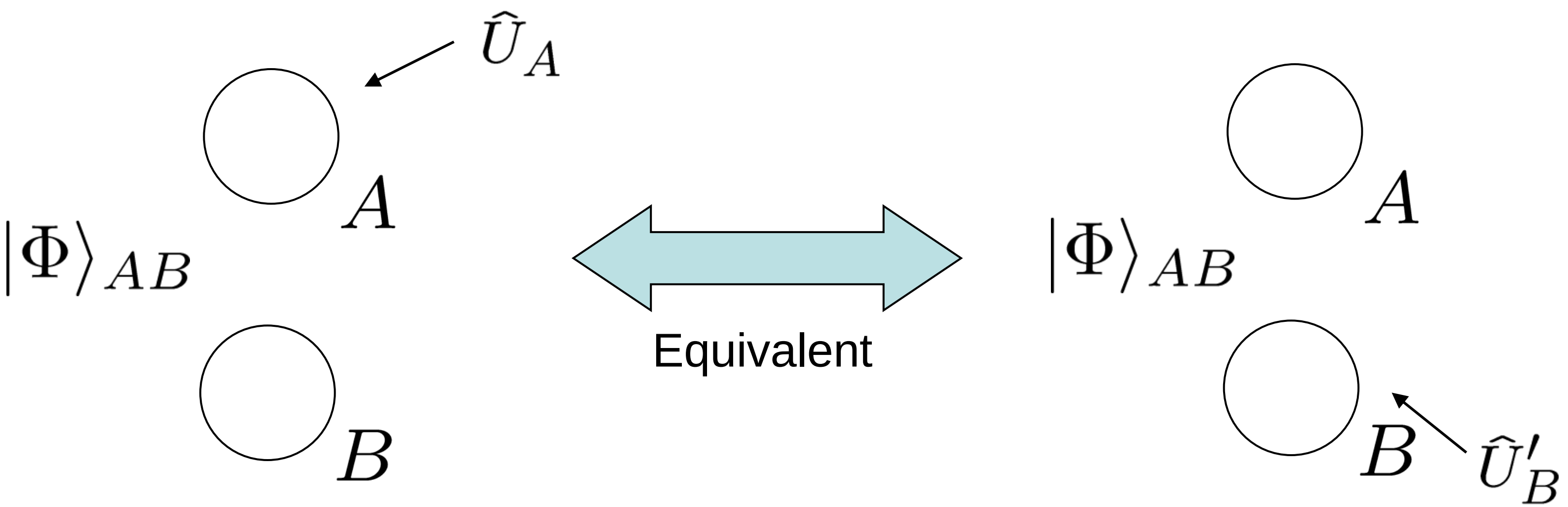
$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A |k\rangle_B$$

$$(\hat{T}_A \otimes \hat{1}_B)|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{T}'_B)|\Phi\rangle_{AB}$$

$${}_A\langle l| \otimes {}_B\langle k|$$

$${}_A\langle l|\hat{T}_A|k\rangle_A = {}_B\langle k|\hat{T}'_B|l\rangle_B$$

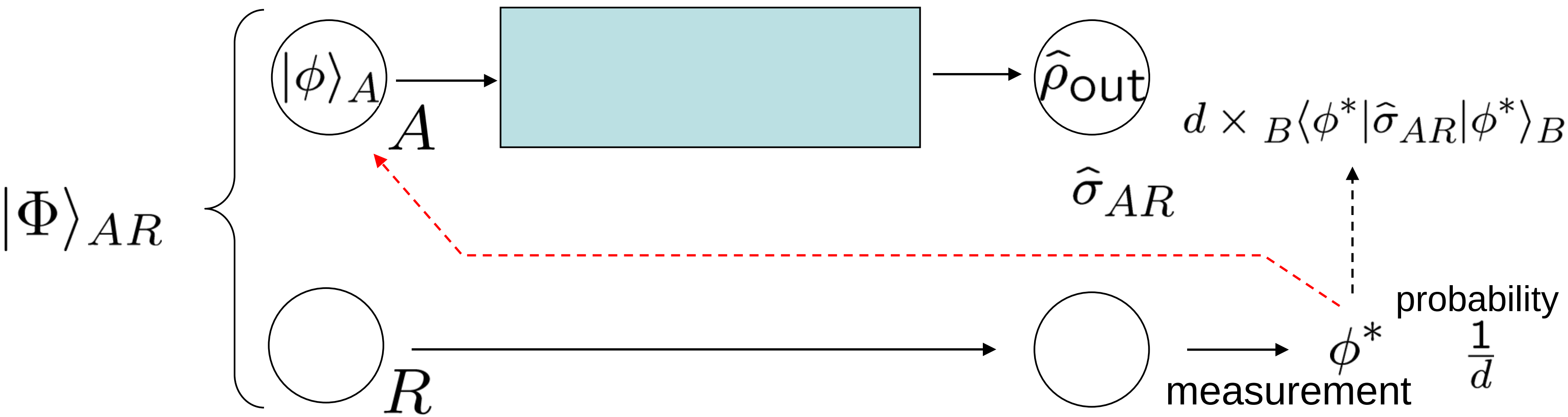
transpose



Quantum operation and bipartite state

We can remotely prepare system A in any state
with a nonzero success probability.

At any time



$\hat{\sigma}_{AR}$:The state obtained when a half of an MES is fed to the channel.

If this state is known,

$$\hat{\rho}_{\mathrm{out}} = {}_B\langle\phi^*|\hat{\sigma}_{AR}|\phi^*\rangle_B d$$

Output for every input state is known!

Characterization of a process = Characterization of a state

Quantum operation and bipartite state

$$\hat{\rho}_{\mathrm{out}} = \sqrt{d}\,{}_R\langle\phi^*|\hat{\sigma}_{AR}|\phi^*\rangle_R\sqrt{d}$$

$${}_R\langle\phi^*| = \sqrt{d}\;{}_{AR}\langle\Phi||\phi\rangle \qquad \hat{\sigma}_{AR} = \sum_j |\Psi_j\rangle_{AR}\;{}_{AR}\langle\Psi_j|$$

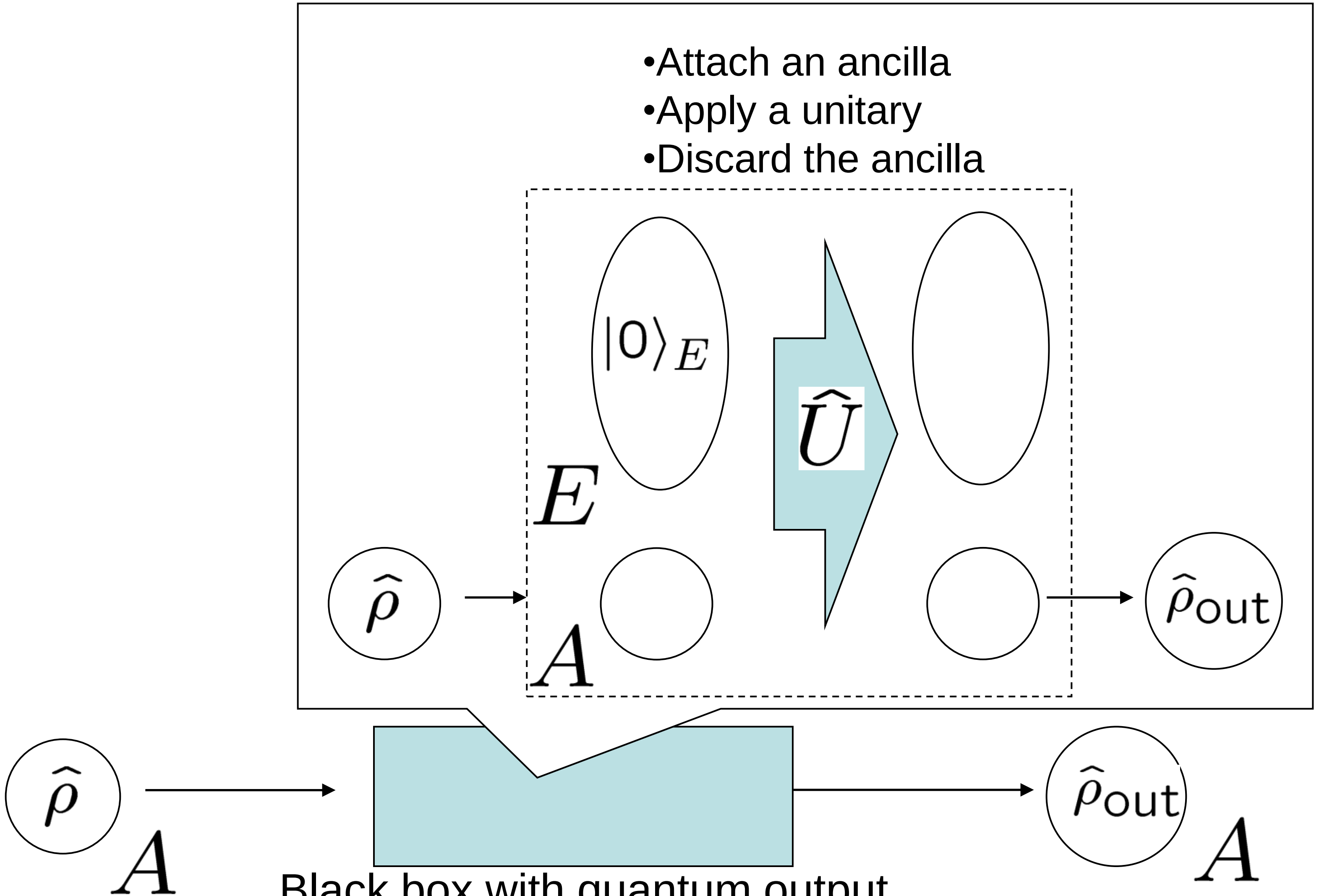
unnormalized

$$\sqrt{d}\,{}_R\langle\phi^*||\Psi_j\rangle_{AR} = \hat{M}^{(j)}|\phi\rangle_A$$ (A linear map)

$$\hat{\rho}_{\mathrm{out}} = \sum_j \hat{M}^{(j)}|\phi\rangle_{A\,A}\langle\phi|\hat{M}^{(j)\dagger}$$

$${}_{AR}\langle\Phi|\,|\phi\rangle_A\,|\Psi_j\rangle_{AR}$$

What we can do in principle

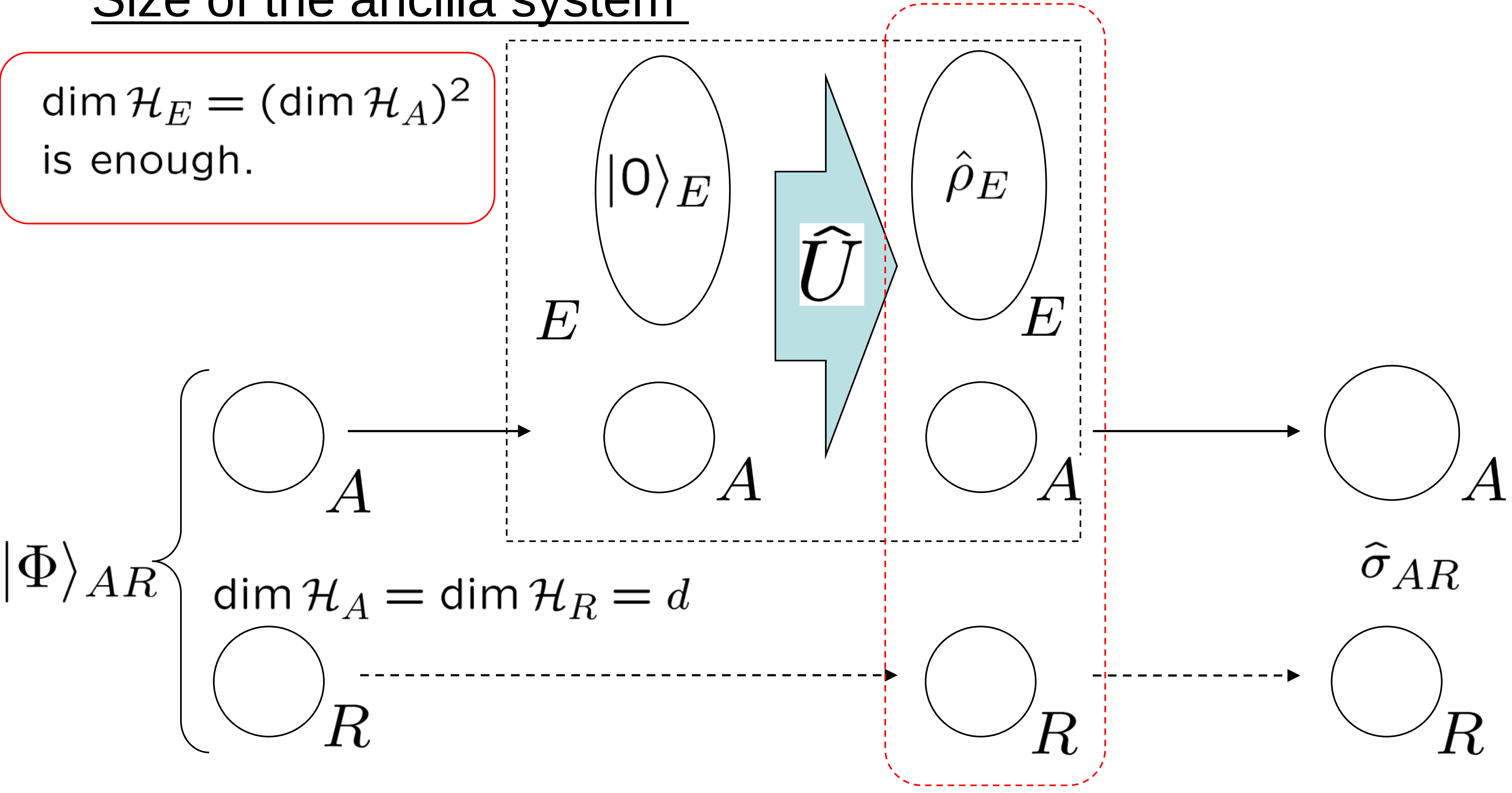


Black box with quantum output
(Quantum channel)

$$\hat{\rho}_{\mathrm{out}} = \sum_j \widehat{M}^{(j)} \hat{\rho} \widehat{M}^{(j)\dagger}$$

Size of the ancilla system

$\dim \mathcal{H}_E = (\dim \mathcal{H}_A)^2$
is enough.



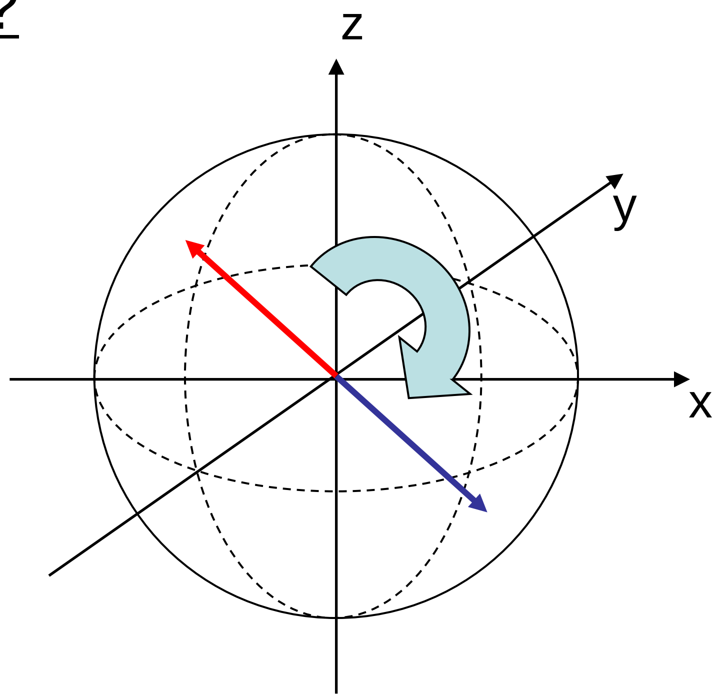
$|\xi\rangle_{ARE} \equiv \hat{U}(|\Psi\rangle_{AR} \otimes |0\rangle_E)$

$$\dim(\mathrm{Ran}\ \hat{\rho}_E) = \dim(\mathrm{Ran}\ \hat{\rho}_{AR}) \leq \dim \mathcal{H}_{AR} = d^2$$

Universal NOT ? Spin reversal ?

Bloch vector

$$\boldsymbol{P} \to -\boldsymbol{P}$$



linear map $\hat{\rho} \to \mathcal{C}(\hat{\rho})$

$$\mathcal{C}(\hat{1}) = \hat{1} \qquad \mathcal{C}(\hat{\sigma}_x) = -\hat{\sigma}_x$$

$$\mathcal{C}(\hat{\sigma}_y) = -\hat{\sigma}_y \qquad \mathcal{C}(\hat{\sigma}_z) = -\hat{\sigma}_z$$

$$\mathcal{C}(|0\rangle\langle 0|) = |1\rangle\langle 1|$$

$$\mathcal{C}(|1\rangle\langle 1|) = |0\rangle\langle 0|$$

$$\mathcal{C}(|0\rangle\langle 1|) = -|0\rangle\langle 1|$$

$$\mathcal{C}(|1\rangle\langle 0|) = -|1\rangle\langle 0|$$

$$\hat{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$\hat{\sigma}_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$

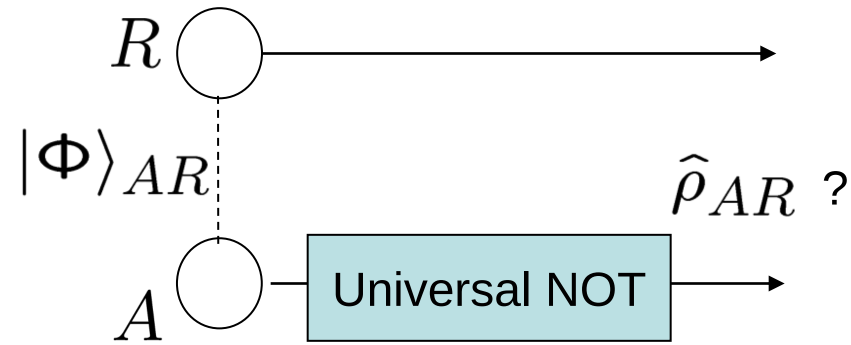
$$\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\hat{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

This map is positive, but...

Universal NOT ? Spin reversal ?

$$\mathcal{C}(|0\rangle\langle 0|) = |1\rangle\langle 1|$$
$$\mathcal{C}(|1\rangle\langle 1|) = |0\rangle\langle 0|$$
$$\mathcal{C}(|0\rangle\langle 1|) = -|0\rangle\langle 1|$$
$$\mathcal{C}(|1\rangle\langle 0|) = -|1\rangle\langle 0|$$



$$2|\Phi\rangle\langle\Phi| = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$
$$= |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|$$

$$2\hat{\rho}_{AR} \equiv 2(\mathcal{C} \otimes \mathcal{I})|\Phi\rangle\langle\Phi| =$$
$$= |10\rangle\langle 10| - |00\rangle\langle 11| - |11\rangle\langle 00| + |01\rangle\langle 01|$$

$$2\hat{\rho}_{AR}(|00\rangle + |11\rangle) = -|11\rangle - |00\rangle = -(|00\rangle + |11\rangle)$$

$\hat{\rho}_{AR}$ has a negative eigenvalue! (The map is not completely positive.)

$\longrightarrow$ Universal NOT is impossible.