5. Communication resources

Classical channel
Quantum channel
Entanglement
- How does the state evolve under LOCC?
- Properties of maximally entangled states
- Bell basis

Quantum dense coding
Quantum teleportation
- Entanglement swapping

Resource conversion protocols and bounds

Classical channel

0 ------------------------------------→ 0
1 ------------------------------------→ 1

Ideal classical channel: faithful transfer of any signal chosen from d symbols

Parallel use of channels

0 ----------→ 0
1 ----------→ 1

a ----------→ a
b ----------→ b
c ----------→ c

}

0a ----------→ 0a
0b ----------→ 0b
0c ----------→ 0c
1a ----------→ 1a
1b ----------→ 1b
1c ----------→ 1c

d-symbol ideal classical channel
d'-symbol ideal classical channel
} (dd')-symbol ideal classical channel

Measure of usefulness

d-symbol ideal classical channel ⟶ (log d) bits

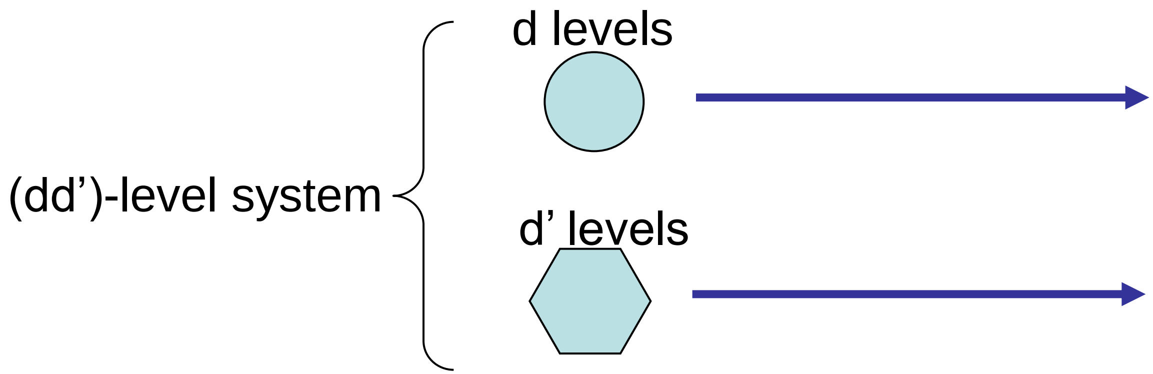
Additive for ideal channels

Quantum channel

$\alpha|0\rangle + \beta|1\rangle$ $\alpha|0\rangle + \beta|1\rangle$

Ideal quantum channel: faithful transfer of any state of an d-level system
(Hilbert space of dimension d)

Parallel use of channels



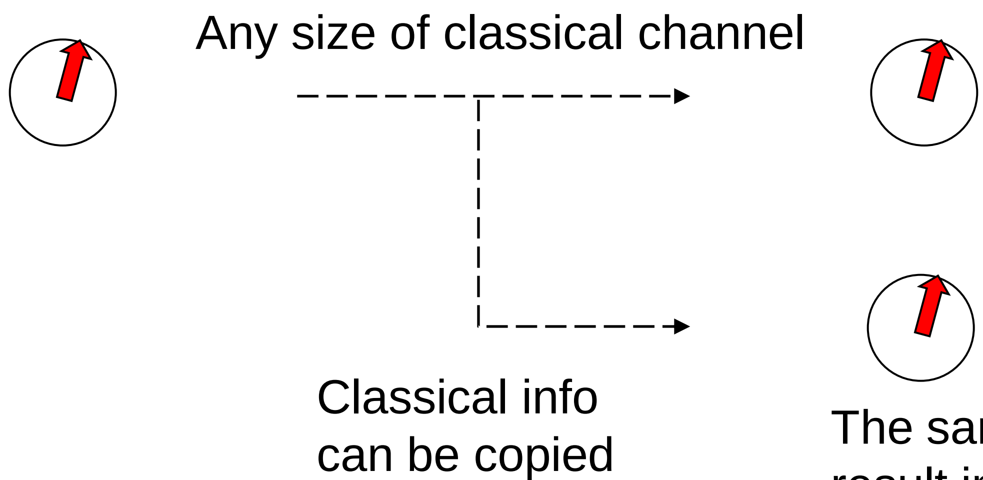
Measure of usefulness

d-level ideal quantum channel → (log d) qubits
Additive for ideal channels

Can classical channels substitute a quantum channel?

NO (with no other resources)

Suppose that it was possible …



The same procedure should result in the same state.

This amounts to the cloning of unknown quantum states, which is forbidden.

Can a quantum channel substitute a classical channel?

Of course yes.

But not so bizarre (with no other resources).

n-qubit ideal quantum channel can only substitute a n-bit classical channel.

(Holevo bound)

Suppose that transfer of an d-level system can convey any signal from s symbols faithfully.

Measurement

$j = 1, 2, \ldots, s$

$\dim \mathcal{H} = d$

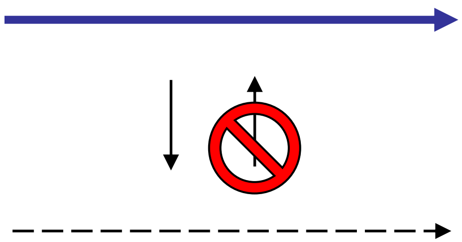
Always $j' = j$

Recall that any measurement must be described by a POVM. $\sum_{j'} \widehat{F}_{j'} = \widehat{1}$

$\mathrm{Tr}(\widehat{F}_j \widehat{\rho}_j) = 1$

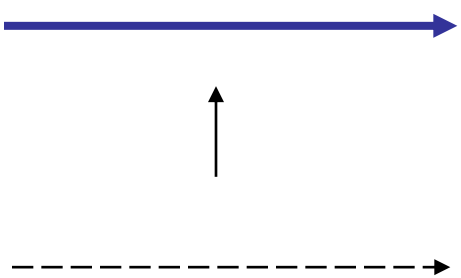
$$s = \sum_j \mathrm{Tr}(\widehat{F}_j \widehat{\rho}_j) \leq \sum_j \mathrm{Tr}(\widehat{F}_j \widehat{1}) = \sum_j \mathrm{Tr}(\widehat{F}_j) \leq \sum_{j'} \mathrm{Tr}(\widehat{F}_{j'}) = \mathrm{Tr}(\widehat{1}) = d$$

Difference between quantum and classical channels



We have seen that a quantum channel is more powerful than a classical channel.

Can we pin down what is missing in a classical channel?



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Other resource

Entanglement

I've already bought a classical channel, but now I want to use a quantum channel. Do I have to buy the quantum channel?

Oh, you can buy this optional package for a cheaper price, and upgrade the classical channel to a quantum channel!

Operational definition of entanglement

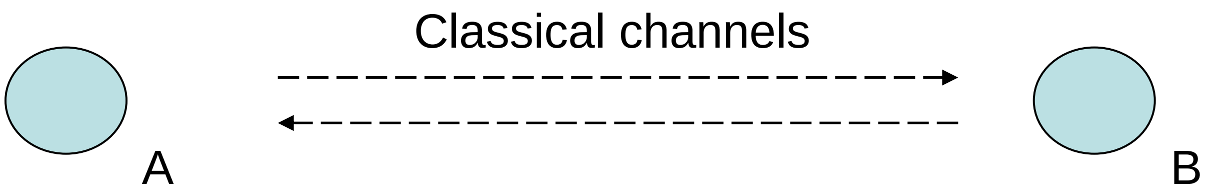
"Correlations that cannot be created over classical channels"

LOCC: Local operations and classical communication

Alice has a subsystem A, and Bob has a subsystem B.

Operations (including measurements) on a local subsystem are free.

Communication between Alice and Bob only uses classical channels.



Separable states: The states that can be created under LOCC from scratch.

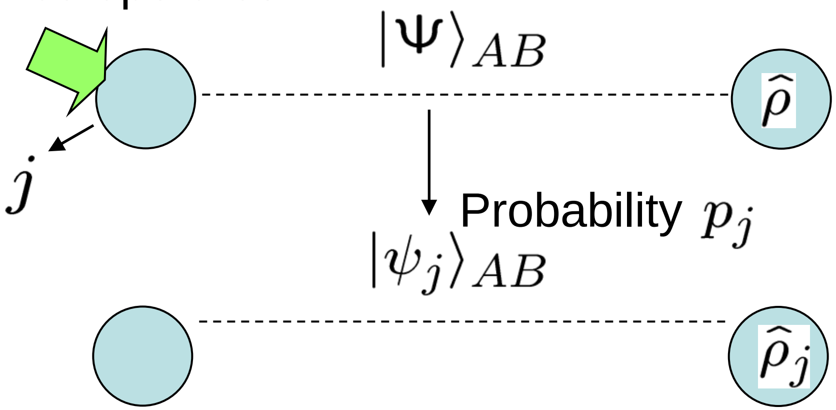
Entangled states: The states that cannot be created under LOCC from scratch.

How does the state evolve under LOCC?

Any LOCC procedure can be made a sequential one:

- Alice appies local operations
- Alice communicates to Bob
- Bob applies local operations
- Bob communicates to Alice
- Alice …..

When Alice operates



$$\sum_j p_j \hat{\rho}_j = \hat{\rho}$$

Ran $\hat{\rho} \supset$ Ran $\hat{\rho}_j$

Schmidt number never increases under LOCC (even probabilistically)

Schmidt number >1 $\longrightarrow$ Impossible under LOCC

If a concave functional S only depends on the eigenvalues,

$$S(\hat{\rho}) \geq \sum_j p_j S(\hat{\rho}_j)$$

Any such functional of the marginal density operator (e.g., von Neumann entropy) is monotone decreasing under LOCC on average.

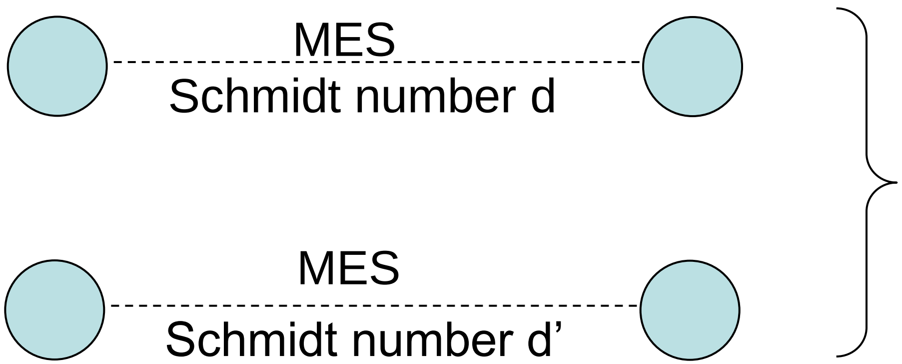
Maximally entangled states (MES)

"ideal" entangled states

$$\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B \qquad p_1 = p_2 = \cdots = p_d = \frac{1}{d}$$

Schmidt number = d

Putting two MESs together



MES with

Schmidt number dd'

$$\left(\sum_{j=1}^{d} \frac{1}{\sqrt{d}} |j\rangle_A \otimes |j\rangle_B \right) \otimes \left(\sum_{k=1}^{d'} \frac{1}{\sqrt{d'}} |k\rangle_{A'} \otimes |k\rangle_{B'} \right) = \sum_{j,k} \frac{1}{\sqrt{dd'}} |jk\rangle_{AA'} \otimes |jk\rangle_{BB'}$$

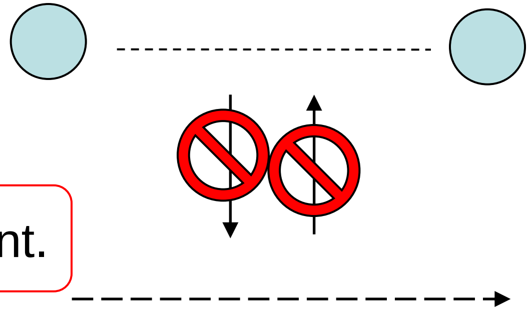
Measure of entanglement

MES with Schmidt number d ⟶ (log d) ebits

Additive for MESs

Ebits and bits are mutually exclusive

Schmidt number never increases under LOCC.

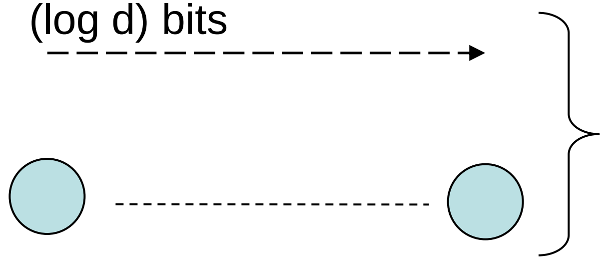


Classical channels cannot increase (ideal) entanglement.

d-symbol ideal classical channel

The outcome can be correctly predicted with probability at least 1/d.

Transfer of s symbols



Success probability 1

Transfer of s symbols

Success probability 1/d

$\longrightarrow \quad s \leq d$

Entanglement cannot assist (ideal) classical channels

Resource conversion protocols

Conversion to ebits

Entanglement sharing

1 qubit ⟶ 1 ebit

Conversion to bits

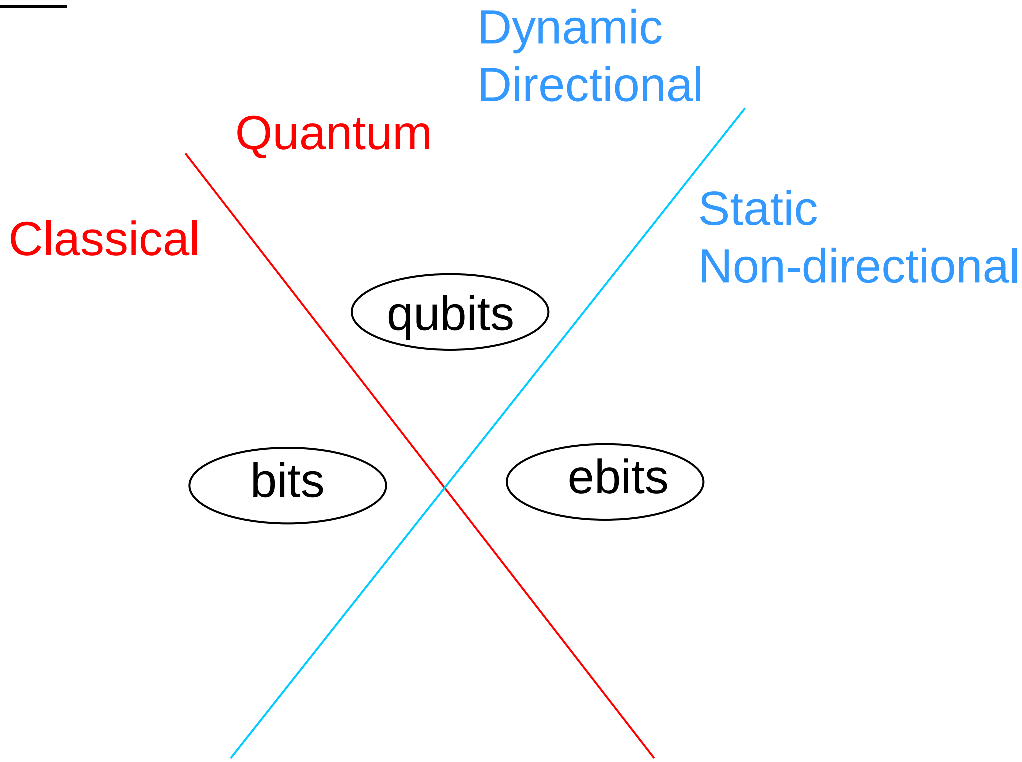
Quantum dense coding

1 qubit + 1 ebit ⟶ 2 bits

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit ⟶ 1 qubit



Restrictions

bits alone ⟶ no ebits

ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit

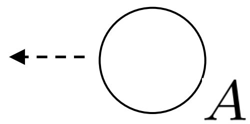
Properties of maximally entangled states

$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

Pair of local states (relative states)

$$\frac{1}{\sqrt{d}}|\phi\rangle_A = {}_B\langle\phi^*||\Phi\rangle_{AB}$$

$$|\phi\rangle_A = \sum_k \alpha_k |k\rangle_A$$



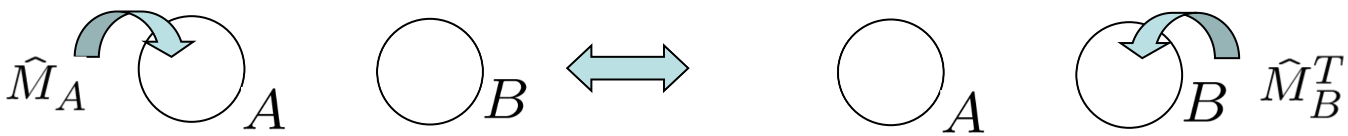
$$|\phi^*\rangle_B = \sum_k \overline{\alpha_k} |k\rangle_B$$

measurement

$p = 1/d$

Pair of local operations

$$(\hat{M}_A \otimes \hat{1}_B)|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{M}_B^T)|\Phi\rangle_{AB}$$



Locally maximally mixed

$$\hat{\rho}_A = \mathrm{Tr}_B |\Phi\rangle\langle\Phi| = \frac{1}{d}\hat{1}_A$$

Convertibility via local unitary

$$|\Phi'\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Phi\rangle_{AB}$$

Orthonormal basis (Bell basis)

$$\langle\Phi_j|\Phi_k\rangle = \delta_{jk} \; (j,k = 1, \ldots d^2)$$

There exists an orthonormal basis composed of MESs.

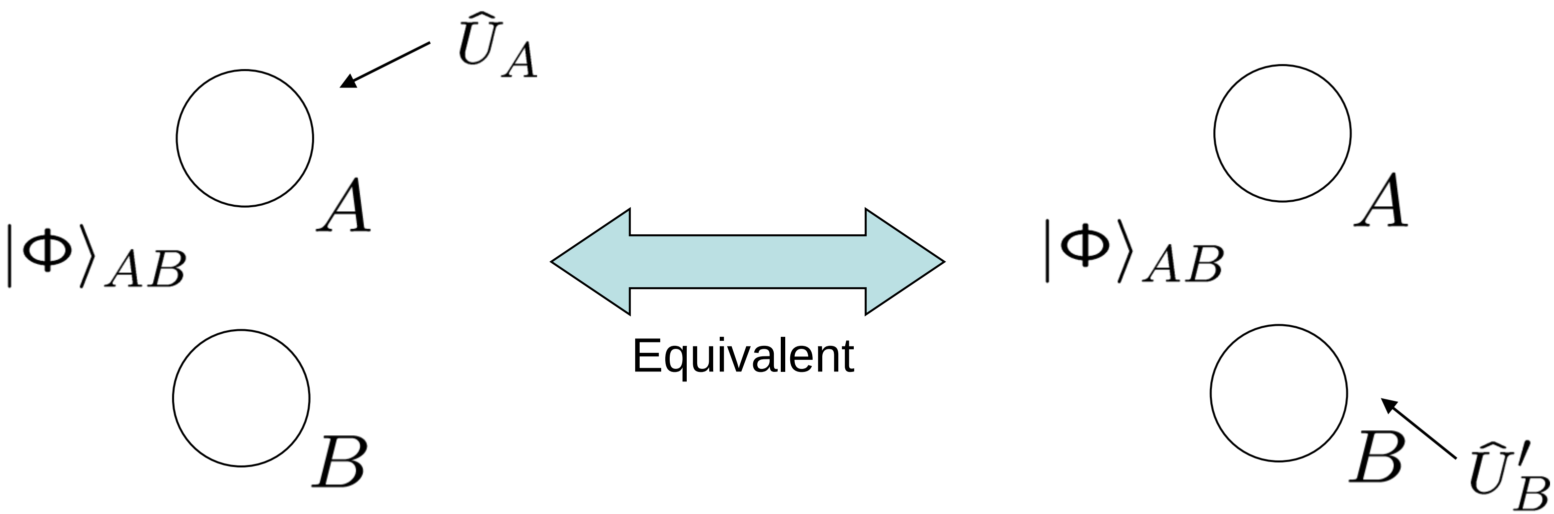
Local operations on a maximally entangled state

$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

$$(\hat{T}_A \otimes \hat{1}_B)|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{T}'_B)|\Phi\rangle_{AB}$$

$${}_A\langle l| \otimes {}_B\langle k|$$

$${}_A\langle l|\hat{T}_A|k\rangle_A = {}_B\langle k|\hat{T}'_B|l\rangle_B$$ transpose



Bell basis for a pair of qubits

$(d = 2)$

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B) = \hat{Z}_B|\Phi_+\rangle$$

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B) = \hat{X}_A|\Phi_+\rangle$$

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B) = (\hat{X}_A \otimes \hat{Z}_B)|\Phi_+\rangle$$

$$\hat{X} \equiv \hat{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$
$$\hat{Z} \equiv \hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Bell basis

$$\beta \equiv \exp[2\pi i/d] \quad (\beta^d = \beta^0 = 1, \beta^{-1} = \overline{\beta})$$

Basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\} \quad (|d\rangle = |0\rangle)$

$$\hat{X} \equiv \sum_{j=0}^{d-1} |j+1\rangle\langle j| \qquad \hat{Z} \equiv \sum_{j=0}^{d-1} \beta^j |j\rangle\langle j| \qquad \text{(Unitary)}$$

$$\hat{X}^T = \hat{X}^{-1} \qquad \hat{Z}^T = \hat{Z}$$

$\hat{Z}^d = \hat{X}^d = \hat{1}$ Eigenvalues: $1, \beta, \beta^2, \ldots, \beta^{d-1}$

$$\hat{Z}\hat{X} = \beta \hat{X}\hat{Z} \qquad \hat{Z}^m \hat{X}^l = \beta^{lm} \hat{X}^l \hat{Z}^m$$

$$|\Phi_{0,0}\rangle \equiv \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

$$(\hat{X}_A \otimes \hat{X}_B)|\Phi_{0,0}\rangle = |\Phi_{0,0}\rangle$$
$$(\hat{Z}_A \otimes \hat{Z}_B^{-1})|\Phi_{0,0}\rangle = |\Phi_{0,0}\rangle$$

Bell basis: $\{|\Phi_{l,m}\rangle\} \; (l = 0, 1, \ldots, d-1; \; m = 0, 1, \ldots, d-1)$

$$|\Phi_{l,m}\rangle \equiv (\hat{X}_A^l \otimes \hat{Z}_B^m)|\Phi_{0,0}\rangle$$

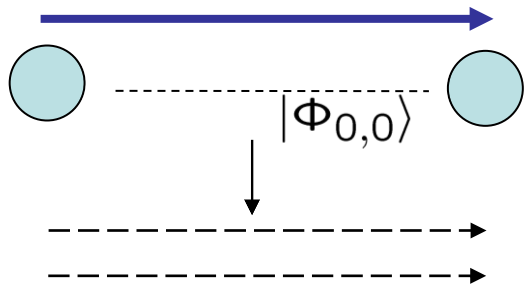
$$\left.\begin{array}{l} (\hat{X}_A \otimes \hat{X}_B)|\Phi_{l,m}\rangle = \beta^{-m}|\Phi_{l,m}\rangle \\ (\hat{Z}_A \otimes \hat{Z}_B^{-1})|\Phi_{l,m}\rangle = \beta^{l}|\Phi_{l,m}\rangle \end{array}\right\} \longrightarrow \text{All states are orthogonal.}$$

Quantum dense coding

1 qubit + 1 ebit ⟶ 2 bits

n qubits + n ebits ⟶ 2n bits

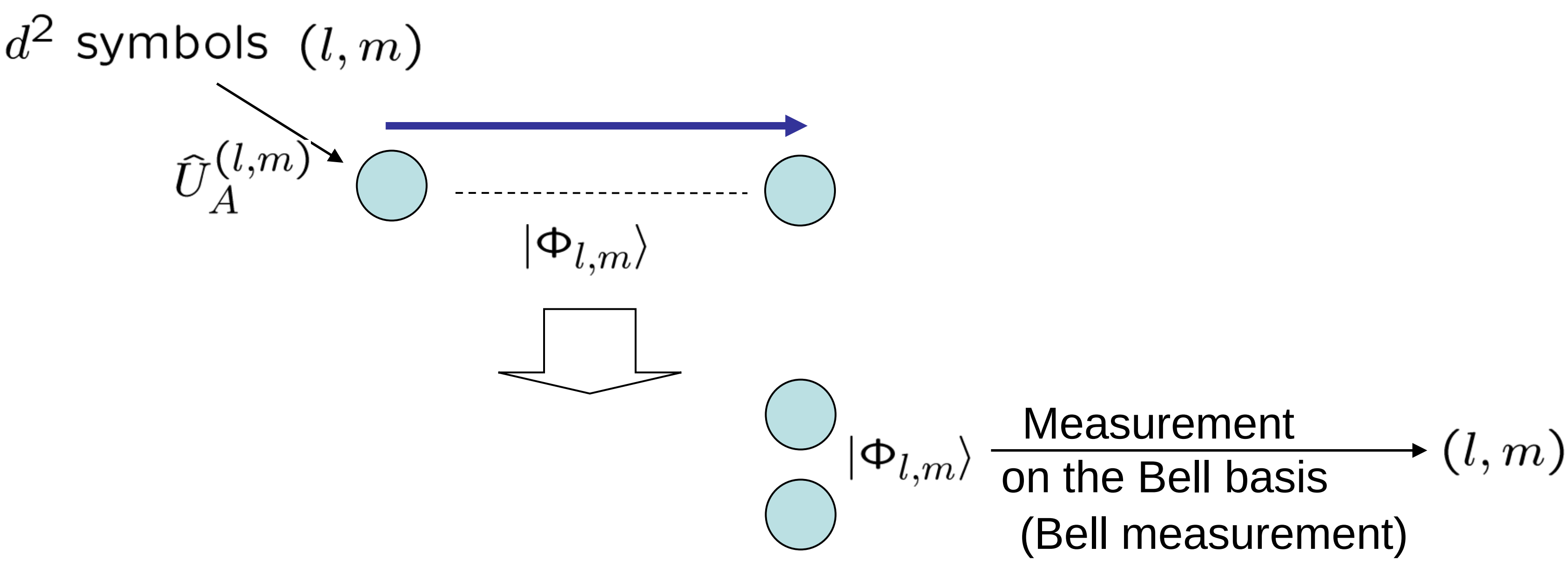
(Dimension d) + (Schmidt number d)
$\to$ (d^2 symbols)



MES

Convertibility via local unitary

Orthonormal basis (Bell basis)



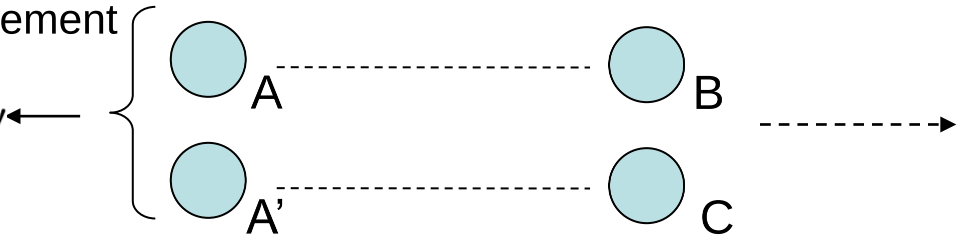
Entanglement swapping

$$|\Phi_{0,0}\rangle \equiv \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle \otimes |k\rangle$$

$$|\Theta\rangle_{AA'BC} = |\Phi_{0,0}\rangle_{AB} |\Phi_{0,0}\rangle_{A'C} = \sum_{j,k} \frac{1}{\sqrt{d^2}} |jk\rangle_{AA'} \otimes |jk\rangle_{BC}$$

Bell measurement

$|\Phi_{0,0}\rangle_{AA'}$

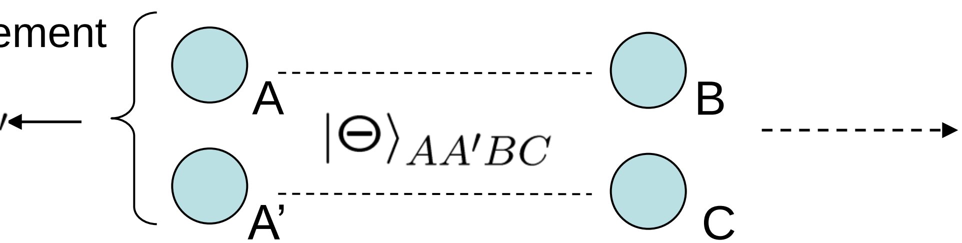


$|\Phi_{0,0}\rangle_{BC}$

$$_{AA'}\langle \Phi_{0,0} | |\Theta\rangle_{AA'BC} = \frac{1}{\sqrt{d^2}} |\Phi_{0,0}\rangle_{BC}$$

Bell measurement

$|\Phi_{l,m}\rangle_{AA'}$

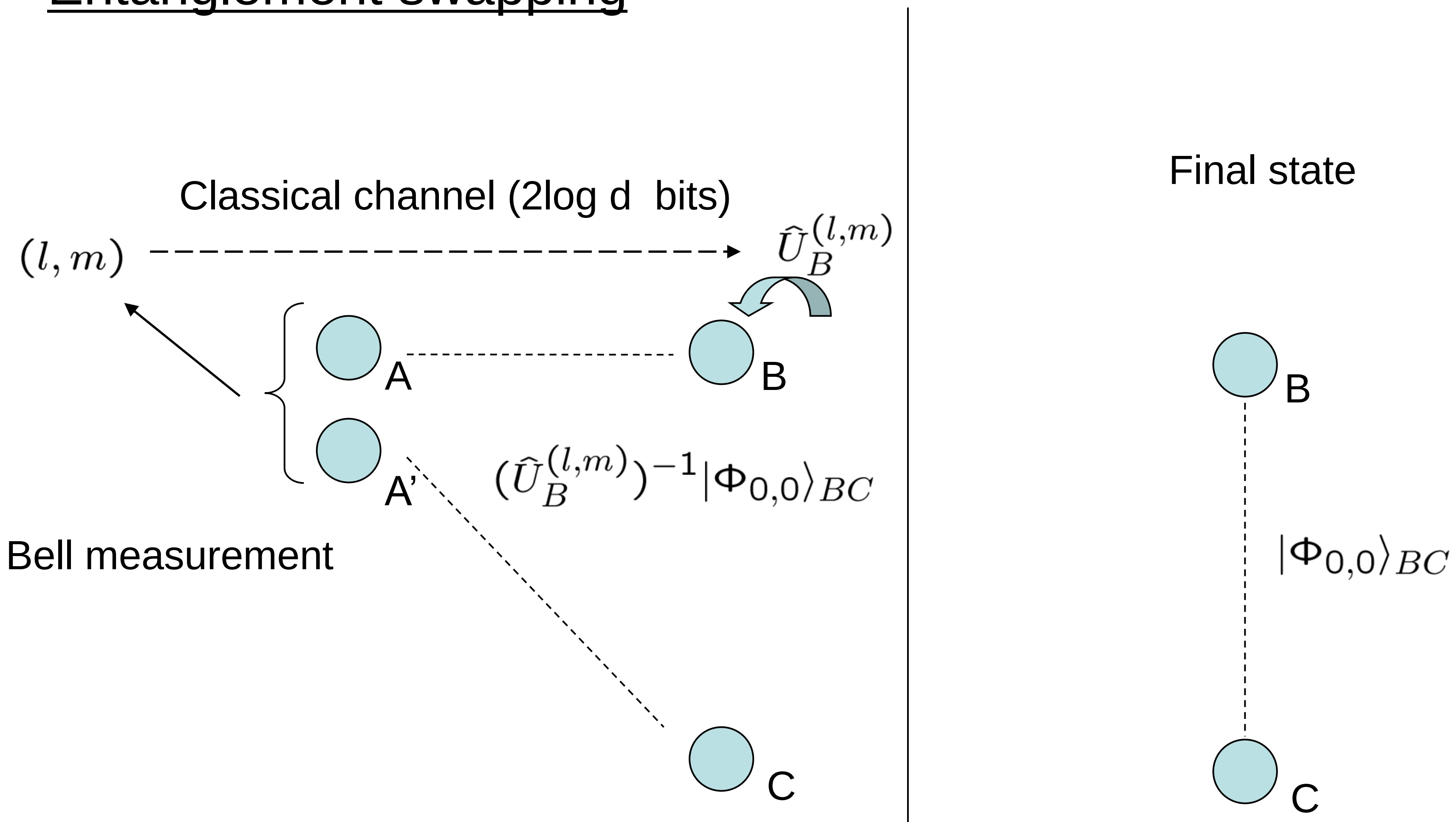


? $\quad (\hat{U}_B^{(l,m)})^{-1} |\Phi_{0,0}\rangle_{BC}$

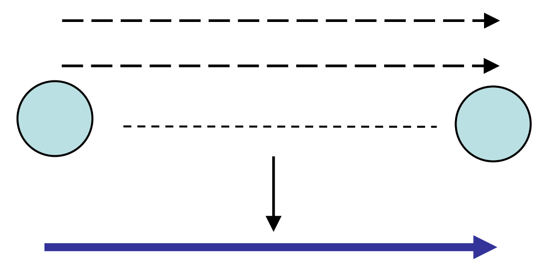
$$|\Phi_{l,m}\rangle_{AA'} = \hat{V}_A |\Phi_{0,0}\rangle_{AA'}$$

$$\begin{aligned} _{AA'}\langle \Phi_{l,m} | |\Theta\rangle_{AA'BC} &= {}_{AA'}\langle \Phi_{0,0} | \hat{V}_A^{\dagger} |\Theta\rangle_{AA'BC} \\ &= {}_{AA'}\langle \Phi_{0,0} | \hat{V}_B^{*} |\Theta\rangle_{AA'BC} \\ &= \hat{V}_B^{*} \left[{}_{AA'}\langle \Phi_{0,0} | |\Theta\rangle_{AA'BC} \right] \end{aligned}$$

Entanglement swapping



Quantum teleportation

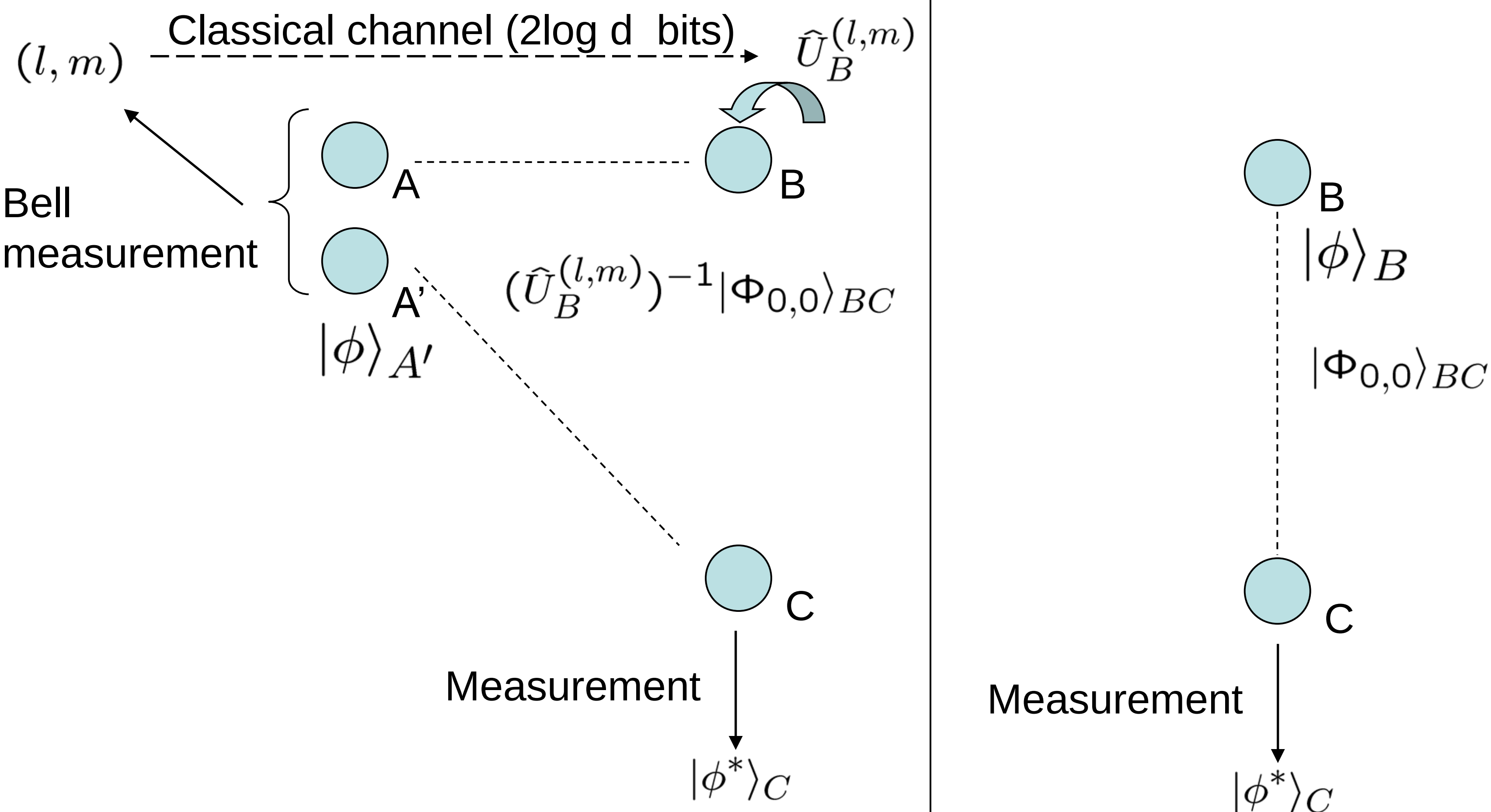


1 ebit + 2 bit → 1 qubit

n ebits + 2n bits → n qubits

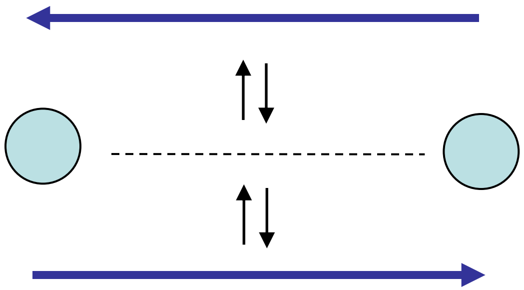
(d^2 symbols) + (Schmidt number d)

→ (Dimension d)



Quantum teleportation

If the cost of classical communication is neglected …

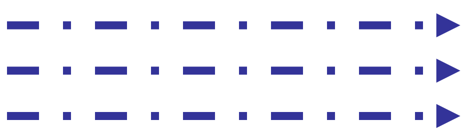


One can reserve the quantum channel by storing a quantum state.

One can use a quantum channel in the opposite direction.

A convenient way of quantum error correction (failure ⟶ retry).

Noisy quantum channel

$|\phi\rangle$  $\widehat{\rho} \rightarrow |\phi\rangle$

Recovering

Failure ⟶ no recovery.

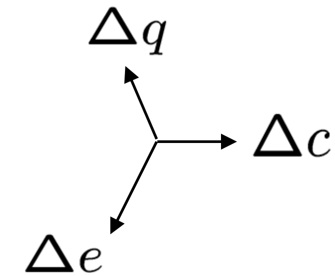
Noisy quantum channel

↓

Noisy entanglement

↓ Recovering

Resource conversion protocols and bounds



We can do the following…

Conversion to ebits

Entanglement sharing

1 qubit ⟶ 1 ebit

$$(\Delta q, \Delta e, \Delta c) = (-1, 1, 0)$$

Conversion to bits

Quantum dense coding

1 qubit + 1 ebit ⟶ 2 bits

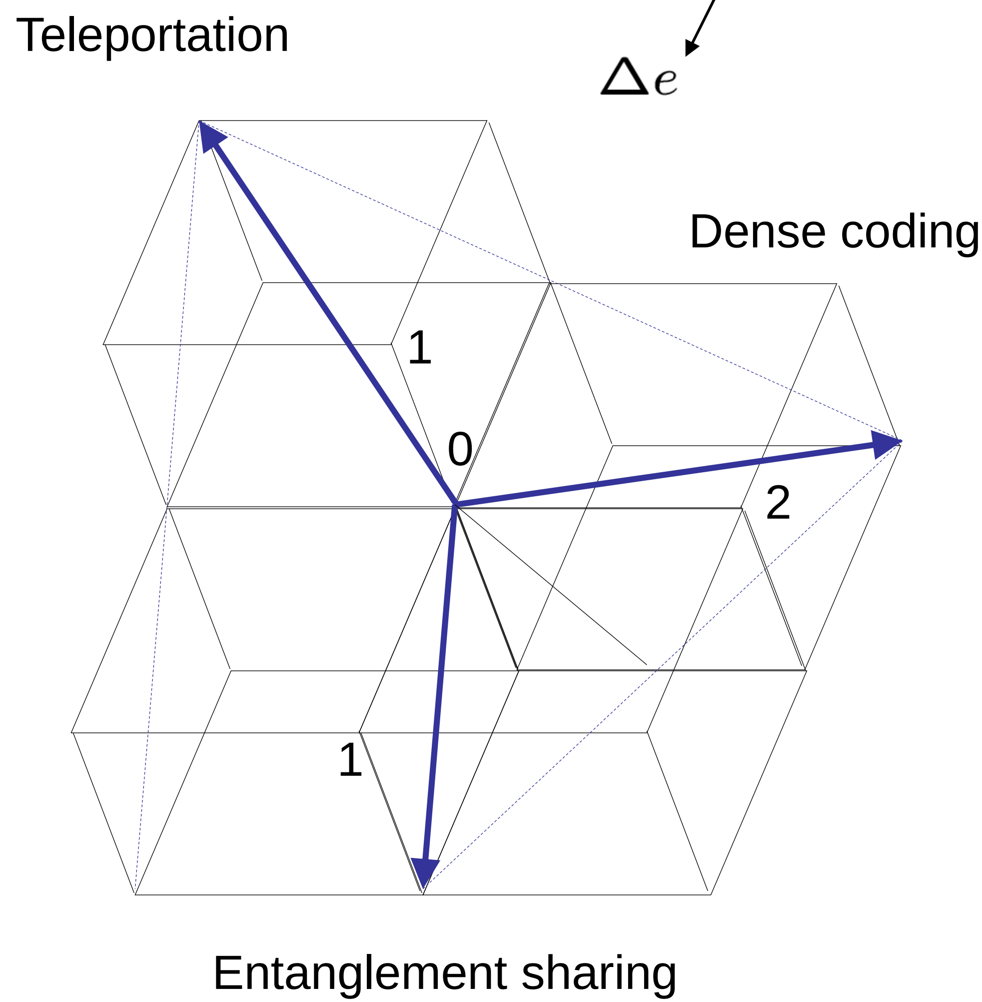
$$(\Delta q, \Delta e, \Delta c) = (-1, -1, 2)$$

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit ⟶ 1 qubit

$$(\Delta q, \Delta e, \Delta c) = (1, -1, -2)$$



Resource conversion protocols and bounds

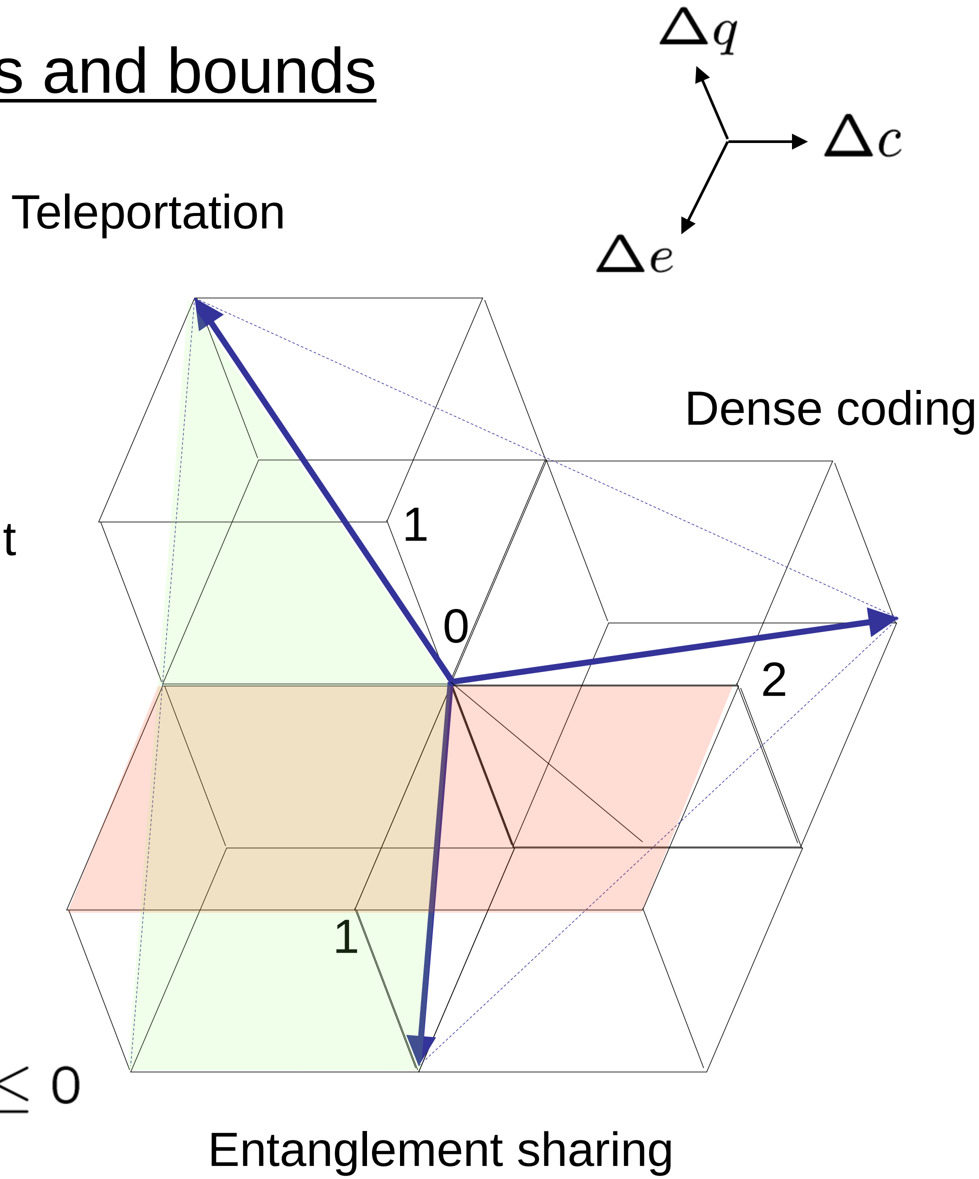
We can do the following…

Restrictions

bits alone ⟶ no ebits

ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit



$$\Delta e + \Delta q \leq 0$$

Resource conversion protocols and bounds

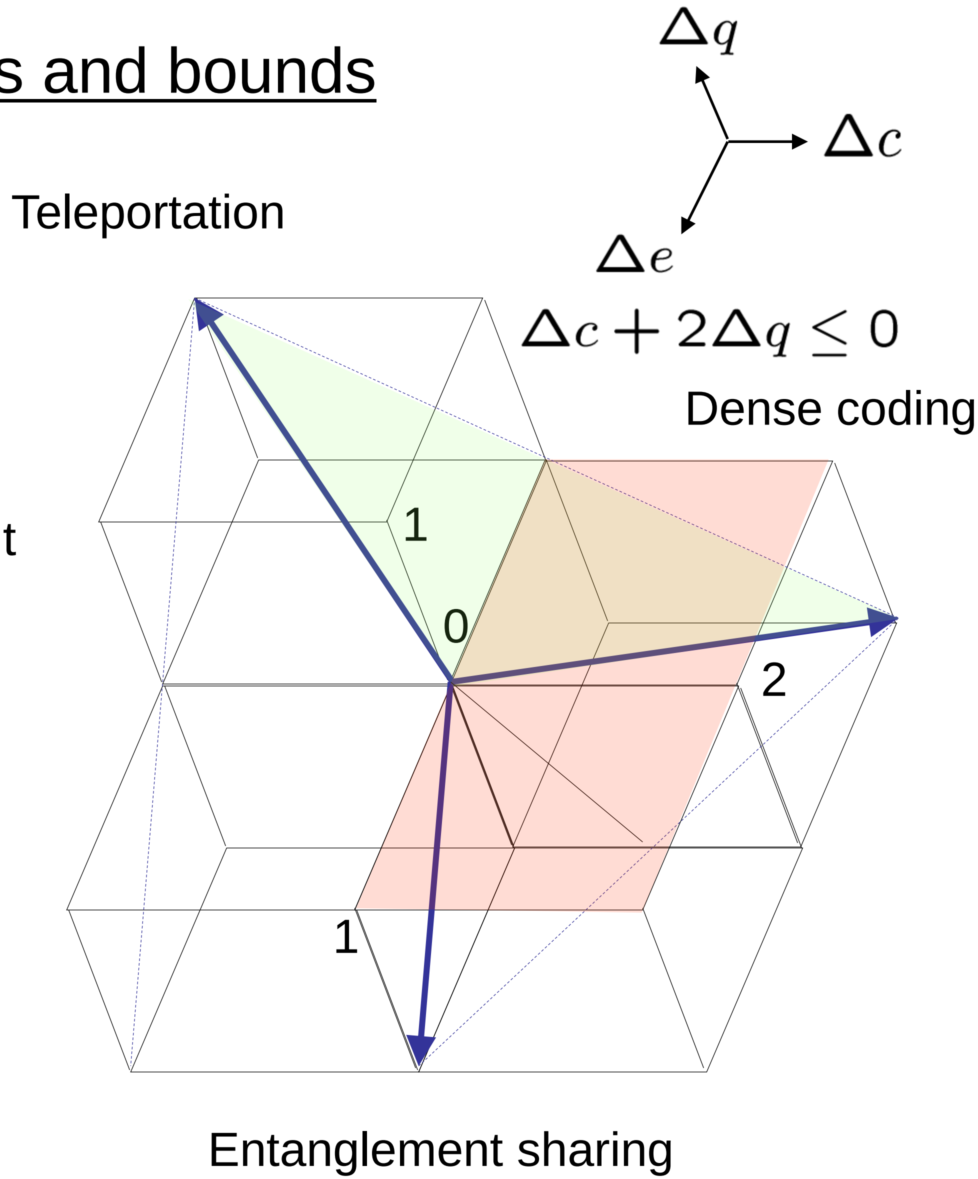
We can do the following…

Restrictions

bits alone ⟶ no ebits

ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit



Resource conversion protocols and bounds

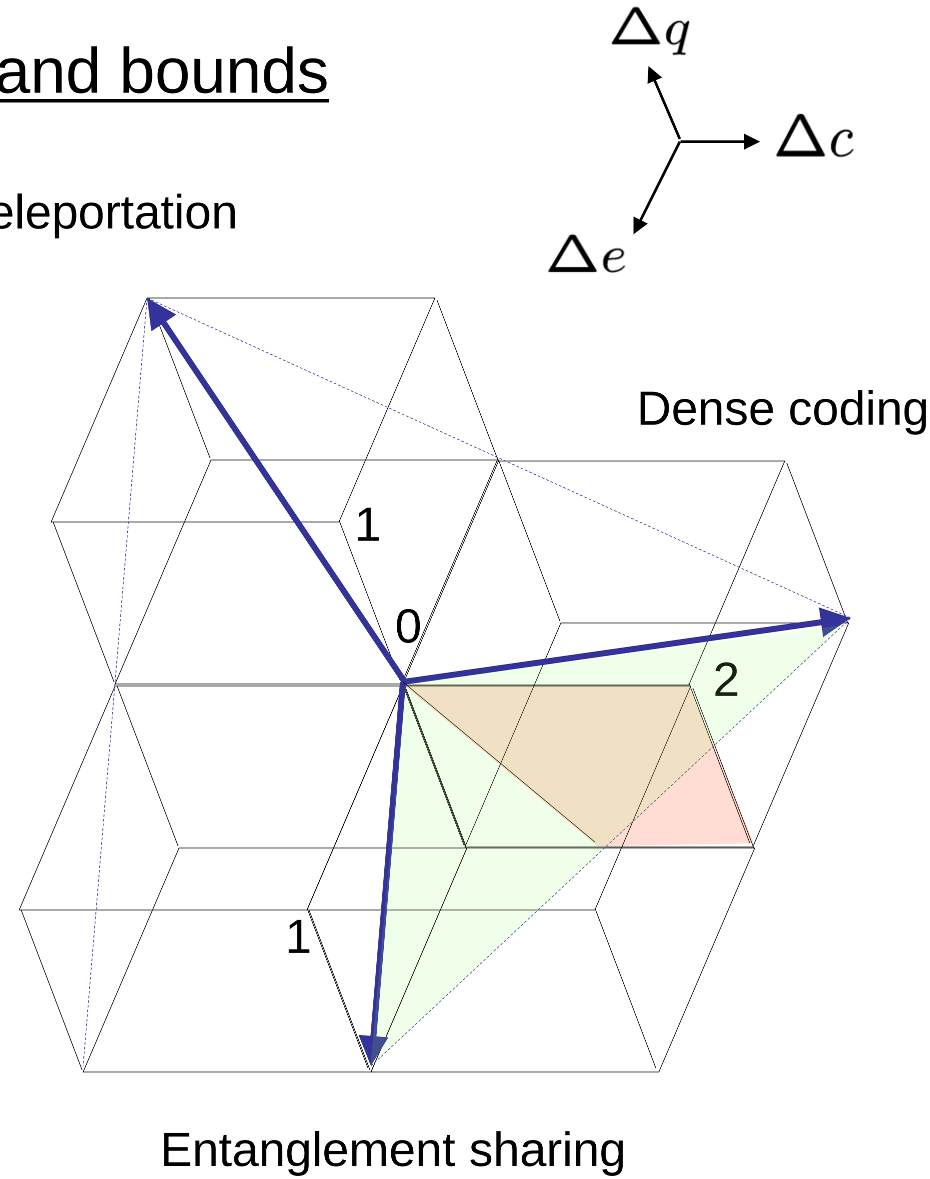
We can do the following…

Restrictions

bits alone ⟶ no ebits

ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit



$$\Delta c + \Delta q + \Delta e \leq 0$$

Resource conversion protocols and bounds

We can do the following…

Conversion to ebits

Entanglement sharing

1 qubit ⟶ 1 ebit

$$(\Delta q, \Delta e, \Delta c) = (-1, 1, 0)$$

Conversion to bits

Quantum dense coding

1 qubit + 1 ebit ⟶ 2 bits

$$(\Delta q, \Delta e, \Delta c) = (-1, -1, 2)$$

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit ⟶ 1 qubit

$$(\Delta q, \Delta e, \Delta c) = (1, -1, -2)$$

We cannot violate the following …

Entanglement never assists classical channels

+ QD,QT

$$\Delta c + 2\Delta q \leq 0$$

Classical channels cannot increase entanglement

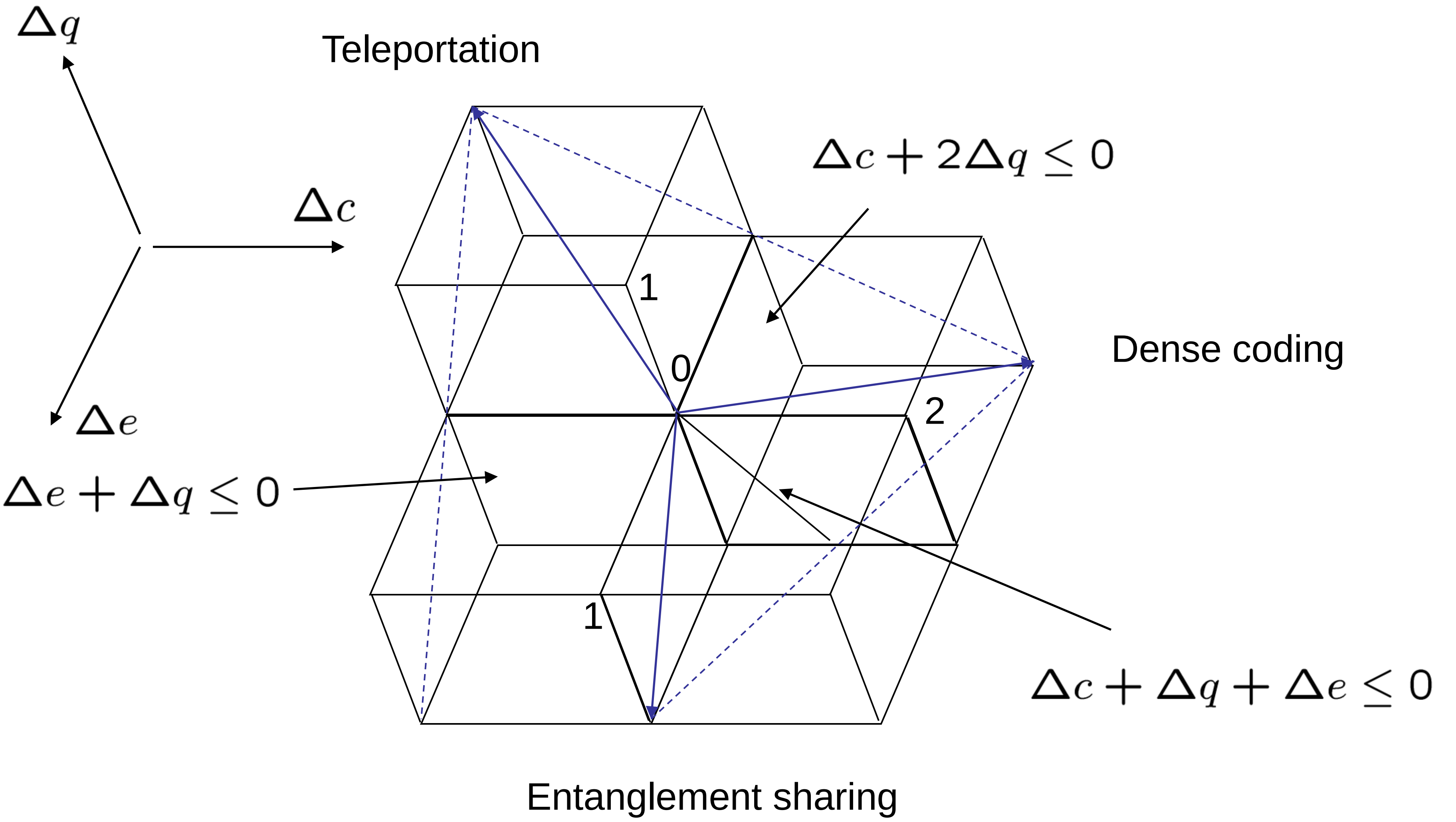
+ QT,ES

$$\Delta e + \Delta q \leq 0$$

Holevo + ES,QD

$$\Delta q + \Delta e + \Delta c \leq 0$$

Resource conversion protocols and bounds



6. Quantum error correcting codes

Error correcting codes (Classical)

Avoid destroying the superposition

- Parity check

Phase errors

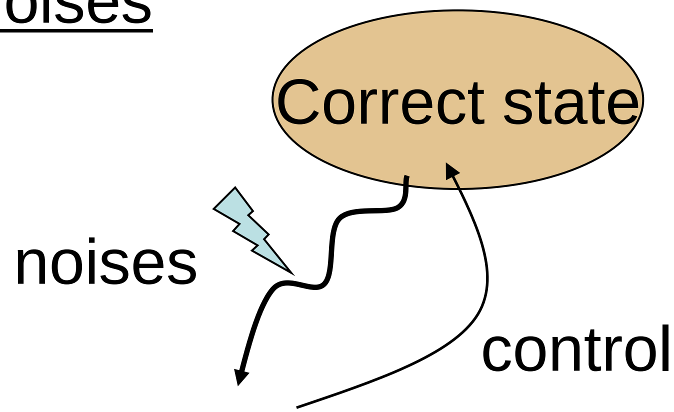
- CSS 7-qubit code (Steane code)

Too many error patterns?

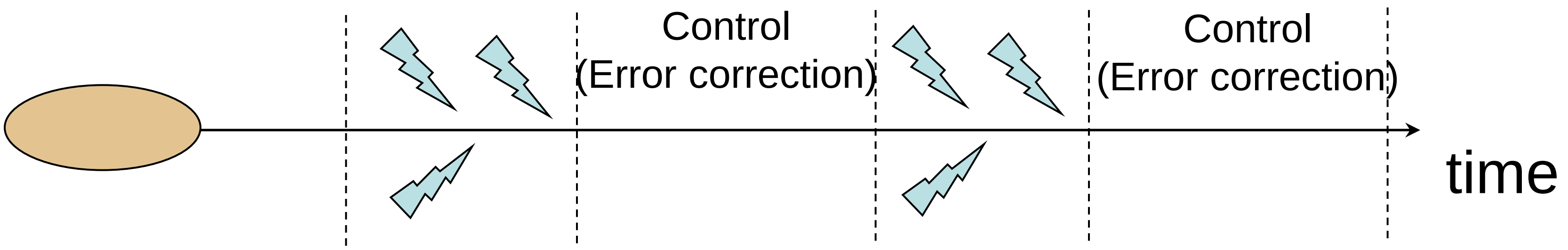
- Syndrome measurement digitizes the error

Codeword states

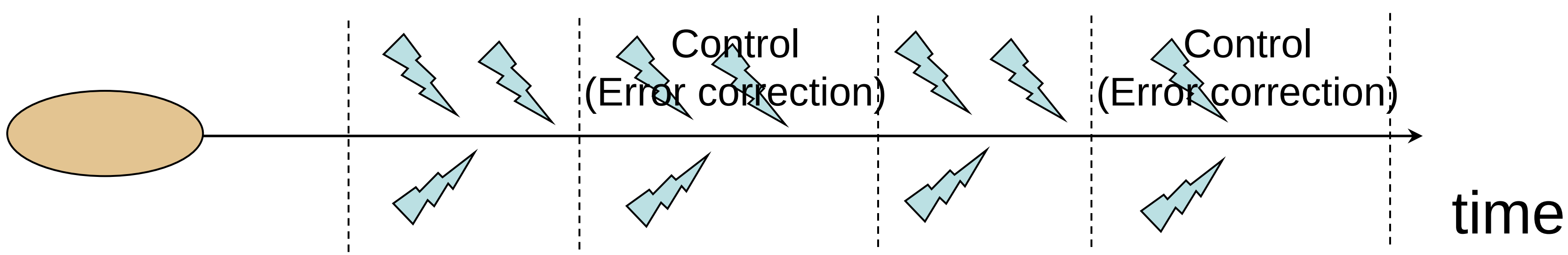
Fighting against noises



Error correcting codes



Fault-tolerant computation



Error correcting codes (Classical)

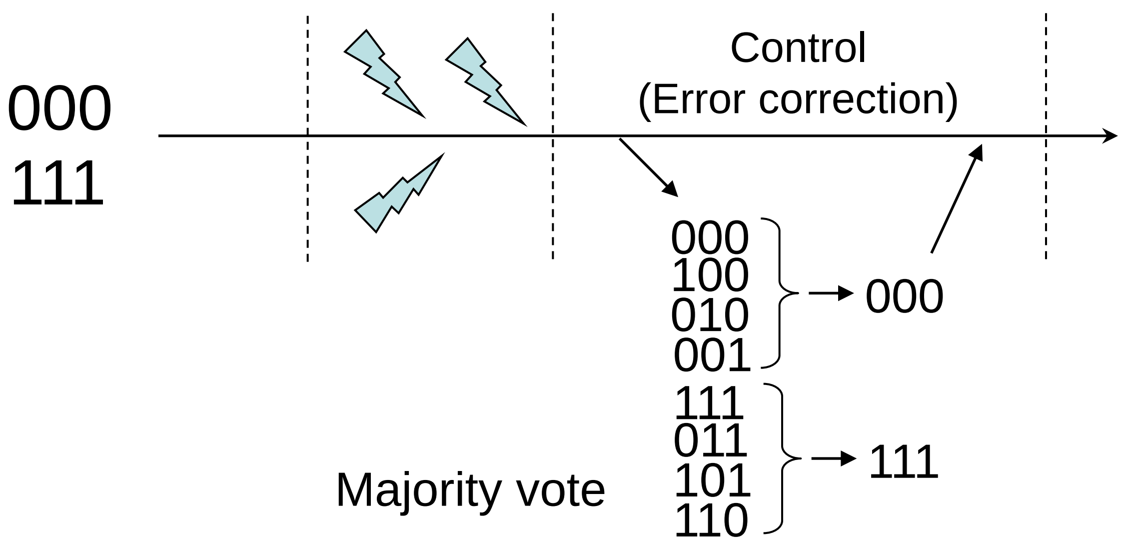
0 ⟶ 000

1 ⟶ 111

Bit error

0 ⟷ 1

No error	1st bit	2nd bit	3rd bit
000	100	010	001
111	011	101	110



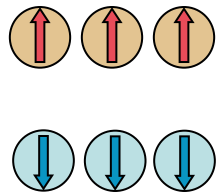
Error rate per bit ϵ

Error rate after the correction $\sim 3\epsilon^2$

Error correcting codes (Quantum)?

$$|0\rangle \longrightarrow |000\rangle$$

$$|1\rangle \longrightarrow |111\rangle$$



"Bit error" (Z error)

$$|0\rangle \xleftrightarrow[\sigma_x]{} |1\rangle$$

Error in observable σ_z

Error caused by unitary σ_x

Problems:

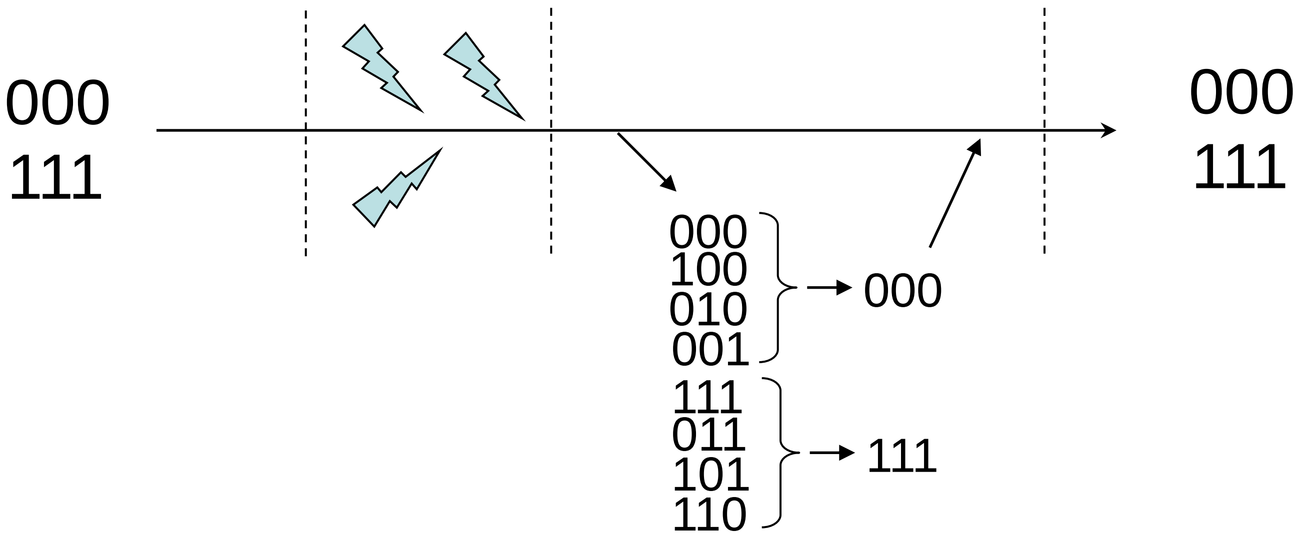
- If we measure the system for the correction, the superposition may collapse.

- Can we correct the phase error? σ_z (X error)

- There are infinite number of error patterns. Can we handle all of them?

Does the majority vote work?

•If we measure the system for the correction, the superposition may collapse.



No error	1st bit	2nd bit	3rd bit
000	100	010	001
111	011	101	110

Distinguish here

States such as $|000\rangle + |111\rangle$ and $|000\rangle - |111\rangle$ will collapse.

Parity check

Parity check matrix

Parity of a subset of bits

XOR

$$s_1 = b_1 \oplus b_2$$
$$s_2 = b_2 \oplus b_3$$

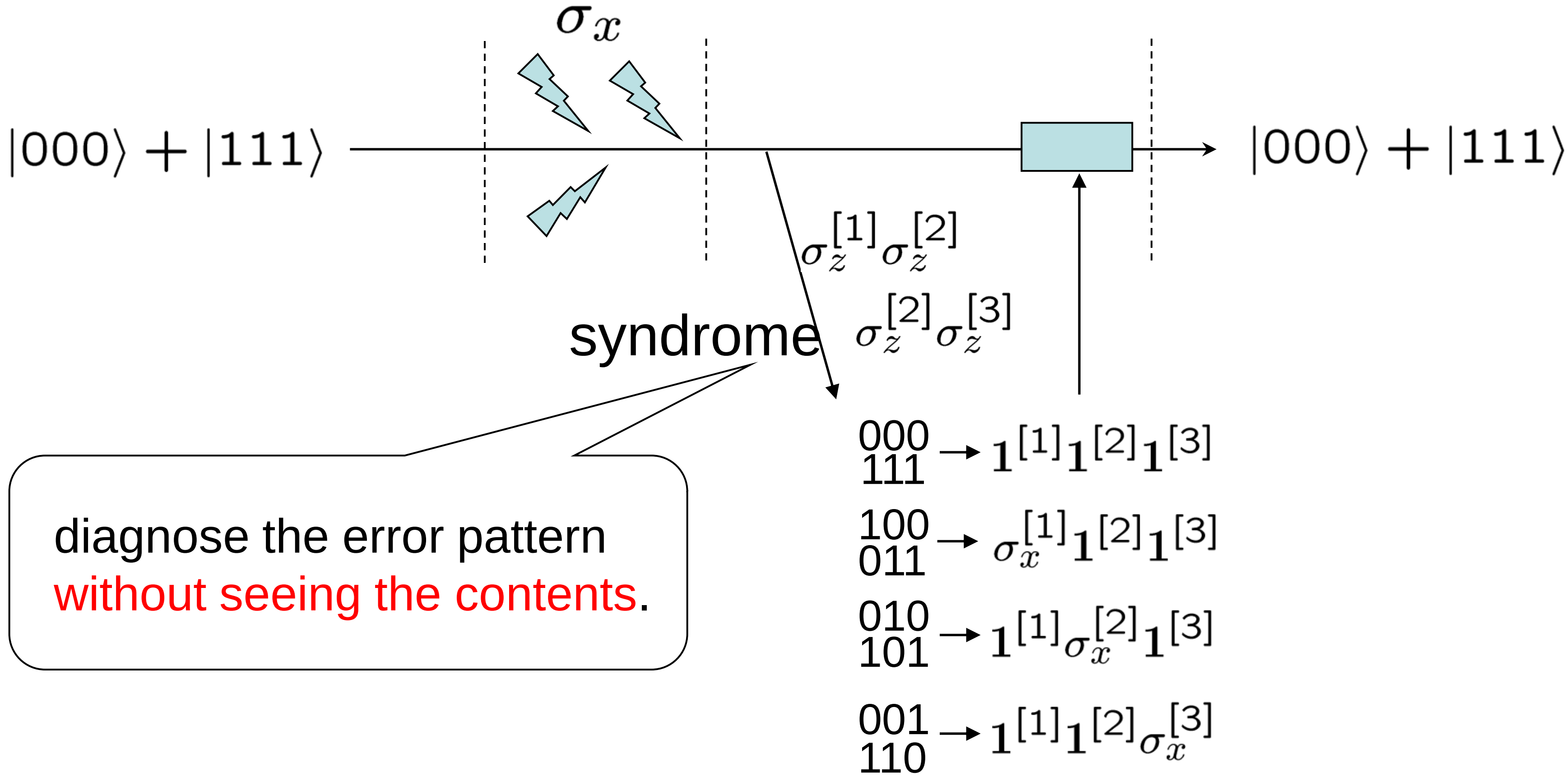
$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

	No error	1st bit	2nd bit	3rd bit	
	000	100	010	001	
	111	011	101	110	
$s_1 s_2$	00	10	11	01	(syndrome)

Distinguish the columns $\sigma_z^{[1]}\sigma_z^{[2]}$ $\sigma_z^{[2]}\sigma_z^{[3]}$

Correction operation $\mathbf{1}^{[1]}\mathbf{1}^{[2]}\mathbf{1}^{[3]}$ $\sigma_x^{[1]}\mathbf{1}^{[2]}\mathbf{1}^{[3]}$ $\mathbf{1}^{[1]}\sigma_x^{[2]}\mathbf{1}^{[3]}$ $\mathbf{1}^{[1]}\mathbf{1}^{[2]}\sigma_x^{[3]}$

Superposition will survive



Any single bit error can be corrected.

Can we correct the phase error?

Problems:

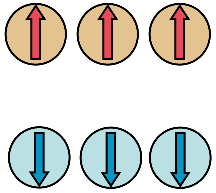
- If we measure the system for the correction, the superposition may collapse. OK

- Can we correct the phase error? σ_z $|0\rangle + |1\rangle \leftrightarrow |0\rangle - |1\rangle$

- There are infinite number of error patterns. Can we handle all of them?

$$|0\rangle \longrightarrow |000\rangle$$

$$|1\rangle \longrightarrow |111\rangle$$



Dimension:

8 in total.

2 for data.

4 different bit-error patterns.

We need more space to correct other errors.

7-bit code

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix}$$

$$\sigma_z^{[1]} \mathbf{1}^{[2]} \sigma_z^{[3]} \mathbf{1}^{[4]} \sigma_z^{[5]} \mathbf{1}^{[6]} \sigma_z^{[7]}$$
$$\mathbf{1}^{[1]} \sigma_z^{[2]} \sigma_z^{[3]} \mathbf{1}^{[4]} \mathbf{1}^{[5]} \sigma_z^{[6]} \sigma_z^{[7]}$$
$$\mathbf{1}^{[1]} \mathbf{1}^{[2]} \mathbf{1}^{[3]} \sigma_z^{[4]} \sigma_z^{[5]} \sigma_z^{[6]} \sigma_z^{[7]}$$

Dimension: $2^7 = 128$ in total.

8 different bit-error patterns.

$$128/8 = 16 = 2^4$$

We can encode 4 qubits of data if only the bit errors occur.

If we use only one qubit of data, we can accommodate 8 more errors.

$$\begin{pmatrix} s_4 \\ s_5 \\ s_6 \end{pmatrix}$$

$$\sigma_x^{[1]} \mathbf{1}^{[2]} \sigma_x^{[3]} \mathbf{1}^{[4]} \sigma_x^{[5]} \mathbf{1}^{[6]} \sigma_x^{[7]}$$
$$\mathbf{1}^{[1]} \sigma_x^{[2]} \sigma_x^{[3]} \mathbf{1}^{[4]} \mathbf{1}^{[5]} \sigma_x^{[6]} \sigma_x^{[7]}$$
$$\mathbf{1}^{[1]} \mathbf{1}^{[2]} \mathbf{1}^{[3]} \sigma_x^{[4]} \sigma_x^{[5]} \sigma_x^{[6]} \sigma_x^{[7]}$$

CSS 7-qubit code (Steane code)

$$\sigma_x \sigma_z = -\sigma_z \sigma_x$$

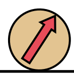
$\sigma_z^{[1]}\mathbf{1}^{[2]}\sigma_z^{[3]}\mathbf{1}^{[4]}\sigma_z^{[5]}\mathbf{1}^{[6]}\sigma_z^{[7]}$	commute	$\sigma_x^{[1]}\mathbf{1}^{[2]}\sigma_x^{[3]}\mathbf{1}^{[4]}\sigma_x^{[5]}\mathbf{1}^{[6]}\sigma_x^{[7]}$
$\mathbf{1}^{[1]}\sigma_z^{[2]}\sigma_z^{[3]}\mathbf{1}^{[4]}\mathbf{1}^{[5]}\sigma_z^{[6]}\sigma_z^{[7]}$	$\Longleftrightarrow$	$\mathbf{1}^{[1]}\sigma_x^{[2]}\sigma_x^{[3]}\mathbf{1}^{[4]}\mathbf{1}^{[5]}\sigma_x^{[6]}\sigma_x^{[7]}$
$\mathbf{1}^{[1]}\mathbf{1}^{[2]}\mathbf{1}^{[3]}\sigma_z^{[4]}\sigma_z^{[5]}\sigma_z^{[6]}\sigma_z^{[7]}$		$\mathbf{1}^{[1]}\mathbf{1}^{[2]}\mathbf{1}^{[3]}\sigma_x^{[4]}\sigma_x^{[5]}\sigma_x^{[6]}\sigma_x^{[7]}$

Dimension: $2^7 = 128$ in total.

6 observables (binary)
$2^6 = 64$ patterns

Each eigenspace has dimension 2.

Bit error

Phase error	no	1	2	3	4	5	6	7
no								
1								
2								
3								
4								
5								
6								
7								

Any single bit error, plus any single phase error can be corrected.

Too many error patterns?

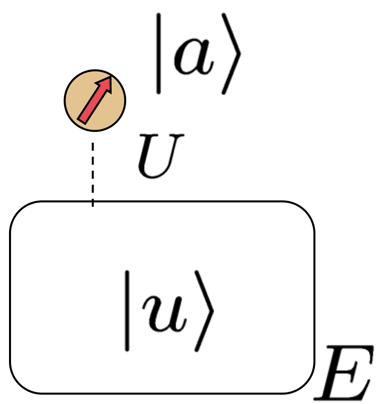
Problems:

- If we measure the system for the correction, the superposition may collapse. OK

- Can we correct the phase error? σ_z OK

- There are infinite number of error patterns. Can we handle all of them?

General errors on a single qubit

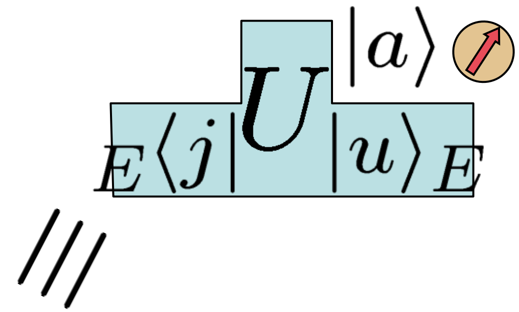


$$U(|a\rangle \otimes |u\rangle_E)$$

Interaction with environment

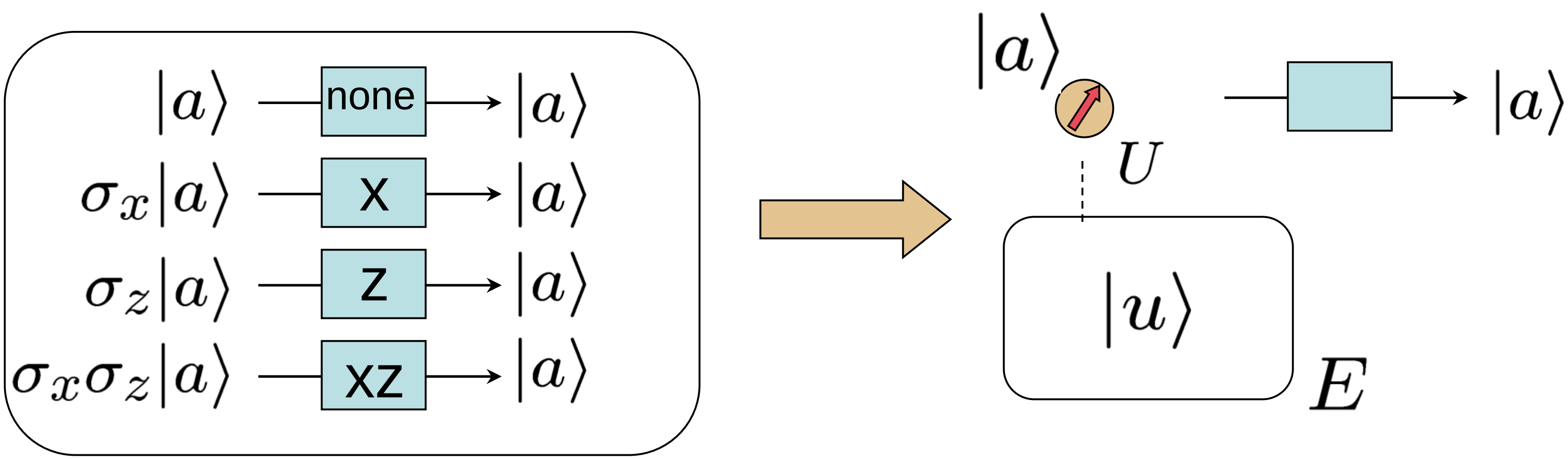
General errors

$$U(|a\rangle \otimes |u\rangle_E)$$
$$= \sum_j |j\rangle_{E\,E}\langle j|U(|a\rangle \otimes |u\rangle_E)$$
$$= \sum_j (A_j|a\rangle \otimes |j\rangle_E)$$
$$= |a\rangle \otimes |u_0\rangle_E + \sigma_x|a\rangle \otimes |u_1\rangle_E$$
$$+\sigma_z|a\rangle \otimes |u_2\rangle_E + \sigma_x\sigma_z|a\rangle \otimes |u_3\rangle_E$$



$$A_j = c_0^{(j)} I + c_1^{(j)}\sigma_x + c_2^{(j)}\sigma_z + c_3^{(j)}\sigma_x\sigma_z$$

$\{|u_i\rangle_E\}$:unnormalized, nonorthogonal



Too many error patterns?

Problems:

- If we measure the system for the correction, the superposition may collapse. OK

- Can we correct the phase error? σ_z OK

- There are infinite number of error patterns. Can we handle all of them? OK

Correcting bit and phase errors is enough.

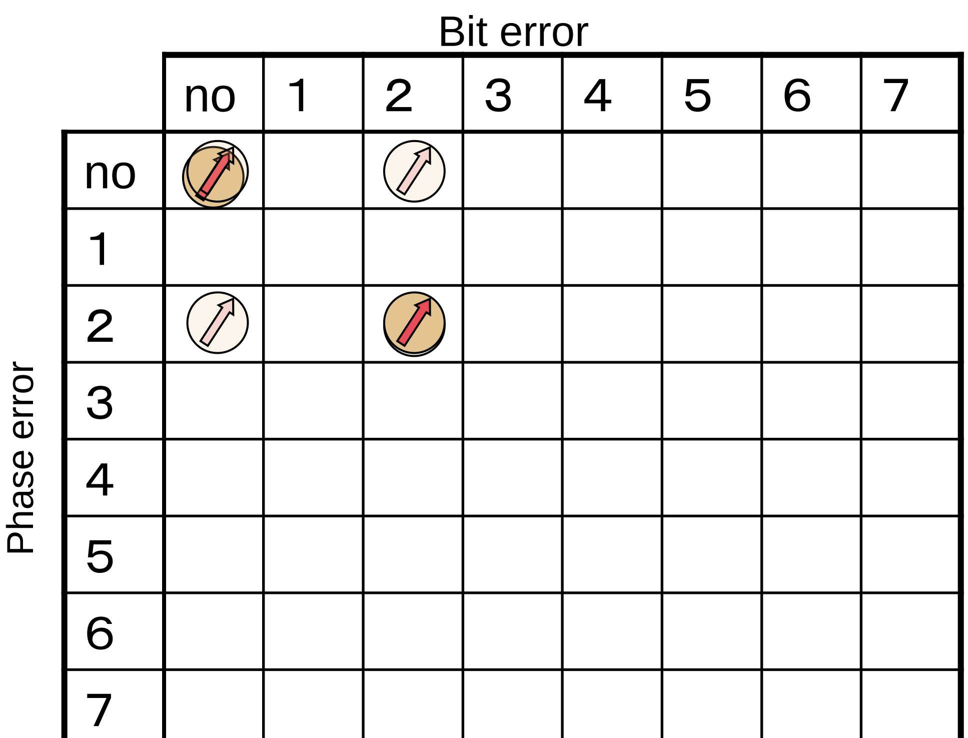
Syndrome measurement projects general errors onto one of these errors.

Syndrome measurement digitizes the error

$$\sigma_z^{[1]}\mathbf{1}^{[2]}\sigma_z^{[3]}\mathbf{1}^{[4]}\sigma_z^{[5]}\mathbf{1}^{[6]}\sigma_z^{[7]}$$
$$\mathbf{1}^{[1]}\sigma_z^{[2]}\sigma_z^{[3]}\mathbf{1}^{[4]}\mathbf{1}^{[5]}\sigma_z^{[6]}\sigma_z^{[7]}$$
$$\mathbf{1}^{[1]}\mathbf{1}^{[2]}\mathbf{1}^{[3]}\sigma_z^{[4]}\sigma_z^{[5]}\sigma_z^{[6]}\sigma_z^{[7]}$$

commute ⟷

$$\sigma_x^{[1]}\mathbf{1}^{[2]}\sigma_x^{[3]}\mathbf{1}^{[4]}\sigma_x^{[5]}\mathbf{1}^{[6]}\sigma_x^{[7]}$$
$$\mathbf{1}^{[1]}\sigma_x^{[2]}\sigma_x^{[3]}\mathbf{1}^{[4]}\mathbf{1}^{[5]}\sigma_x^{[6]}\sigma_x^{[7]}$$
$$\mathbf{1}^{[1]}\mathbf{1}^{[2]}\mathbf{1}^{[3]}\sigma_x^{[4]}\sigma_x^{[5]}\sigma_x^{[6]}\sigma_x^{[7]}$$



Any error on a single qubit can be corrected.

CSS QECC

Calderbank & Shor (1996)
Steane (1996)

Codeword states

$$\sigma_x \sigma_z = -\sigma_z \sigma_x$$

$$\sigma_z^{[1]} \mathbf{1}^{[2]} \sigma_z^{[3]} \mathbf{1}^{[4]} \sigma_z^{[5]} \mathbf{1}^{[6]} \sigma_z^{[7]}$$
$$\mathbf{1}^{[1]} \sigma_z^{[2]} \sigma_z^{[3]} \mathbf{1}^{[4]} \mathbf{1}^{[5]} \sigma_z^{[6]} \sigma_z^{[7]}$$
$$\mathbf{1}^{[1]} \mathbf{1}^{[2]} \mathbf{1}^{[3]} \sigma_z^{[4]} \sigma_z^{[5]} \sigma_z^{[6]} \sigma_z^{[7]}$$

commute

$$\sigma_x^{[1]} \mathbf{1}^{[2]} \sigma_x^{[3]} \mathbf{1}^{[4]} \sigma_x^{[5]} \mathbf{1}^{[6]} \sigma_x^{[7]}$$
$$\mathbf{1}^{[1]} \sigma_x^{[2]} \sigma_x^{[3]} \mathbf{1}^{[4]} \mathbf{1}^{[5]} \sigma_x^{[6]} \sigma_x^{[7]}$$
$$\mathbf{1}^{[1]} \mathbf{1}^{[2]} \mathbf{1}^{[3]} \sigma_x^{[4]} \sigma_x^{[5]} \sigma_x^{[6]} \sigma_x^{[7]}$$

independent

independent

$$\sigma_z^{[1]} \sigma_z^{[2]} \sigma_z^{[3]} \sigma_z^{[4]} \sigma_z^{[5]} \sigma_z^{[6]} \sigma_z^{[7]}$$

$$\sigma_x^{[1]} \sigma_x^{[2]} \sigma_x^{[3]} \sigma_x^{[4]} \sigma_x^{[5]} \sigma_x^{[6]} \sigma_x^{[7]}$$

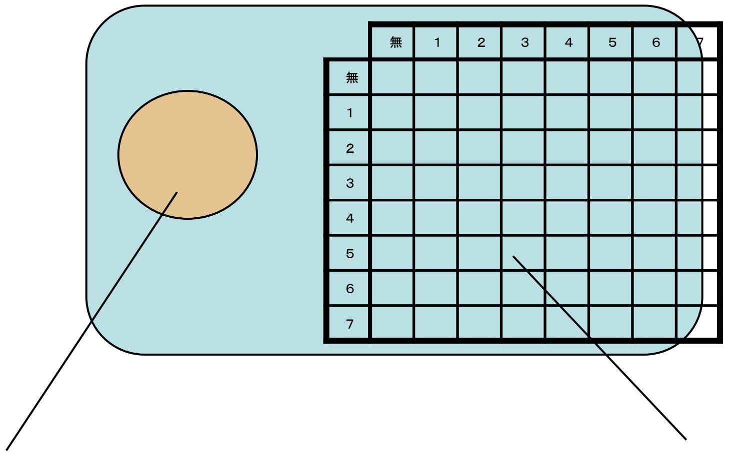
Anti-commute

$$|\mathbf{0}\rangle = |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle$$

$$|\mathbf{1}\rangle = \sigma_x^{[1]} \sigma_x^{[2]} \sigma_x^{[3]} \sigma_x^{[4]} \sigma_x^{[5]} \sigma_x^{[6]} \sigma_x^{[7]} |\mathbf{0}\rangle$$

Quantum error correcting codes

Special state with quantum correlation



Data

Quantum

Do not touch!

Error patterns

Changes are allowed, as long as we can keep track of them.

Measurement is OK.

It makes infinite error patterns shrink to finite ones.