Summer School 2011, Aug. 12-17, Kyoto

量子情報基礎

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1. Basic rules of quantum mechanics
2. State of subsystems
3. Qubits
4. Power of ancilla system
5. Communication resources
6. Quantum error correcting codes

1. Basic rules of quantum mechanics

How to describe the states of an ideally controlled system?

How to describe changes in an ideally controlled system?

How to describe measurements on an ideally controlled system?

How to treat composite systems?

How to describe the states of an ideally controlled system?

(Basic rule I)

A physical system $\leftrightarrow$ a **Hilbert space** $\mathcal{H}$

A state $\leftrightarrow$ a **ray** in the Hilbert space

Usually, we use a normalized vector ϕ satisfying $(\phi, \phi) = 1$ as a representative of the ray.

Distinguishability —— Inner product

For normalized vectors ϕ and ψ,
$|(\phi, \psi)| = 0$ — perfectly distinguishable
$|(\phi, \psi)| = 1$ — completely indistinguishable (the same state)

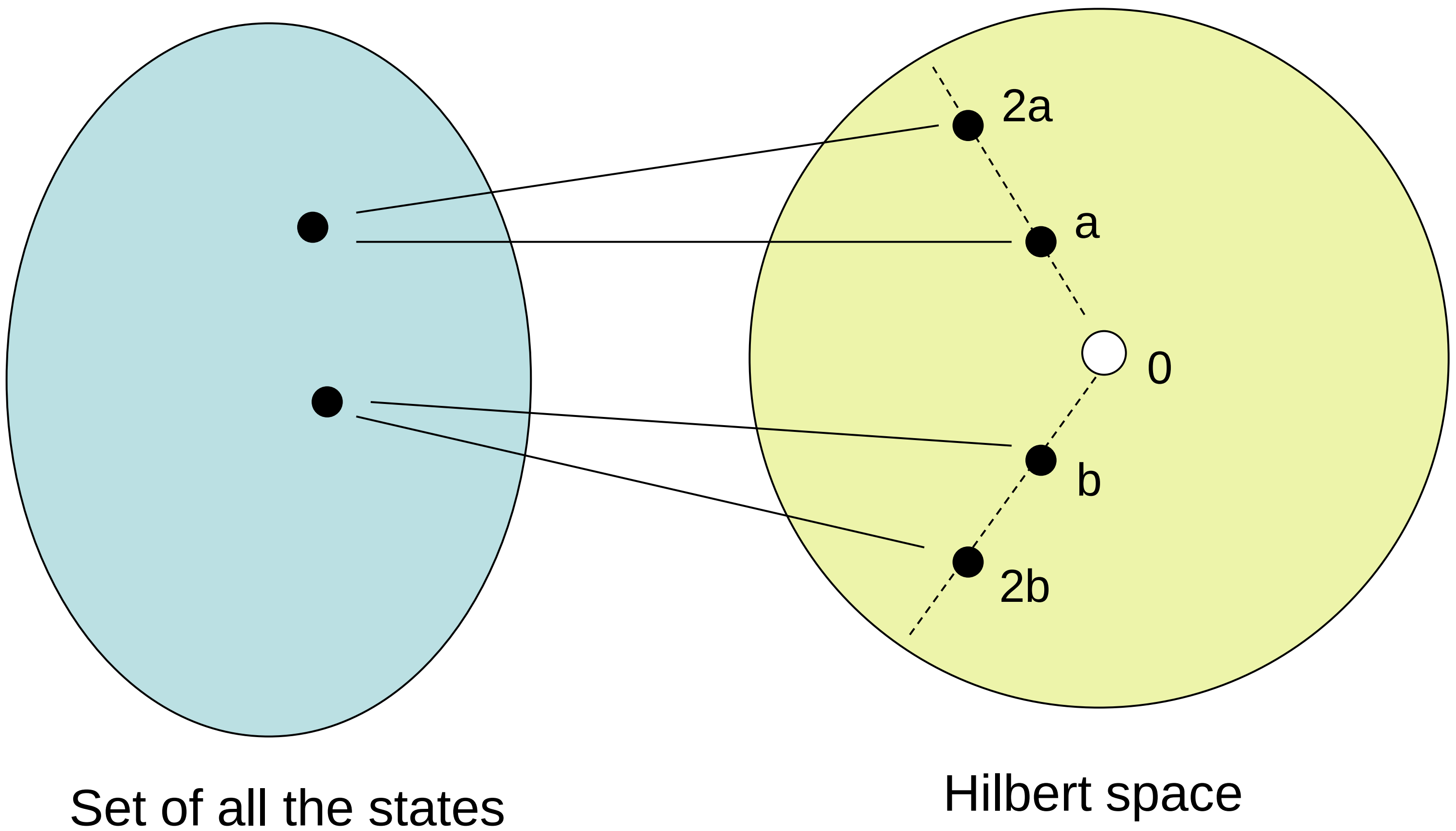
Dirac notation

'ket' $|\phi\rangle$ — vector $\phi \in \mathcal{H}$.
'bra' $\langle\phi|$ — linear functional $(\phi, \cdot) : \mathcal{H} \to \mathbb{C}$.

$$\langle\phi|\psi\rangle \text{ — } (\phi, \psi)$$

How to describe the states of an ideally controlled system?

(Basic rule I)



A state $\leftrightarrow$ a **ray** in the Hilbert space

ray including vector $a \neq 0$ is

$\{\alpha a | \alpha \in \mathbb{C}, \alpha \neq 0\}$.

How to describe changes in an ideally controlled system?

(Basic rule II)

Reversible evolution

A unitary operator $\hat{U}$:

$$|\phi_{\text{out}}\rangle = \hat{U}|\phi_{\text{in}}\rangle$$

Inner products are preserved by unitary operations.

Distinguishability should never be improved by any operation.

↓

Distinguishability should be unchanged by any reversible operation.

↓

Inner products will be preserved in any reversible operation.

Infinitesimal change

$$|\phi(t_2)\rangle = \hat{U}(t_2, t_1)|\phi(t_1)\rangle$$

$$|\phi(t+dt)\rangle = \hat{U}(t+dt, t)|\phi(t)\rangle$$

$$\hat{U}(t+dt, t) \cong \hat{1} - (i/\hbar)\hat{H}(t)dt$$

Self-adjoint operator $\hat{H}(t)$: Hamiltonian of the system

Schrödinger equation:

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = \hat{H}(t)|\phi(t)\rangle$$

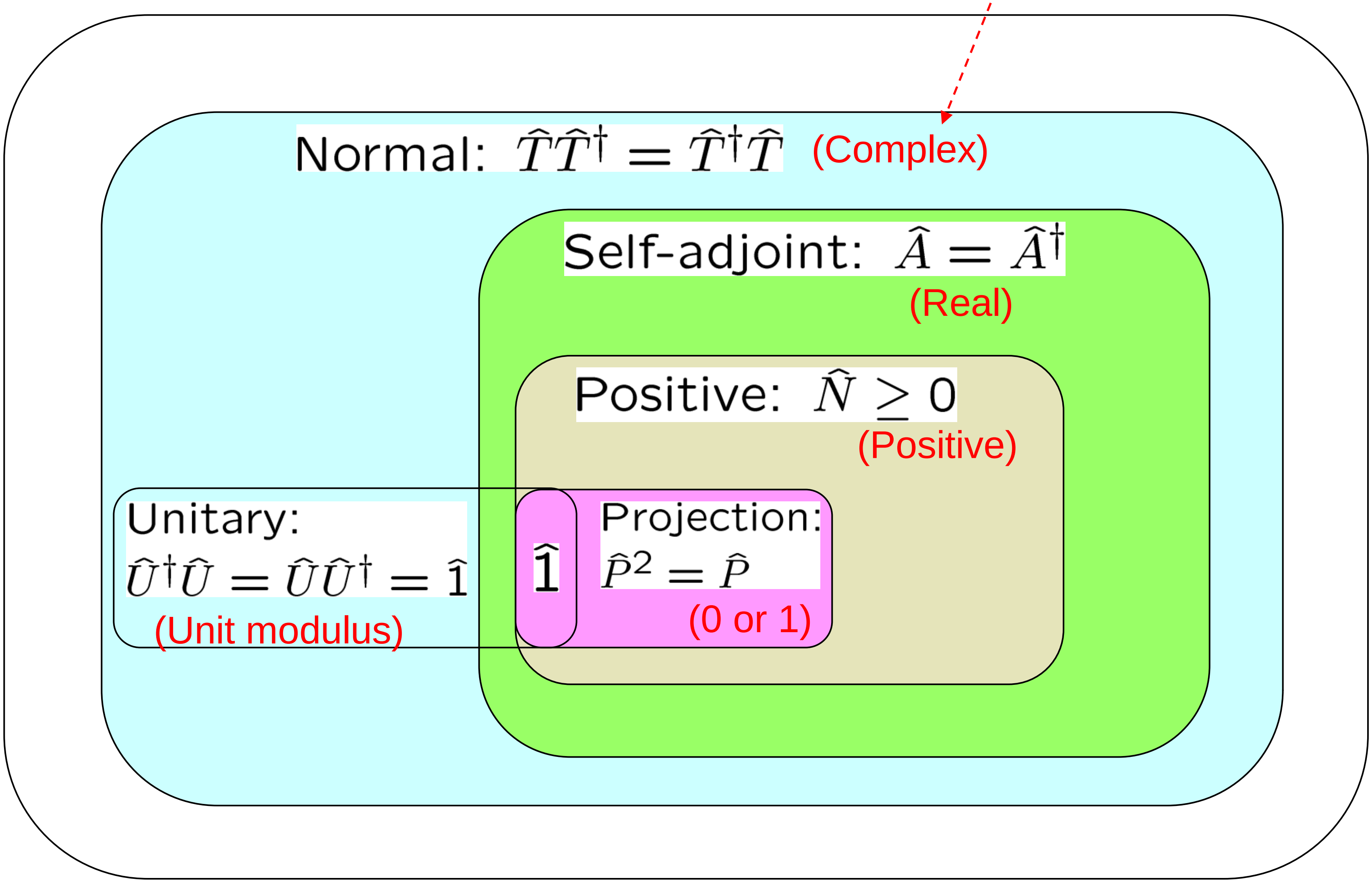
Linear operators: $\mathcal{H} \to \mathcal{H}$.

$\hat{T}$ is normal $\leftrightarrow$ $\hat{T}$ is diagonalizable.

$$\hat{T} = \sum_j \lambda_j |u_j\rangle\langle u_j|$$

Eigenvalues

An orthonormal basis



How to describe measurements on an ideally controlled system?

(Basic rule III)

Orthogonal measurement on an orthonormal basis $\{|a_j\rangle\}_{j=1,\cdots,d}$
(von Neumann measurement, projection measurement)

Input state $|\phi\rangle = \sum_j |a_j\rangle\langle a_j|\phi\rangle$

Probability of outcome j $\quad P(j) = |\langle a_j|\phi\rangle|^2$

Note: This is not the unique way of defining the 'best' measurement. We'll see later.

$d = \dim \mathcal{H}.$

Closure relation

$\sum_j |a_j\rangle\langle a_j| = \hat{1}$

Measurement of an observable

Self-adjoint operator $\hat{A}$

$$\hat{A} = \sum_j \lambda_j |a_j\rangle\langle a_j|$$

Measurement on $\{|a_j\rangle\}_{j=1,\cdots,d}$ Assign $j \to \lambda_j$

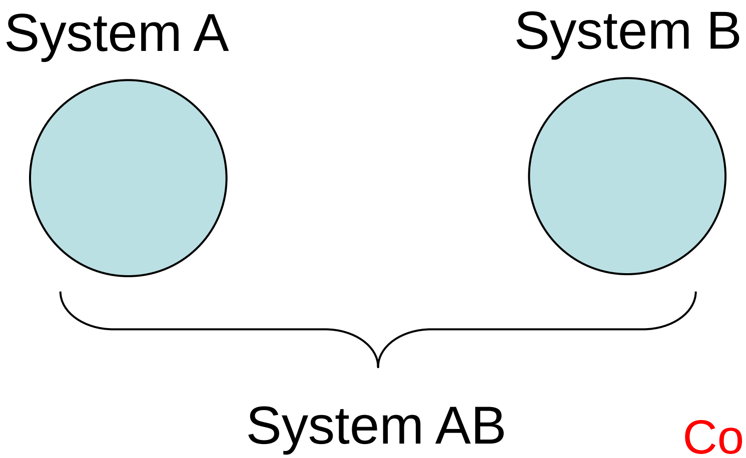
$$\langle \hat{A}\rangle \equiv \sum_j P(j)\lambda_j = \sum_j \langle\phi|a_j\rangle\langle a_j|\phi\rangle\lambda_j = \langle\phi|\hat{A}|\phi\rangle$$

How to treat composite systems?

(Basic rule IV)

We know how to describe each of the systems A and B.

How to describe AB as a single system?



Subsystems

Composite system

System A: Hilbert space $\mathcal{H}_A$ Basis $\{|a_i\rangle\}_{i=1,\cdots,d_A}$

System B: Hilbert space $\mathcal{H}_B$ Basis $\{|b_j\rangle\}_{j=1,\cdots,d_B}$

Composite system AB:

Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$

Tensor product

Basis

$\{|a_i\rangle \otimes |b_j\rangle\}_{i=1,\cdots,d_A;j=1,\cdots,d_B}$

$$\mathrm{dim}(\mathcal{H}_A \otimes \mathcal{H}_B) = \mathrm{dim}\,\mathcal{H}_A\,\mathrm{dim}\,\mathcal{H}_B$$

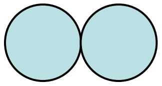
How to treat composite systems?

(Basic rule IV)

When system A and system B are independently accessed …

	State preparation	Unitary evolution	Orthogonal measurement
System A	$\vert\phi\rangle_A$	$\hat{U}_A$	$\{\vert a_i\rangle_A\}_{i=1,\cdots,d_A}$
System B	$\vert\psi\rangle_B$	$\hat{V}_B$	$\{\vert b_j\rangle_B\}_{j=1,\cdots,d_B}$
System AB	$\vert\phi\rangle_A \otimes \vert\psi\rangle_B$	$\hat{U}_A \otimes \hat{V}_B$	$\{\vert a_i\rangle_A \otimes \vert b_j\rangle_B\}_{i=1,\cdots,d_A}^{j=1,\cdots,d_B}$
	Separable states	Local unitary operations	Local measurements

When system A and system B are directly interacted …



$\vert\Psi\rangle_{AB} \in \mathcal{H}_{AB}$ $\sum_k \alpha_k \vert\phi_k\rangle_A \otimes \vert\psi_k\rangle_B$	$\hat{U}_{AB} : \mathcal{H}_{AB} \to \mathcal{H}_{AB}$	$\{\vert\Psi_k\rangle_{AB}\}_{k=1,2,\ldots,d_A d_B}$
Entangled states	Global unitary operations	Global measurements

2. State of a subsystem

Rule for a local measurement

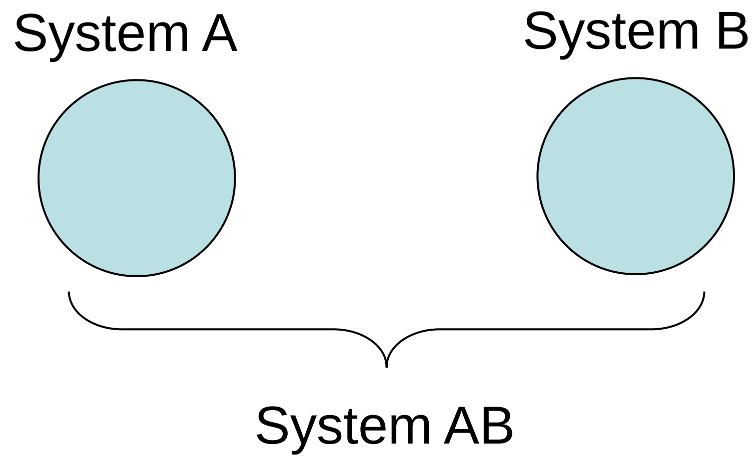
State after discarding a subsystem (marginal state)

Density operator
- Properties of density operators
- Rules in terms of density operators

Why is the density operator sufficient for description ?

- Schmidt decomposition
- Pure states with the same marginal state
- Ensembles with the same density operator

Entanglement



Suppose that the whole system (AB) is ideally controlled (prepared in a definite state).

state: $|\Phi\rangle_{AB}$

Intuition in a ‘classical’ world:

If the whole is under a good control, so are the parts.

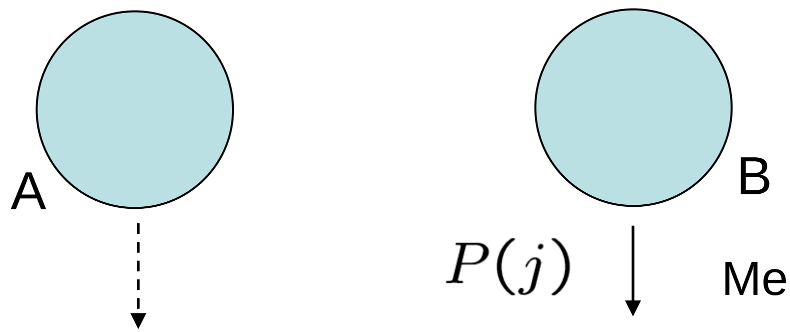
But ….

It is not always possible to assign a state vector to subsystem A.

What is the state of subsystem A?

Rule for a local measurement

Initial state: $|\Phi\rangle_{AB}$



Measurement on $\{|b_j\rangle_B\}_{j=1,\cdots,d_B}$

State $|\phi_j\rangle_A$ Outcome j

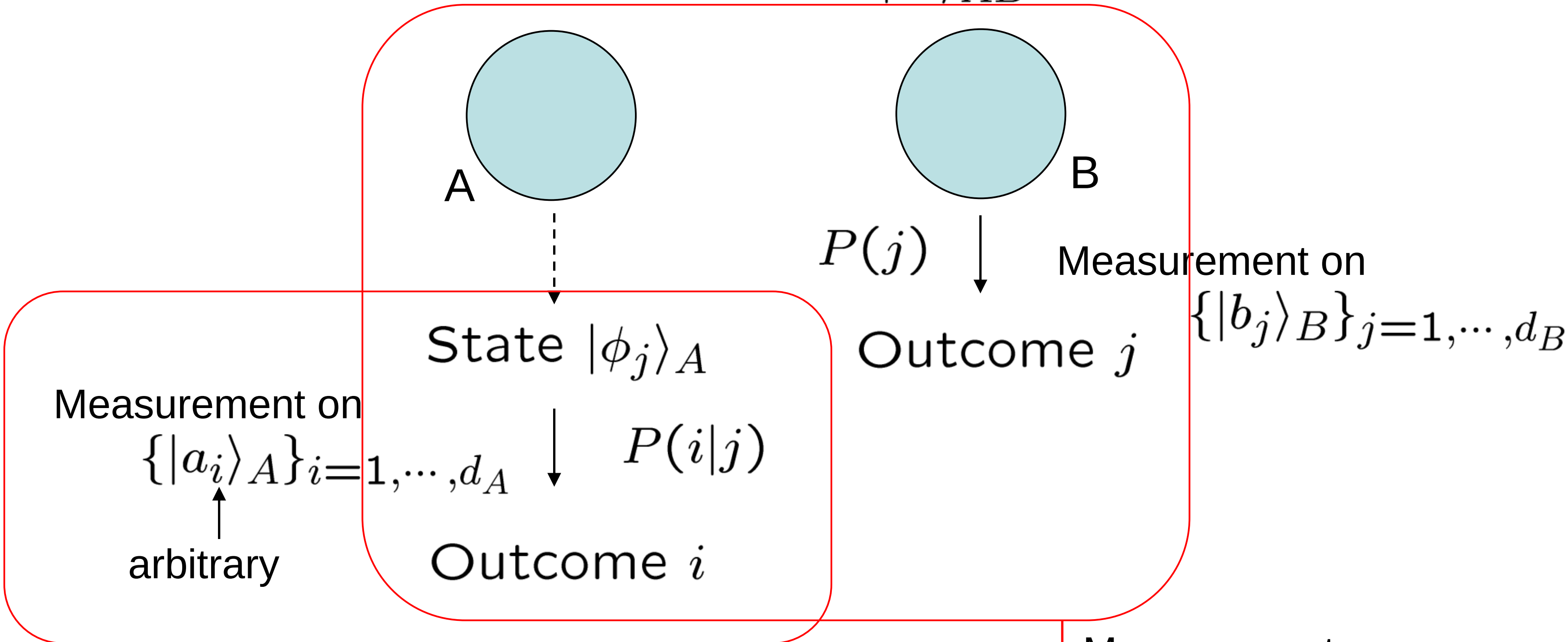
$$\boxed{\sqrt{P(j)}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}}$$

$$P(j) = \|{}_B\langle b_j||\Phi\rangle_{AB}\|^2$$

$$|\phi_j\rangle_A = \frac{{}_B\langle b_j||\Phi\rangle_{AB}}{\|{}_B\langle b_j||\Phi\rangle_{AB}\|}$$

Rule for a local measurement

Initial state: $|\Phi\rangle_{AB}$



Measurement on $\{|a_i\rangle_A \otimes |b_j\rangle_B\}_{i=1,\cdots,d_A}^{j=1,\cdots,d_B}$

$$P(i|j) = |_A\langle a_i|\phi_j\rangle_A|^2$$

$$P(i,j) = \left|_A\langle a_i|_B\langle b_j||\Phi\rangle_{AB}\right|^2$$

$$P(i,j) = P(i|j)P(j) = |_A\langle a_i|\sqrt{P(j)}|\phi_j\rangle_A|^2$$

A remark on notations

$$
{}_A\langle a_i| \otimes {}_B\langle b_j||\Phi\rangle_{AB}
$$

$$
\begin{array}{c} {}_A\langle a_i| \\ {}_B\langle b_j| \end{array} \Big|\Phi\Big\rangle_{AB}
$$

$$
= {}_A\langle a_i|\underbrace{(\hat{1}_A \otimes {}_B\langle b_j|)}|\Phi\rangle_{AB}
$$

$$
\begin{array}{ll} {}_A\langle a_i| & \\ & {}_B\langle b_j| \end{array} \Big|\Phi\Big\rangle_{AB}
$$

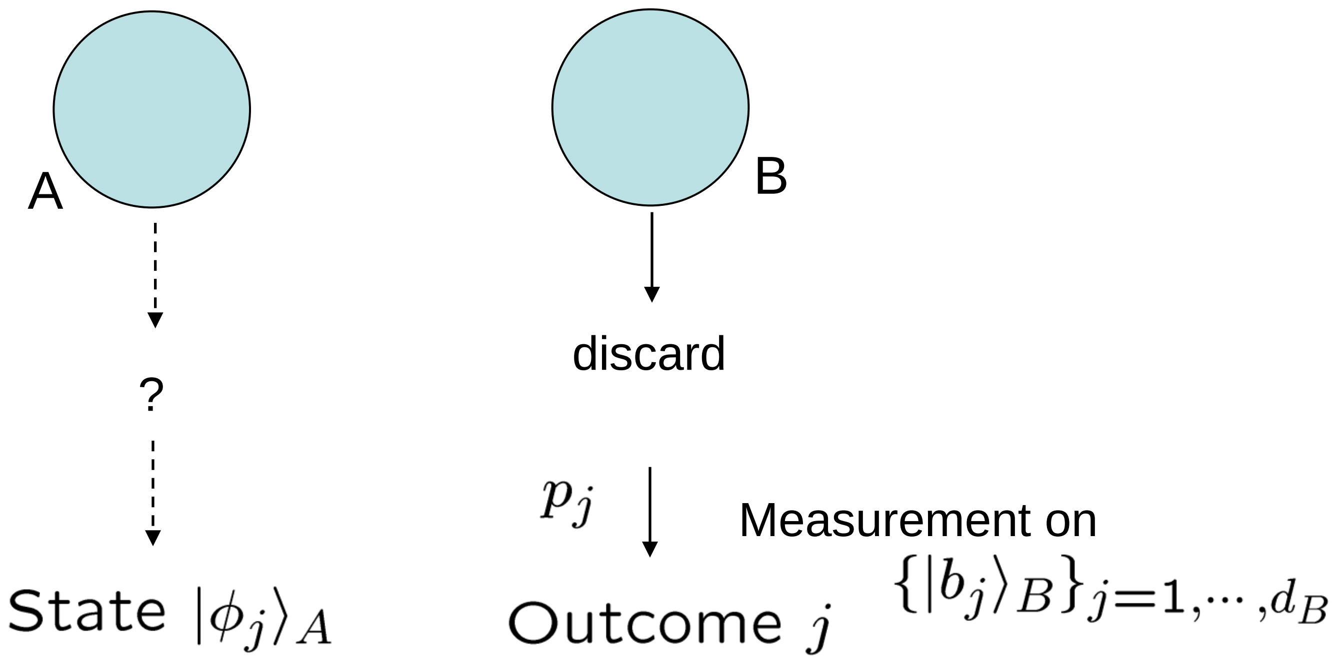
$\downarrow$ abbreviation

$$
= {}_A\langle a_i| \; {}_B\langle b_j| \; |\Phi\rangle_{AB}
$$

$$
\begin{aligned}
{}_B\langle b_j| &: \mathcal{H}_B \to \mathbb{C} \\
\hat{1}_A &: \mathcal{H}_A \to \mathcal{H}_A \\
\hat{1}_A \otimes {}_B\langle b_j| &: \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A
\end{aligned}
$$

State after discarding a subsystem (marginal state)

Initial state: $|\Phi\rangle_{AB}$



State of system A: $|\phi_j\rangle_A$ with probability p_j $\longrightarrow$ $\{p_j, |\phi_j\rangle_A\}$

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

This description is correct, but dependence on the fictitious measurement is weird…

Alternative description: density operator

$\{p_j, |\phi_j\rangle_A\}$ $\qquad$ $|\phi_j\rangle_A$ with probability p_j

$$\hat{\rho}_A \equiv \sum_j p_j |\phi_j\rangle_{AA}\langle\phi_j|$$

Cons

$\{q_k, |\psi_k\rangle_A\}$

$\{p_j, |\phi_j\rangle_A\}$

Same $\hat{\rho}_A$

Two different physical states could have the same density operator.
(The description could be insufficient.)

Pros

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

$$\hat{\rho}_A = \sum_j p_j|\phi_j\rangle_{AA}\langle\phi_j| = \sum_j \sqrt{p_j}|\phi_j\rangle_{AA}\langle\phi_j|\sqrt{p_j}$$

$$= \sum_j {}_B\langle b_j||\Phi\rangle\langle\Phi||b_j\rangle_B = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

Independent of the choice of the fictitious measurement

Properties of density operators

$$\hat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

For any $|\psi\rangle$, $\langle\psi|\hat{\rho}|\psi\rangle = \sum_j p_j |\langle\psi|\phi_j\rangle|^2 \geq 0$ Positive

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}) &= \sum_j p_j \mathrm{Tr}(|\phi_j\rangle\langle\phi_j|) \\ &= \sum_j p_j \langle\phi_j|\phi_j\rangle = \sum_j p_j = 1\end{aligned}$$

Unit trace

Positive & Unit trace $\longrightarrow$ $\hat{\rho} = \sum_j p_j |\phi_j\rangle\langle\phi_j|$

probability

This decomposition is by no means unique!

Mixed state $\hat{\rho} = \sum_j p_j |\phi_j\rangle\langle\phi_j|$

Pure state $\hat{\rho} = |\phi\rangle\langle\phi|$ (One eigenvalue is 1)

Rules in terms of density operators

Prepare $|\phi_j\rangle$ with probability p_j

$$\hat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

Prepare $\hat{\rho}_j$ with probability p_j

$$\hat{\rho} = \sum_j p_j \hat{\rho}_j$$

Unitary evolution

$$|\phi_{\text{out}}\rangle = \hat{U}|\phi_{\text{in}}\rangle$$

$$\hat{\rho}_{\text{out}} = \hat{U}\hat{\rho}_{\text{in}}\hat{U}^\dagger$$

Hint: $|\phi_{\text{out}}\rangle\langle\phi_{\text{out}}| = \hat{U}|\phi_{\text{in}}\rangle\langle\phi_{\text{in}}|\hat{U}^\dagger$

Orthogonal measurement on basis $\{|a_j\rangle\}$

$$P(j) = |\langle a_j|\phi\rangle|^2$$

$$P(j) = \langle a_j|\hat{\rho}|a_j\rangle$$

Hint: $P(j) = \langle a_j|\phi\rangle\langle\phi|a_j\rangle$

Expectation value of an observable $\hat{A}$

$$\langle\hat{A}\rangle = \langle\phi|\hat{A}|\phi\rangle$$

$$\langle\hat{A}\rangle = \mathrm{Tr}(\hat{A}\hat{\rho})$$

Hint: $\langle\hat{A}\rangle = \mathrm{Tr}(\hat{A}|\phi\rangle\langle\phi|)$

Rules in terms of density operators

Independently prepared systems A and B

$$|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\psi\rangle_B \qquad\qquad \hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$$

Local measurement on system B on basis $\{|b_j\rangle_B\}$

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB} \qquad\qquad p_j\hat{\rho}_A^{(j)} = {}_B\langle b_j|\hat{\rho}_{AB}|b_j\rangle_B$$

Discarding system B

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|) \qquad\qquad \hat{\rho}_A = \mathrm{Tr}_B[\hat{\rho}_{AB}]$$

All the rules so far can be written in terms of density operators.

Which is the better description?

$\{p_j, |\phi_j\rangle\}$

This looks natural. The system is in one of the pure states, but we just don't know. Quantum mechanics may treat just the pure states, and leave mixed states to statistical mechanics or probability theory.

$\widehat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$ *Best description*

All the rules so far can be written in terms of density operators.

Which description has one-to-one correspondence to physical states?

Theorem: Two states $\{p_j, |\phi_j\rangle\}$ and $\{q_k, |\psi_k\rangle\}$ with the same density operator are physically indistinguishable (hence are the same state).

Schmidt decomposition

Bipartite pure states have a very nice standard form.

Any orthonormal basis $\{|a_i\rangle_A\}$ $\{|b_j\rangle_B\}$

$$|\Phi\rangle_{AB} = \sum_{ij} \alpha_{ij} |a_i\rangle_A |b_j\rangle_B$$

We can always choose the two bases such that

$$|\Phi\rangle_{AB} = \sum_i \sqrt{p_i} |a_i\rangle_A |b_i\rangle_B$$ Schmidt decomposition

$\{|a_i\rangle_A\}$: Diagonalizes $\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$

Proof: $|\Phi\rangle_{AB} = \sum_i |a_i\rangle_A |\tilde{b}_i\rangle_B$ $\quad |\tilde{b}_i\rangle_B \equiv {}_A\langle a_i||\Phi\rangle_{AB}$

unnormalized

$$\begin{aligned} {}_B\langle\tilde{b}_j|\tilde{b}_i\rangle_B &= \mathrm{Tr}[{}_A\langle a_i||\Phi\rangle_{AB\,AB}\langle\Phi||a_j\rangle_A] \\ &= {}_A\langle a_i|\mathrm{Tr}_B[|\Phi\rangle_{AB\,AB}\langle\Phi|]|a_j\rangle_A \\ &= {}_A\langle a_i|\hat{\rho}_A|a_j\rangle_A = p_j\delta_{ij}. \end{aligned}$$

$$\sqrt{p_j}|b_j\rangle \equiv |\tilde{b}_j\rangle_B$$

Entangled states and separable states

$|\phi\rangle_A \otimes |\psi\rangle_B$ $\sum_k \alpha_k |\phi_k\rangle_A \otimes |\psi_k\rangle_B$

Separable states Entangled states

Are there any procedure to distinguish between the two classes?

$\longrightarrow$ Schmidt decomposition $$|\Phi\rangle_{AB} = \sum_{i=1}^{s} \sqrt{p_i} |a_i\rangle_A |b_i\rangle_B$$

$$p_1 \geq p_2 \geq \cdots \geq p_s > 0$$

Schmidt number

Number of nonzero coefficients in Schmidt decomposition

= The rank of the marginal density operators

$\{p_j\}$:The eigenvalues of the marginal density operators (the same for A and B)

'Symmetry' between A and B

$\hat{\rho}_A, \hat{\rho}_B$ The same set of eigenvalues

$$\mathrm{Rank}(\hat{\rho}_A) = \mathrm{Rank}(\hat{\rho}_B) = s$$

Separable states Schmidt number = 1

$p_1 = 1$

Entangled states Schmidt number > 1

$p_1 \geq p_2 > 0$

Range and Kernel of $\hat{\rho}$

$$\mathrm{Ran}\ \hat{\rho} \equiv \{\hat{\rho}|x\rangle \mid |x\rangle \in \mathcal{H}\}$$

Subspace in which $\hat{\rho} > 0$

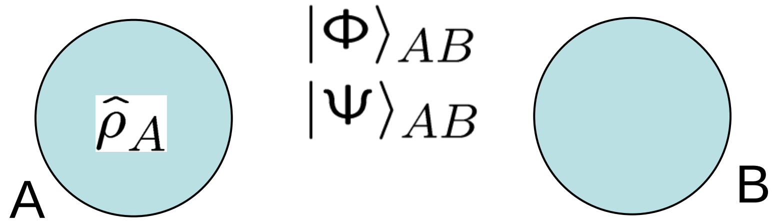
$$\mathrm{Ker}\ \hat{\rho} \equiv \{|y\rangle \mid \hat{\rho}|y\rangle = 0\}$$

Subspace in which $\hat{\rho} = 0$

$$\mathcal{H} = (\mathrm{Ran}\ \hat{\rho}) \oplus (\mathrm{Ker}\ \hat{\rho})$$

$$\mathrm{Rank}(\hat{\rho}) \equiv \dim \mathrm{Ran}\ \hat{\rho}$$

Pure states with the same marginal state



$$\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$
$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$$

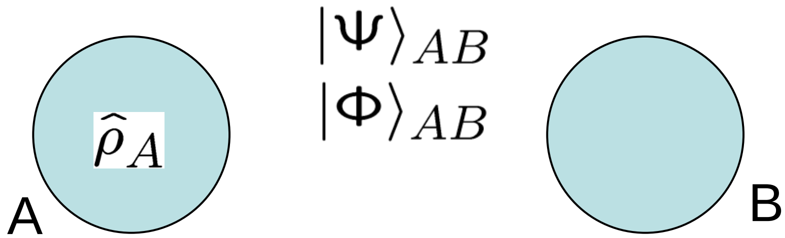
$|\Phi\rangle_{AB} \longrightarrow \hat{\rho}_A$ Marginal state (unique)

$\hat{\rho}_A \longrightarrow |\Phi\rangle_{AB}$, $\hat{\rho}_A \longrightarrow |\Psi\rangle_{AB}$ Purification
Pure Extension (not unique)

$$\boxed{|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}}$$

Theorem: If $|\Psi\rangle_{AB}$ and $|\Phi\rangle_{AB}$ are purifications of the same state $\hat{\rho}_A$, state $|\Psi\rangle_{AB}$ can be physically converted to state $|\Phi\rangle_{AB}$ without touching system A.

Pure states with the same marginal state



$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

Schmidt decomposition

Orthonormal basis $\{|a_i\rangle_A\}$ that diagonalizes $\hat{\rho}_A$

$$|\Psi\rangle_{AB} = \sum_i \sqrt{p_i}|a_i\rangle_A|\mu_i\rangle_B$$

$$|\Phi\rangle_{AB} = \sum_i \sqrt{p_i}|a_i\rangle_A|\nu_i\rangle_B$$

$\{|\mu_i\rangle_B\}$ Orthonormal basis

$\{|\nu_i\rangle_B\}$ Orthonormal basis

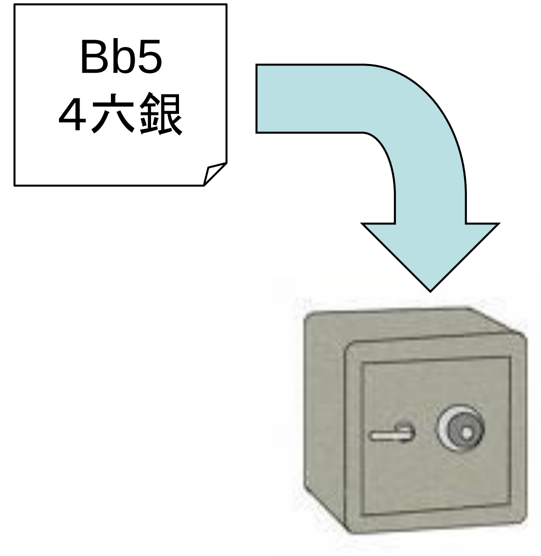
$$|\nu_i\rangle_B = \hat{U}_B|\mu_i\rangle_B$$

unitary

$$|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}$$

Sealed move　(封じ手)

Chess, Go, Shogi …



Let us call it a day and shall we start over tomorrow, with Bob’s move.

While they are (suppose to be) sleeping...

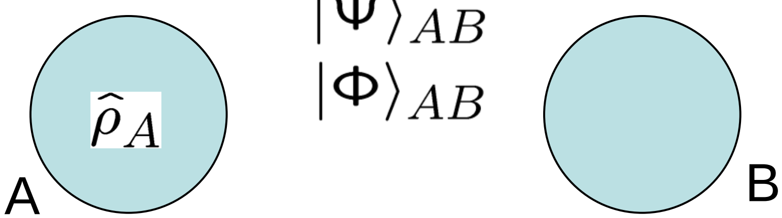
- Alice should not learn the sealed move.
- Bob should not alter the sealed move.

Sealed move

- Alice should not learn the sealed move.
- Bob should not alter the sealed move.

If there is no reliable safe available …

(If there is no system out of both Alice's and Bob's reach …)



$$|\Psi\rangle_{AB}$$
$$|\Phi\rangle_{AB}$$

Alice has no knowledge

↓

Bob can alter the states

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

$$\boxed{|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}}$$

Function of the "safe" cannot be realized.

Impossibility of unconditionally secure quantum bit commitment

(Lo, Mayers)

Ensembles with the same density operator

$\{p_j, |\phi_j\rangle_A\}$ $\quad$ $|\phi_j\rangle_A$ with probability p_j

$\{q_k, |\psi_k\rangle_A\}$ $\quad$ $|\psi_k\rangle_A$ with probability q_k

$$\hat{\rho}_A \equiv \sum_j p_j |\phi_j\rangle_{AA}\langle\phi_j| = \sum_k q_k |\psi_k\rangle_{AA}\langle\psi_k|$$

A scheme to realize the ensemble $\{p_j, |\phi_j\rangle_A\}$

Prepare system AB in state

$\{|b_j\rangle_B\}$ Orthonormal basis

$$|\Phi\rangle_{AB} \equiv \sum_j \sqrt{p_j} |\phi_j\rangle_A |b_j\rangle_B$$

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

Measure system B on basis $\{|b_j\rangle_B\}$

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

$|\phi_j\rangle_A$ with probability p_j

Ensembles with the same density operator

Prepare system AB in state

$$|\Psi\rangle_{AB} \equiv \sum_k \sqrt{q_k}|\psi_k\rangle_A|b_k\rangle_B$$

Apply unitary operation $\hat{U}_B$ to system B

$$|\Phi\rangle_{AB} \equiv \sum_j \sqrt{p_j}|\phi_j\rangle_A|b_j\rangle_B$$

Measure system B on basis $\{|b_j\rangle_B\}$

$|\phi_j\rangle_A$ with probability p_j

$$\{p_j, |\phi_j\rangle_A\}$$

$$|\Psi\rangle_{AB} \equiv \sum_k \sqrt{q_k}|\psi_k\rangle_A|b_k\rangle_B$$

Measure system B on basis $\{|b_k\rangle_B\}$

$|\psi_k\rangle_A$ with probability q_k

$$\{q_k, |\psi_k\rangle_A\}$$

$$\hat{\rho}_A = \text{Tr}_B(|\Psi\rangle\langle\Psi|) = \text{Tr}_B(|\Phi\rangle\langle\Phi|)$$

$$|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}$$

Ensembles with the same density operator

$|\Psi\rangle_{AB}$

A $\{p_j, |\phi_j\rangle_A\}$

$\{q_k, |\psi_k\rangle_A\}$

B

Alice

Bob

Can Alice distinguish the two states even partially?

NO!

Bob can remotely decide which of the states the system A is in.

Bob can postpone his decision indefinitely.

Theorem: Two states $\{p_j, |\phi_j\rangle\}$ and $\{q_k, |\psi_k\rangle\}$ with the same density operator are physically indistinguishable (hence are the same state).

Density operator

$\updownarrow$ One-to-one

Physical state

Example

$\{|0\rangle_A, |1\rangle_A\}$: an orthonormal basis $\qquad |\pm\rangle_A \equiv \frac{1}{\sqrt{2}}(|0\rangle_A \pm |1\rangle_A)$

$\{|+\rangle_A, |-\rangle_A\}$: an orthonormal basis

Recipe I: $\{p_j, |\phi_j\rangle_A\}$ $\quad p_0 = p_1 = \frac{1}{2},\ |\phi_0\rangle_A = |0\rangle_A, |\phi_1\rangle_A = |1\rangle_A$

Recipe II: $\{q_k, |\psi_k\rangle_A\}$ $\quad q_0 = q_1 = \frac{1}{2},\ |\psi_0\rangle_A = |+\rangle_A, |\psi_1\rangle_A = |-\rangle_A$

$$\frac{1}{2}|0\rangle_{AA}\langle 0| + \frac{1}{2}|1\rangle_{AA}\langle 1| \;=\; \frac{1}{2}|+\rangle_{AA}\langle +| + \frac{1}{2}|-\rangle_{AA}\langle -| \;=\; \frac{1}{2}\hat{1}$$

$$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \xrightarrow[\{|0\rangle_B, |1\rangle_B\}]{\text{meas.}} \text{Recipe I:}$$

$$\Big\downarrow\ \hat{U} = |+\rangle_{BB}\langle 0| \;+\; |-\rangle_{BB}\langle 1|$$

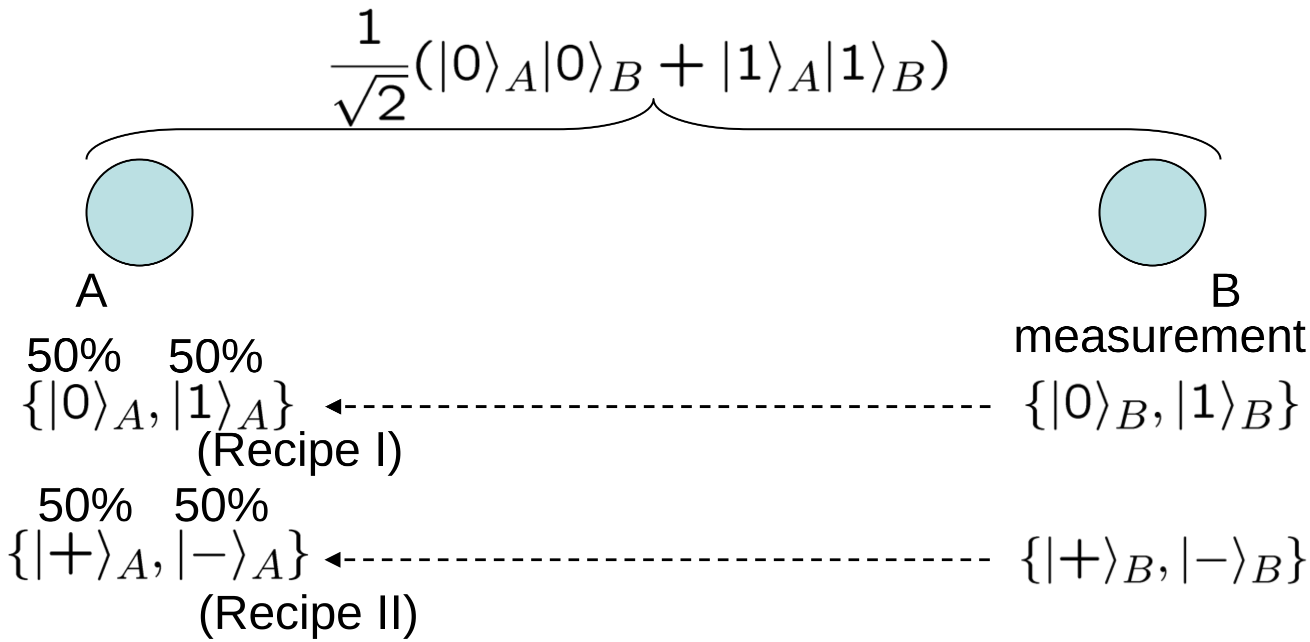
$$\frac{1}{\sqrt{2}}(|0\rangle_A|+\rangle_B + |1\rangle_A|-\rangle_B)$$

$$||$$

$$\frac{1}{\sqrt{2}}(|+\rangle_A|0\rangle_B + |-\rangle_A|1\rangle_B) \xrightarrow[\{|0\rangle_B, |1\rangle_B\}]{\text{meas.}} \text{Recipe II:}$$

meas.
$\{|+\rangle_B, |-\rangle_B\}$

Example



$$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \xrightarrow[\{|0\rangle_B, |1\rangle_B\}]{\text{meas.}}$$ Recipe I:

$$\Big\downarrow \; \hat{U} = |+\rangle_{BB}\langle 0| \; + \; |-\rangle_{BB}\langle 1|$$

$$\frac{1}{\sqrt{2}}(|0\rangle_A|+\rangle_B + |1\rangle_A|-\rangle_B)$$

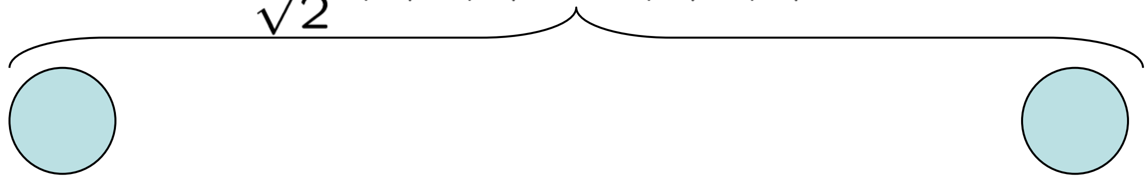
$$||$$

$$\frac{1}{\sqrt{2}}(|+\rangle_A|0\rangle_B + |-\rangle_A|1\rangle_B) \xrightarrow[\{|0\rangle_B, |1\rangle_B\}]{\text{meas.}}$$ Recipe II:

$$\text{meas.} \; \{|+\rangle_B, |-\rangle_B\}$$

Example

$$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$



A

B measurement

50% 50%

$\{|0\rangle_A, |1\rangle_A\}$ ←------------------ $\{|0\rangle_B, |1\rangle_B\}$

(Recipe I)

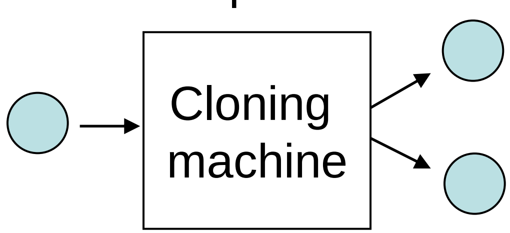
50% 50%

$\{|+\rangle_A, |-\rangle_A\}$ ←------------------ $\{|+\rangle_B, |-\rangle_B\}$

(Recipe II)

If Recipes I and II were distinguishable even partially, the causality would be violated.

For example...



$$\begin{cases} |0\rangle \to |0\rangle|0\rangle \\ |1\rangle \to |1\rangle|1\rangle \\ |+\rangle \to |+\rangle|+\rangle \\ |-\rangle \to |-\rangle|-\rangle \end{cases}$$

Such a machine should not exist.

3. Qubits

Pauli operators (Pauli matrices)

Bloch representation (Bloch sphere)

Orthogonal measurement

Unitary operation

Qubit

$$\dim \mathcal{H} = 2$$

Take a standard basis $\{|0\rangle, |1\rangle\}$

Linear operator $\hat{A}$

Matrix representation (for $\{|0\rangle, |1\rangle\}$)

$$\hat{A} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

$$A_{ij} = \langle i|\hat{A}|j\rangle$$

$$\hat{A} = \sum_{ij} A_{ij}|i\rangle\langle j|$$

4 complex parameters

$$\hat{A} = \alpha_0\hat{\sigma}_0 + \alpha_1\hat{\sigma}_1 + \alpha_2\hat{\sigma}_2 + \alpha_3\hat{\sigma}_3$$

Pauli operators (Pauli matrices)

Take a standard basis $\{|0\rangle, |1\rangle\}$

$$\hat{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_x = \hat{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{\sigma}_y = \hat{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \hat{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Unitary and self-adjoint

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k$$

$$\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{i,j}\hat{1}$$

$$\mathrm{Tr}(\hat{\sigma}_i) = 0, \ \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j) = 2\delta_{i,j}.$$

$$i, j = 1, 2, 3$$

Levi-Civita symbol

$$\begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \\ \text{Otherwise } \epsilon_{ijk} = 0 \end{cases}$$

Einstein notation

$\sum_k$ is omitted.

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z$$

$$\hat{\sigma}_x^2 = \hat{1}$$

$$\{\hat{\sigma}_x, \hat{\sigma}_z\} \equiv \hat{\sigma}_x\hat{\sigma}_z + \hat{\sigma}_z\hat{\sigma}_x = 0$$

$$\mathrm{Tr}(\hat{\sigma}_\mu\hat{\sigma}_\nu) = 2\delta_{\mu,\nu}$$

$$(\mu, \nu = 0, 1, 2, 3; \ \sigma_0 \equiv \hat{1})$$

'Orthogonality' with respect to

$$(\hat{A}, \hat{B}) \equiv \mathrm{Tr}(\hat{A}^\dagger \hat{B})$$

Pauli operators (Pauli matrices)

$$
\begin{aligned}
[\hat{\sigma}_i, \hat{\sigma}_j] &= 2i\epsilon_{ijk}\hat{\sigma}_k \\
\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i &= 2\delta_{i,j}\hat{1} \\
\mathrm{Tr}(\hat{\sigma}_i) = 0,\ \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j) &= 2\delta_{i,j}.
\end{aligned}
$$

Linear operator $\hat{A}$ 4 complex parameters (P_0, P_x, P_y, P_z)

$$
\hat{A} = \frac{1}{2}\left(P_0\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) = \frac{1}{2}\begin{pmatrix} P_0 + P_z & P_x - iP_y \\ P_x + iP_y & P_0 - P_z \end{pmatrix}
$$

$$
\begin{aligned}
\boldsymbol{P} &= (P_x, P_y, P_z) \\
\hat{\boldsymbol{\sigma}} &= (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)
\end{aligned}
$$

$$
P_0 = \mathrm{Tr}(\hat{A}) \qquad \boldsymbol{P} = \mathrm{Tr}(\hat{\boldsymbol{\sigma}}\hat{A})
$$

Pauli operators (Pauli matrices)

$$\hat{A} = \frac{1}{2}\left(P_0\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) = \frac{1}{2}\begin{pmatrix} P_0 + P_z & P_x - iP_y \\ P_x + iP_y & P_0 - P_z \end{pmatrix}$$

$\hat{A}$ is self-adjoint. $\longleftrightarrow$ P_0 and $\boldsymbol{P}$ are real.

Eigenvalues λ_+, λ_-

$$\det(\hat{A}) = \lambda_+\lambda_- = \tfrac{1}{4}(P_0^2 - |\boldsymbol{P}|^2)$$
$$\mathrm{Tr}(\hat{A}) = \lambda_+ + \lambda_- = P_0$$
$$\downarrow$$
$$\lambda_\pm = (P_0 \pm |\boldsymbol{P}|)/2$$

$\hat{A}$ is positive. $\longleftrightarrow$ P_0 and $\boldsymbol{P}$ are real, $P_0 \geq |\boldsymbol{P}|$

Bloch representation (Bloch sphere)

Density operator

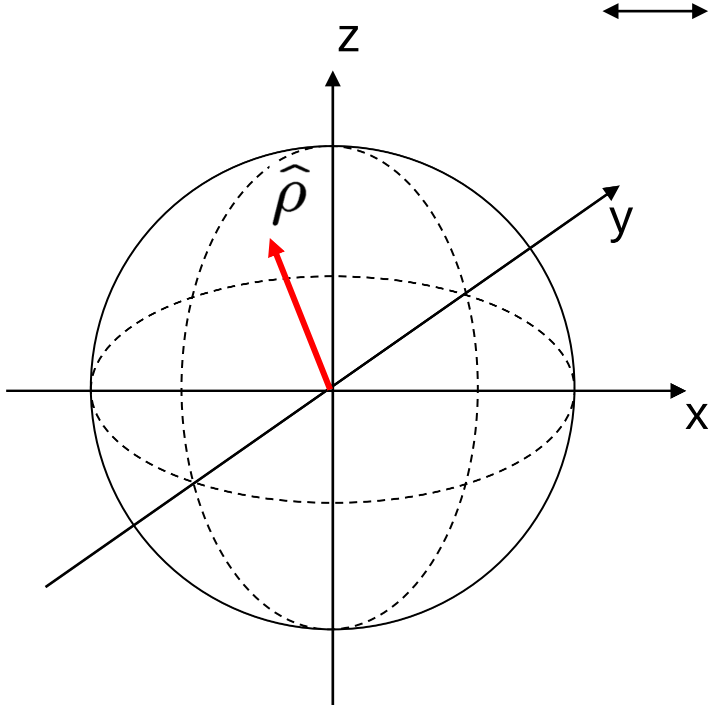
Positive & Unit trace

$P_0 \geq |\boldsymbol{P}| \qquad P_0 = 1$

$$\hat{\rho} = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) \qquad |\boldsymbol{P}| \leq 1$$

Density operator for a qubit system

$\longleftrightarrow$ A 3D real vector of length no greater than 1



A point inside or on the sphere of radius 1

$$\boldsymbol{P} = (P_x, P_y, P_z)$$

Bloch vector

$$\lambda_{\pm} = (P_0 \pm |\boldsymbol{P}|)/2$$

Pure states $\longleftrightarrow \lambda_+ = 1, \lambda_- = 0$

$\longleftrightarrow |\boldsymbol{P}| = 1$

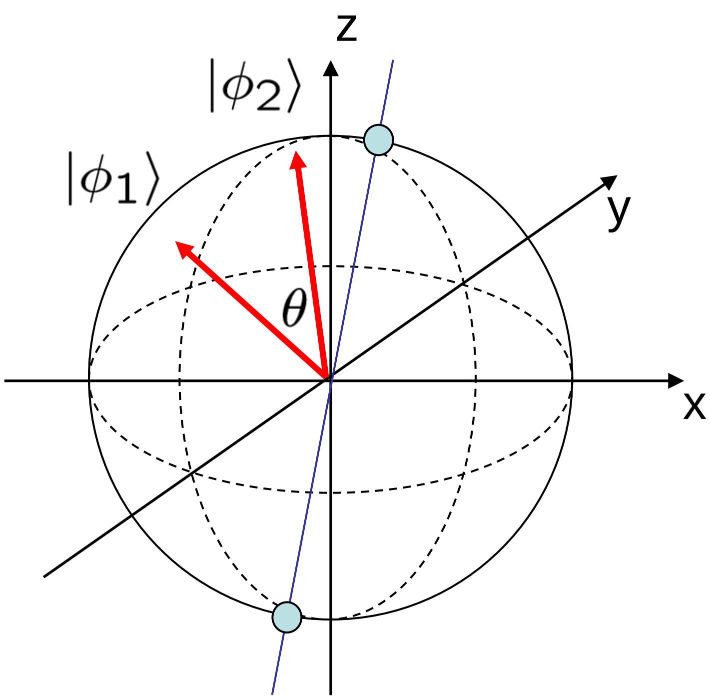
$\longleftrightarrow$ On the sphere

Pure states $\hat{\rho}_j = |\phi_j\rangle\langle\phi_j|$

$$\hat{\rho}_j = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}_j \cdot \hat{\boldsymbol{\sigma}}\right)$$

$$|\langle\phi_1|\phi_2\rangle|^2 = \mathrm{Tr}[\hat{\rho}_1\hat{\rho}_2]$$

$$= \frac{1 + \boldsymbol{P}_1 \cdot \boldsymbol{P}_2}{2} = \cos^2\frac{\theta}{2}$$

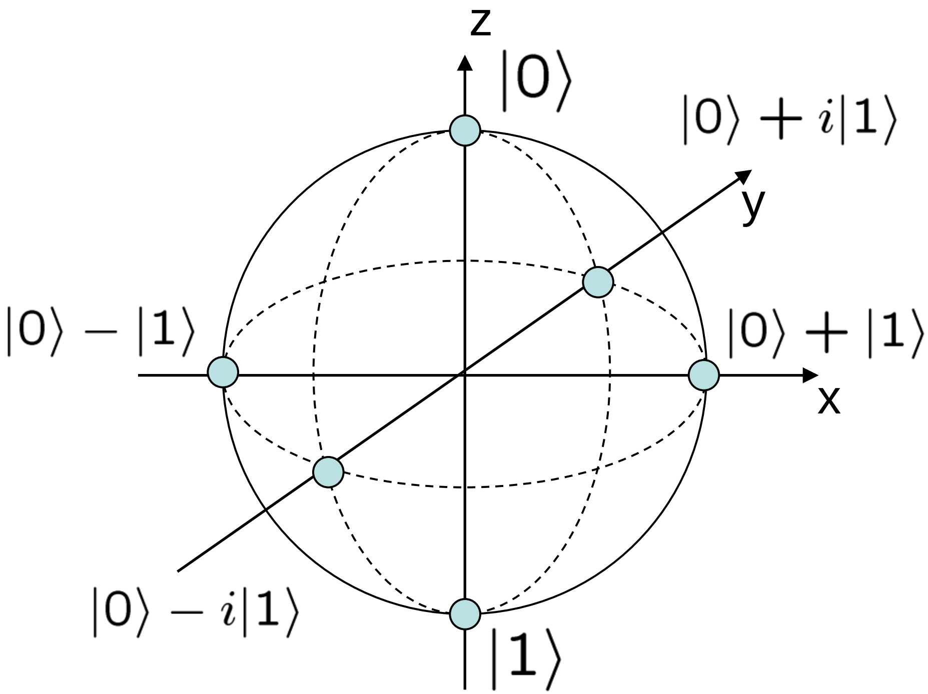


$$\boldsymbol{P}_1 \cdot \boldsymbol{P}_2 = \cos\theta$$

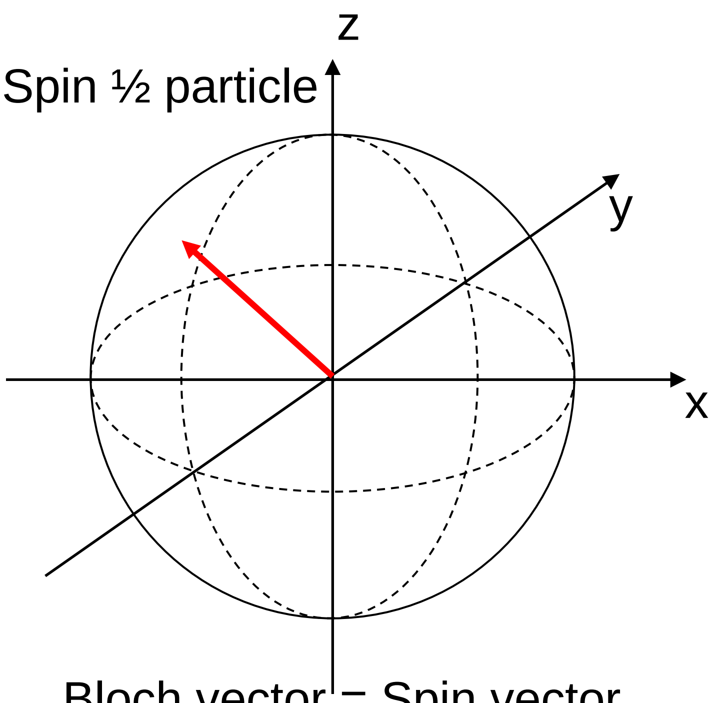
Orthogonal states $\longleftrightarrow$ $\theta = \pi$

Orthonormal basis $\longleftrightarrow$ A line through the origin

Examples

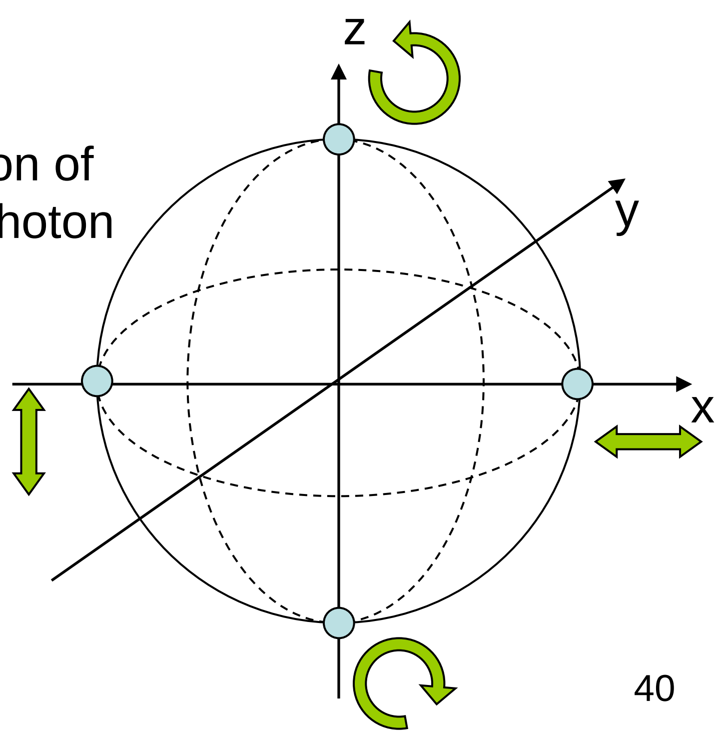


Spin ½ particle



Bloch vector = Spin vector

Polarization of
a single photon

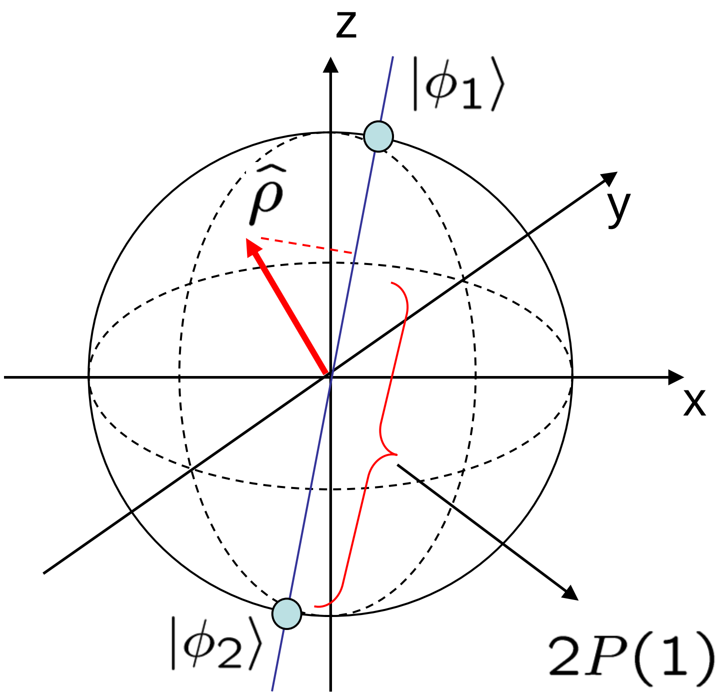


Orthogonal measurement

Orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle\}$ $\longleftrightarrow$ A line through the origin

$$P(1) = \langle\phi_1|\hat{\rho}|\phi_1\rangle = \mathrm{Tr}(\hat{\rho}_1\hat{\rho}) = \frac{1 + \boldsymbol{P}_1 \cdot \boldsymbol{P}}{2}$$

$$P(2) = \frac{1 - \boldsymbol{P}_1 \cdot \boldsymbol{P}}{2}$$



Example

Measurement of observable $\hat{\sigma}_z$

$\downarrow$

Z axis

Unitary operation

$|\psi\rangle, e^{i\theta}|\psi\rangle$ The same physical state

$\hat{U}, \quad e^{i\theta}\hat{U}$ The same physical operation

$$\det(e^{i\theta}\hat{U}) = e^{2i\theta}\det\hat{U}$$

group $SU(2)$: Set of $\hat{U}$ with $\det\hat{U} = 1$ $\hat{U} \in SU(2) \leftrightarrow -\hat{U} \in SU(2)$

(2 to 1 correspondence to the physical unitary operations)

$$\hat{U} = \exp[i\hat{S}]$$

Self-adjoint, traceless

$$\hat{U} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

$$\hat{S} = \frac{1}{2}(\boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}})$$

$$\hat{S} = \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}$$

We can parameterize the elements of SU(2) as

$$\hat{U}(\boldsymbol{n},\varphi) \equiv \exp[-i(\varphi/2)\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}]$$

Unit vector

Unitary operation

$$\hat{\rho} = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) \xrightarrow{\hat{U}(\boldsymbol{n},\varphi)} \hat{\rho}' = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}'\cdot\hat{\boldsymbol{\sigma}}\right)$$

How does the Bloch vector change?

Infinitesimal change $\hat{U}(\boldsymbol{n},\delta\varphi) \sim \hat{1} - i(\delta\varphi/2)\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}$

$$\delta\boldsymbol{P} \equiv \boldsymbol{P}' - \boldsymbol{P} = \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}'] - \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}]$$

$$\begin{aligned}
&= \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{U}(\boldsymbol{n},\delta\varphi)\hat{\rho}\hat{U}^{\dagger}(\boldsymbol{n},\delta\varphi)] - \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}] \\
&= \mathrm{Tr}[\hat{U}^{\dagger}(\boldsymbol{n},\delta\varphi)\hat{\boldsymbol{\sigma}}\hat{U}(\boldsymbol{n},\delta\varphi)\hat{\rho}] - \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}] \\
&\sim \mathrm{Tr}\{(i\delta\varphi/2)[(\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}),\hat{\boldsymbol{\sigma}}]\hat{\rho}\} = -\delta\varphi\mathrm{Tr}[n_i\epsilon_{ijk}\hat{\sigma}_k\hat{\rho}] \\
&= \delta\varphi\mathrm{Tr}[(\boldsymbol{n}\times\hat{\boldsymbol{\sigma}})\hat{\rho}] = \delta\varphi\boldsymbol{n}\times\boldsymbol{P}.
\end{aligned}$$

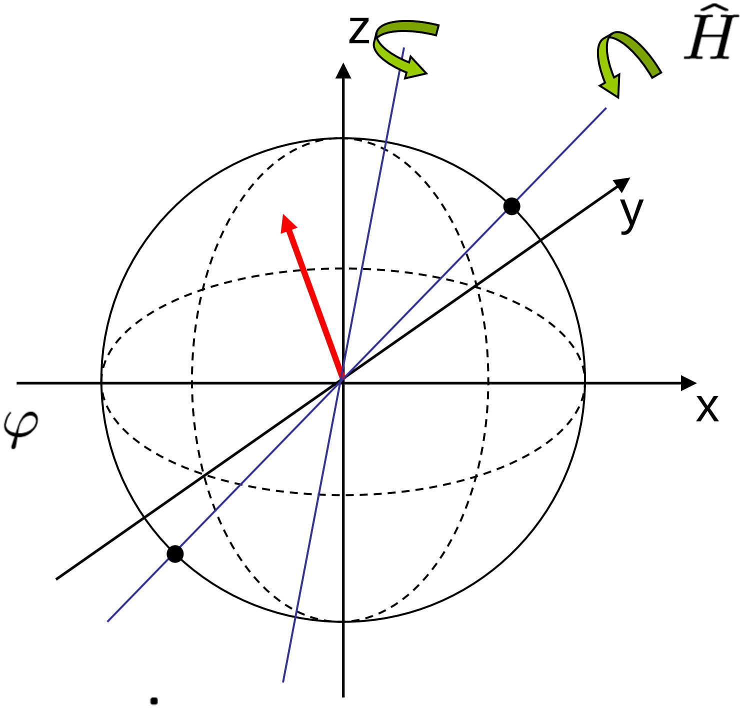
Rotation around axis $\boldsymbol{n}$ by angle $\delta\varphi$

Unitary operation

$$\hat{U} \in SU(2)$$

$$\hat{U} = \exp[-i(\varphi/2)\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}}]$$

Rotation around axis $\boldsymbol{n}$ by angle φ



Examples

$\hat{\sigma}_z$: π rotation around z axis

$\hat{\sigma}_x$: π rotation around x axis

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hadamard transform

π rotation (interchanges z and x axes)

4. Power of an ancillary system

Kraus representation (Operator-sum rep.)

Generalized measurement

Unambiguous state discrimination

Quantum operation (Quantum channel, CPTP map)

Relation between quantum operations and bipartite states

A maximally entangled state and relative states

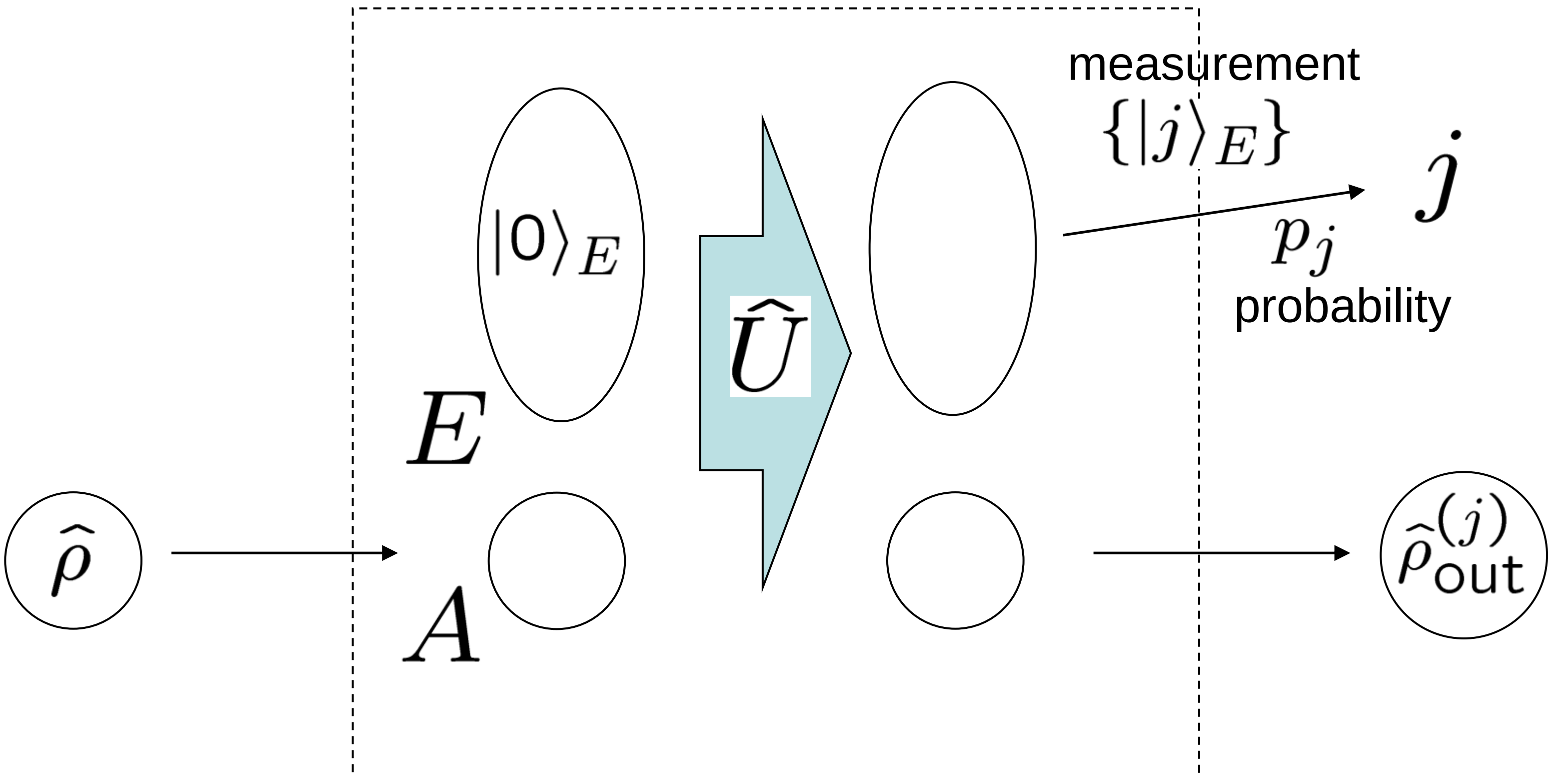
What can we do in principle?

Power of an ancilla system

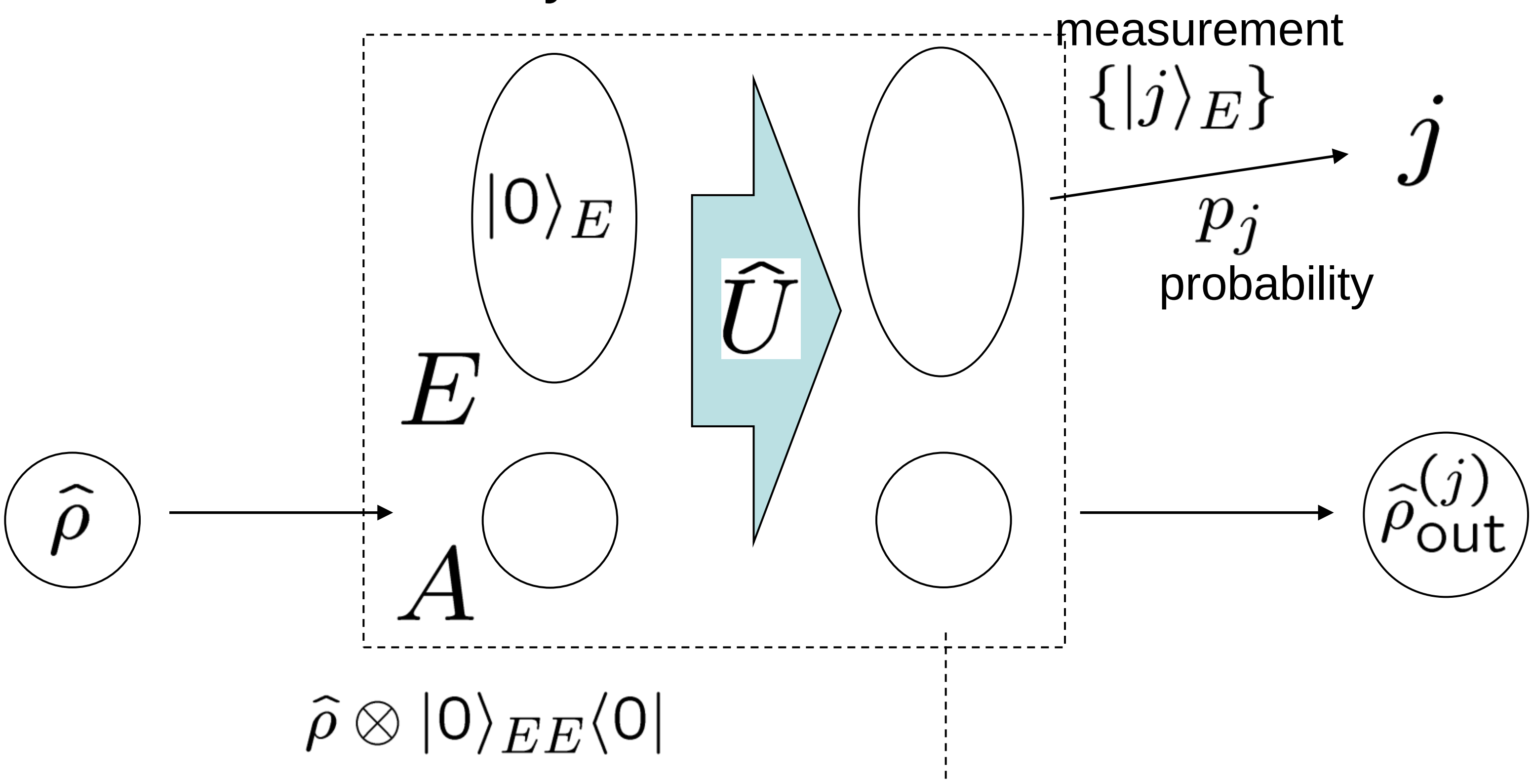
Basic operations
Unitary operations
Orthogonal measurements

+

An auxiliary system
(ancilla)



Power of an ancilla system



$$\widehat{U}(\widehat{\rho}\otimes|0\rangle_{EE}\langle 0|)\widehat{U}^\dagger$$

$$p_j\widehat{\rho}_{\mathrm{out}}^{(j)} = {}_E\langle j|\widehat{U}(\widehat{\rho}\otimes|0\rangle_{EE}\langle 0|)\widehat{U}^\dagger|j\rangle_E$$

$$= \widehat{M}^{(j)}\widehat{\rho}\widehat{M}^{(j)\dagger}$$

$$\widehat{M}^{(j)} \equiv {}_E\langle j|\widehat{U}|0\rangle_E$$

$${}_E\langle j|\ \widehat{U}\ |0\rangle_E$$

$$\widehat{M}^{(j)}:\mathcal{H}_A\to\mathcal{H}_A$$

Kraus representation (Operator-sum rep.)

$$p_j \hat{\rho}^{(j)}_{\mathrm{out}} = {}_E\langle j|\hat{U}(\hat{\rho} \otimes |0\rangle_{EE}\langle 0|)\hat{U}^\dagger |j\rangle_E$$

$$\downarrow \quad \hat{M}^{(j)} \equiv {}_E\langle j|\hat{U}|0\rangle_E$$ Kraus operators

$$p_j \hat{\rho}^{(j)}_{\mathrm{out}} = \hat{M}^{(j)}\hat{\rho}\hat{M}^{(j)\dagger} \ \text{ with } \sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$$

Representation with no reference to the ancilla system

$$\begin{aligned}
\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} &= \sum_j {}_E\langle 0|\hat{U}^\dagger|j\rangle_{EE}\langle j|\hat{U}|0\rangle_E \\
&= {}_E\langle 0|\hat{U}^\dagger\hat{U}|0\rangle_E \\
&= {}_E\langle 0|\hat{1}_A \otimes \hat{1}_E|0\rangle_E \\
&= \hat{1}_A
\end{aligned}$$

Kraus operators $\longrightarrow$ Physical realization

$$p_j \hat{\rho}^{(j)}_{\text{out}} = {}_E\langle j|\hat{U}(\hat{\rho}\otimes|0\rangle_{EE}\langle 0|)\hat{U}^\dagger|j\rangle_E$$

$$\hat{M}^{(j)} \equiv {}_E\langle j|\hat{U}|0\rangle_E$$ Kraus operators

$$p_j \hat{\rho}^{(j)}_{\text{out}} = \hat{M}^{(j)}\hat{\rho}\hat{M}^{(j)\dagger} \text{ with } \textstyle\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$$

Arbitrary set $\{\hat{M}^{(j)}\}$ satisfying $\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$

$|\phi\rangle_A \otimes |0\rangle_E \mapsto \sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E$ is linear.

preserves inner products.

For any two states $|\phi\rangle_A$ and $|\psi\rangle_A$,

$$\left(\sum_{j'} \hat{M}^{(j')}|\psi\rangle_A \otimes |j'\rangle_E\right)^\dagger \left(\sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E\right)$$
$$= {}_A\langle\psi|\phi\rangle_A = (|\psi\rangle_A \otimes |0\rangle_E)^\dagger (|\phi\rangle_A \otimes |0\rangle_E)\,.$$

There exists a unitary satisfying
$\hat{U}(|\phi\rangle_A \otimes |0\rangle_E) = \sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E$

Generalized measurement

$$p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

$$p_j = \text{Tr}[\hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger}] = \text{Tr}[\hat{F}^{(j)} \hat{\rho}]$$

$$\hat{F}^{(j)} \equiv \hat{M}^{(j)\dagger} \hat{M}^{(j)} \underset{\text{positive}}{\geq} 0$$

$$\boxed{p_j = \text{Tr}[\hat{F}^{(j)} \hat{\rho}] \text{ with } \sum_j \hat{F}^{(j)} = \hat{1}}$$

$\{\hat{F}^{(j)}\}$ POVM

Positive operator valued measure

Generalized measurement

$$p_j = \mathrm{Tr}[\widehat{F}^{(j)}\widehat{\rho}] \;\text{ with } \textstyle\sum_j \widehat{F}^{(j)} = \hat{1}$$

Examples

Orthogonal measurement on basis $\{|a_j\rangle\}$

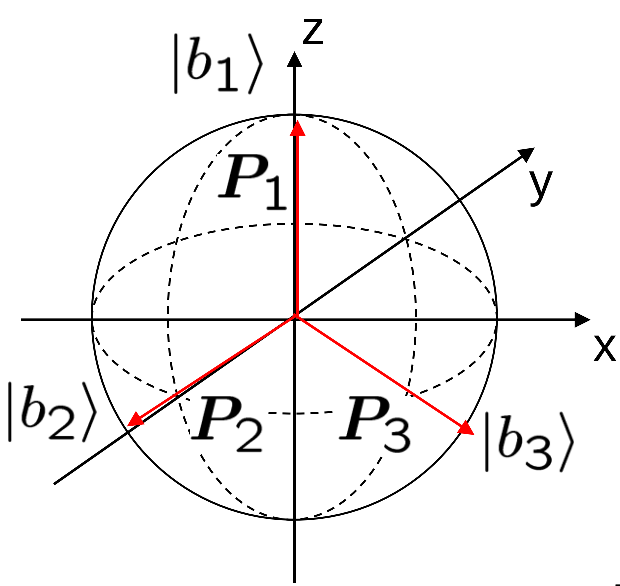
$$\widehat{F}^{(j)} = |a_j\rangle\langle a_j|$$

Trine measurement on a qubit

$$\widehat{F}^{(j)} = \tfrac{2}{3}|b_j\rangle\langle b_j|$$

$$|b_j\rangle\langle b_j| = \tfrac{1}{2}\left(\hat{1} + \boldsymbol{P}_j \cdot \hat{\boldsymbol{\sigma}}\right)$$

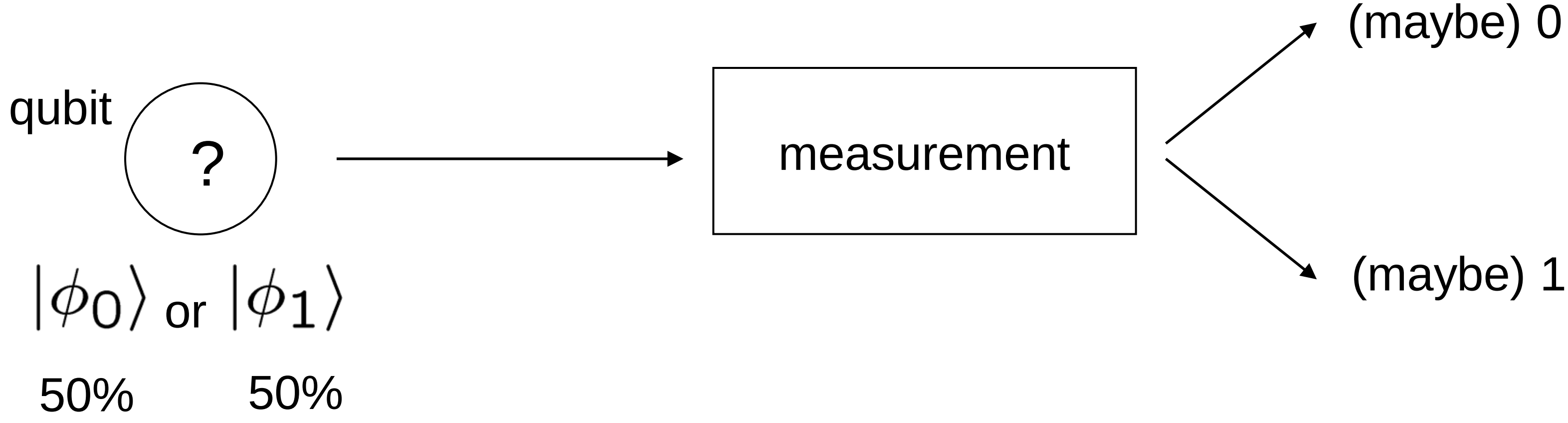
$$\textstyle\sum_j \boldsymbol{P}_j = 0 \longrightarrow \textstyle\sum_j \widehat{F}^{(j)} = \hat{1}$$



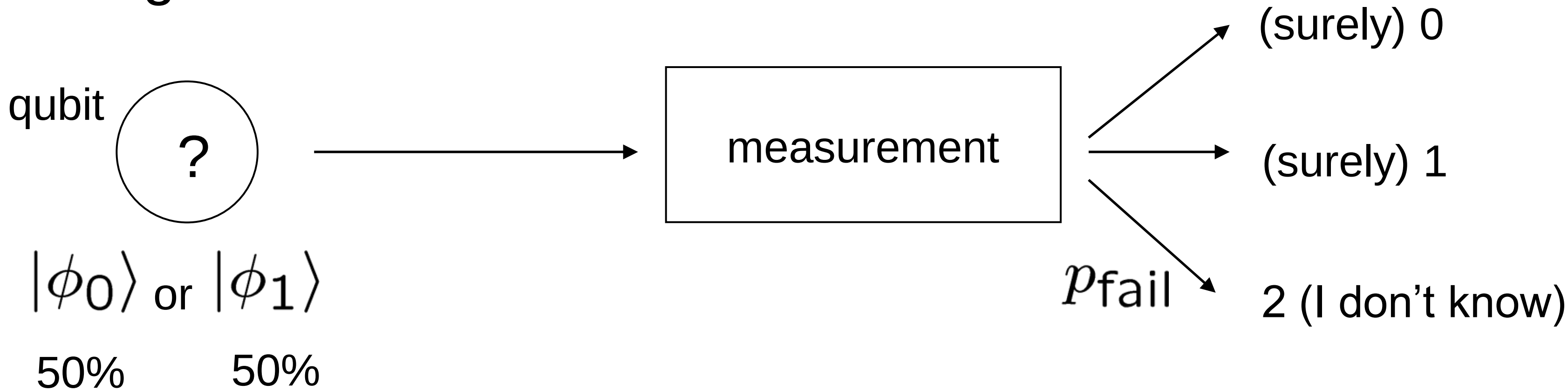
Distinguishing two nonorthogonal states

$$\langle \phi_0 | \phi_1 \rangle = s > 0$$

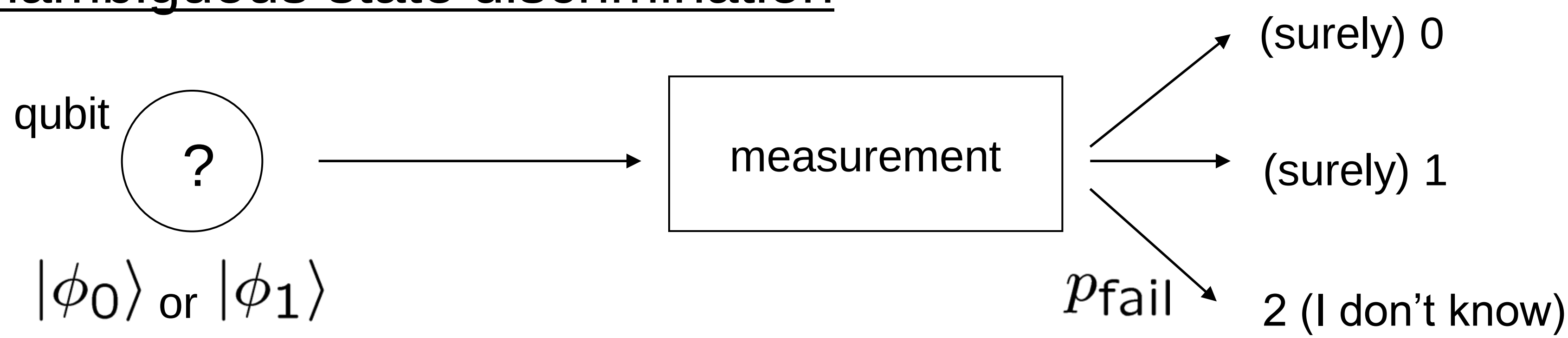
Minimum-error discrimination



Unambiguous state discrimination



Unambiguous state discrimination



$|\phi_0\rangle$ or $|\phi_1\rangle$

50% 50%

$$\langle\phi_0|\phi_1\rangle = s > 0$$

Orthogonal measurement

$$\{|\phi_0\rangle, |\phi_0^\perp\rangle\}$$

2 (I don't know) (surely) 1

If the initial state is $|\phi_0\rangle$

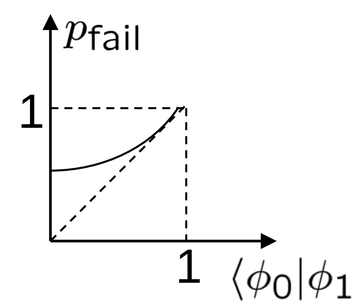
it always fails.

If the initial state is $|\phi_1\rangle$

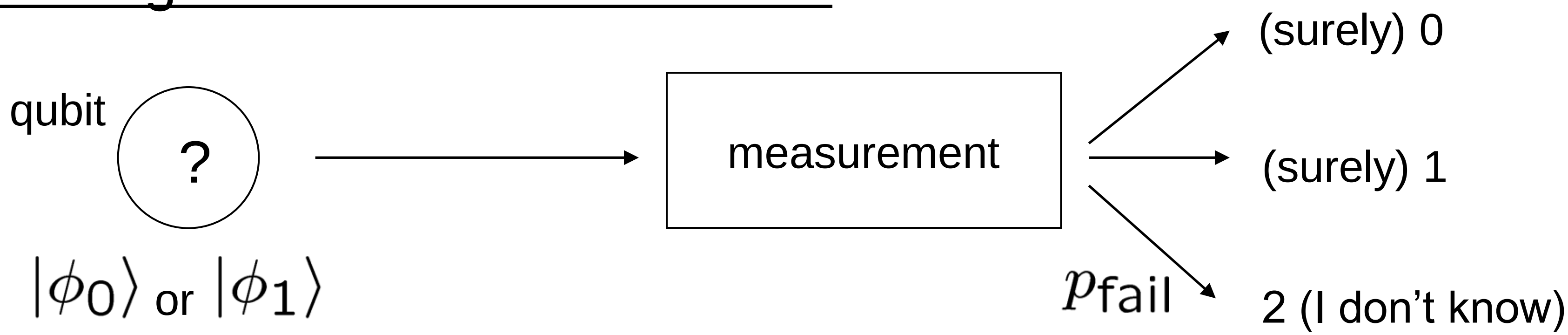
it fails with prob. $|\langle\phi_0|\phi_1\rangle|^2 = s^2$

$$\{|\phi_1\rangle, |\phi_1^\perp\rangle\}$$

$$p_{\text{fail}} = \frac{1+s^2}{2}$$



Unambiguous state discrimination



$|\phi_0\rangle$ or $|\phi_1\rangle$

50% 50%

$$\langle \phi_0 | \phi_1 \rangle = s > 0$$

Generalized measurement

$$\hat{F}_0 := \mu |\phi_1^{\perp}\rangle\langle\phi_1^{\perp}|$$

$$\hat{F}_1 := \mu |\phi_0^{\perp}\rangle\langle\phi_0^{\perp}|$$

$$\hat{F}_2 := \hat{1} - \hat{F}_0 - \hat{F}_1$$

The only constraint on μ comes from $\hat{F}_2 \geq 0$

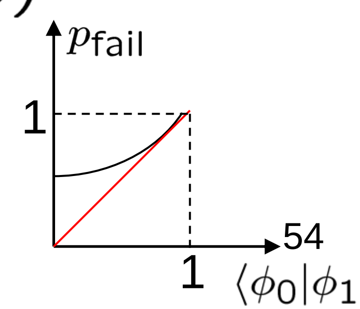
$$\langle \phi_0^{\perp} | \phi_1^{\perp} \rangle = s \qquad (\hat{F}_0 + \hat{F}_1 \leq \hat{1})$$

$$(\hat{F}_0 + \hat{F}_1)(|\phi_0^{\perp}\rangle \pm |\phi_1^{\perp}\rangle)$$
$$= \mu(1 \pm s)(|\phi_0^{\perp}\rangle \pm |\phi_1^{\perp}\rangle)$$

The optimum: $\mu = (1+s)^{-1}$

$$p_{\mathrm{fail}} = 1 - \frac{\mu}{2}|\langle \phi_0 | \phi_1^{\perp} \rangle|^2 - \frac{\mu}{2}|\langle \phi_1 | \phi_0^{\perp} \rangle|^2$$
$$= 1 - \mu(1 - s^2)$$

$$p_{\mathrm{fail}} = s$$



Quantum operation (Quantum channel, CPTP map)

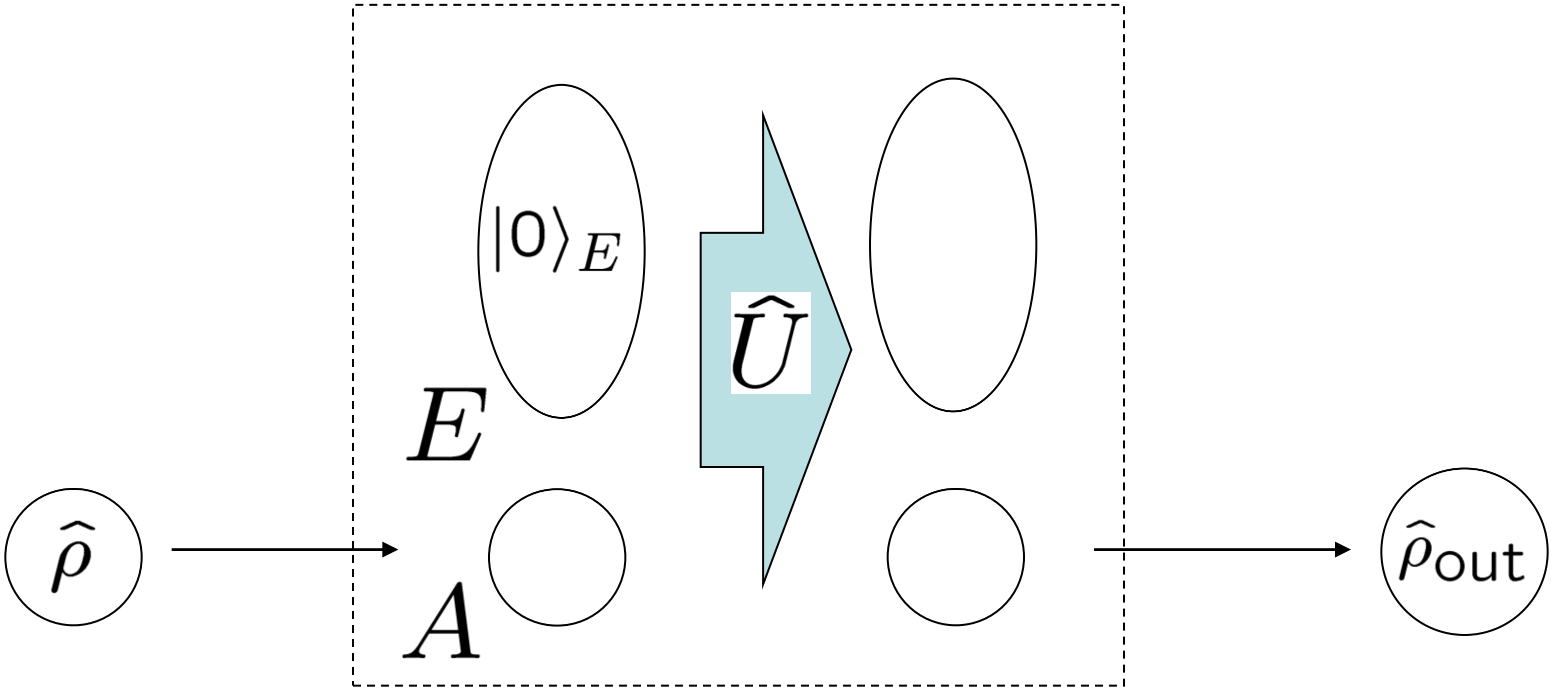
$$p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

$$\begin{aligned}\hat{\rho}_{\text{out}} = \sum_j p_j \hat{\rho}_{\text{out}}^{(j)} &= \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \\ &= \sum_j {}_E\langle j| \hat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \hat{U}^\dagger |j\rangle_E \\ &= \mathrm{Tr}_E[\hat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \hat{U}^\dagger]\end{aligned}$$

$$\boxed{\begin{aligned}\hat{\rho}_{\text{out}} &= \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \\ &= \mathrm{Tr}_E[\hat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \hat{U}^\dagger]\end{aligned}}$$

$\hat{\rho}_{\text{out}} = \mathcal{C}(\hat{\rho})$ completely-positive trace-preserving map
CPTP map

Quantum operation (Quantum channel, CPTP map)



$$\hat{\rho}_{\mathrm{out}} = \sum_j \widehat{M}^{(j)} \hat{\rho} \widehat{M}^{(j)\dagger} \quad \text{with } \sum_j \widehat{M}^{(j)\dagger} \widehat{M}^{(j)} = \hat{1}$$
$$= \mathrm{Tr}_E[\widehat{U}(\hat{\rho} \otimes |0\rangle_{EE}\langle 0|)\widehat{U}^\dagger]$$

$\hat{\rho}_{\mathrm{out}} = \mathcal{C}(\hat{\rho})$ completely-positive trace-preserving map
CPTP map

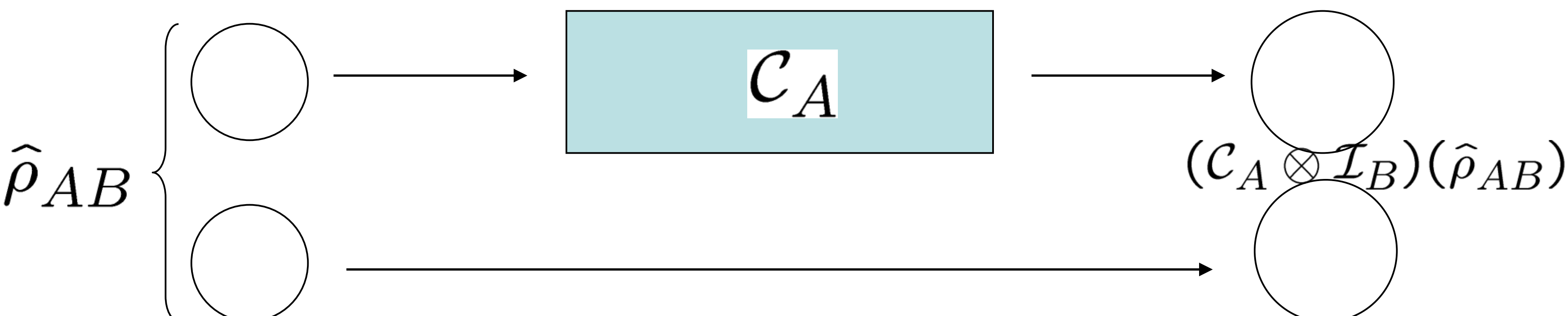
Positive maps and completely-positive maps

Linear map

$$\hat{\rho}_A \mapsto \mathcal{C}_A(\hat{\rho}_A)$$

“positive”: $\mathcal{C}_A(\hat{\rho}_A)$ is positive whenever $\hat{\rho}_A$ is positive

“completely-positive”: $(\mathcal{C}_A \otimes \mathcal{I}_B)(\hat{\rho}_{AB})$ is positive whenever $\hat{\rho}_{AB}$ is positive



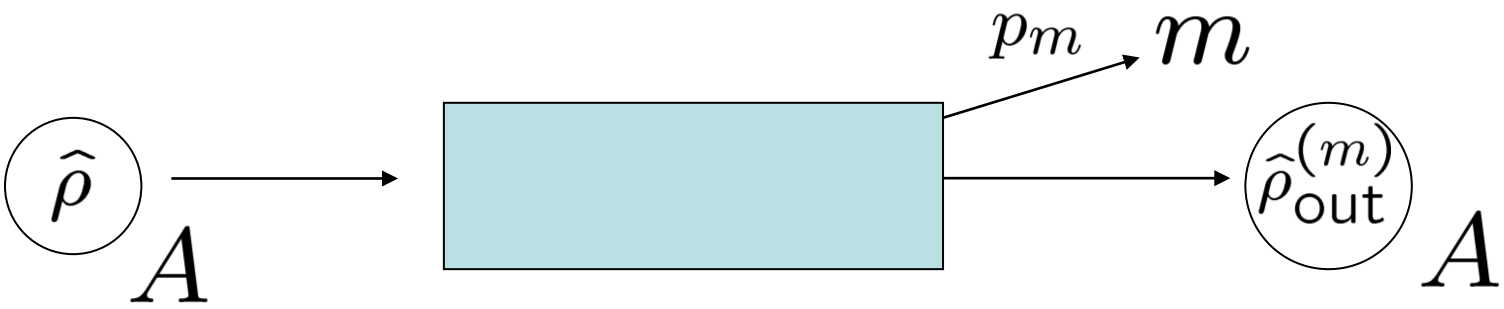
$$(\mathcal{C}_A \otimes \mathcal{I}_B)(\hat{\rho}_{AB}) = \sum_j \hat{M}_A^{(j)} \hat{\rho}_{AB} \hat{M}_A^{(j)\dagger}$$

What can we do in principle?

We have seen what we can (at least) do by using an ancilla system.

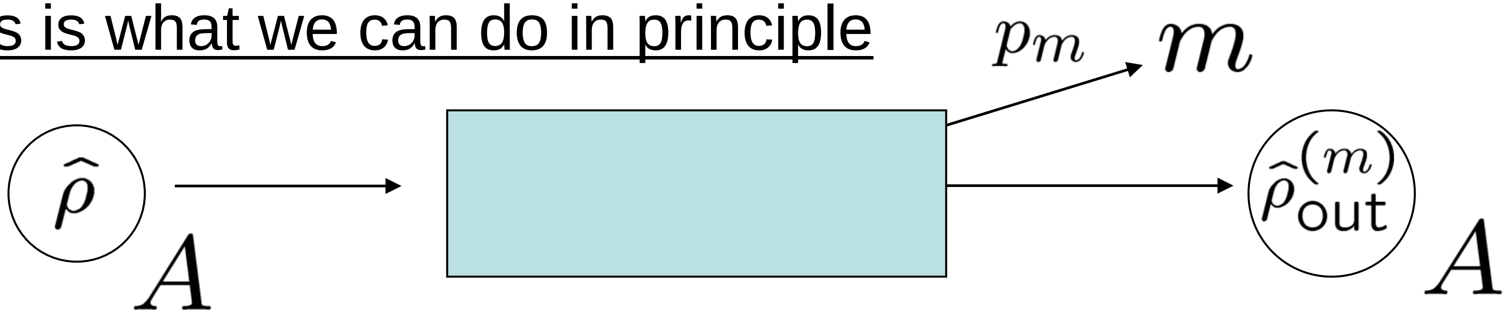
$$p_j \hat{\rho}^{(j)}_{\text{out}} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

We also want to know what we cannot do.



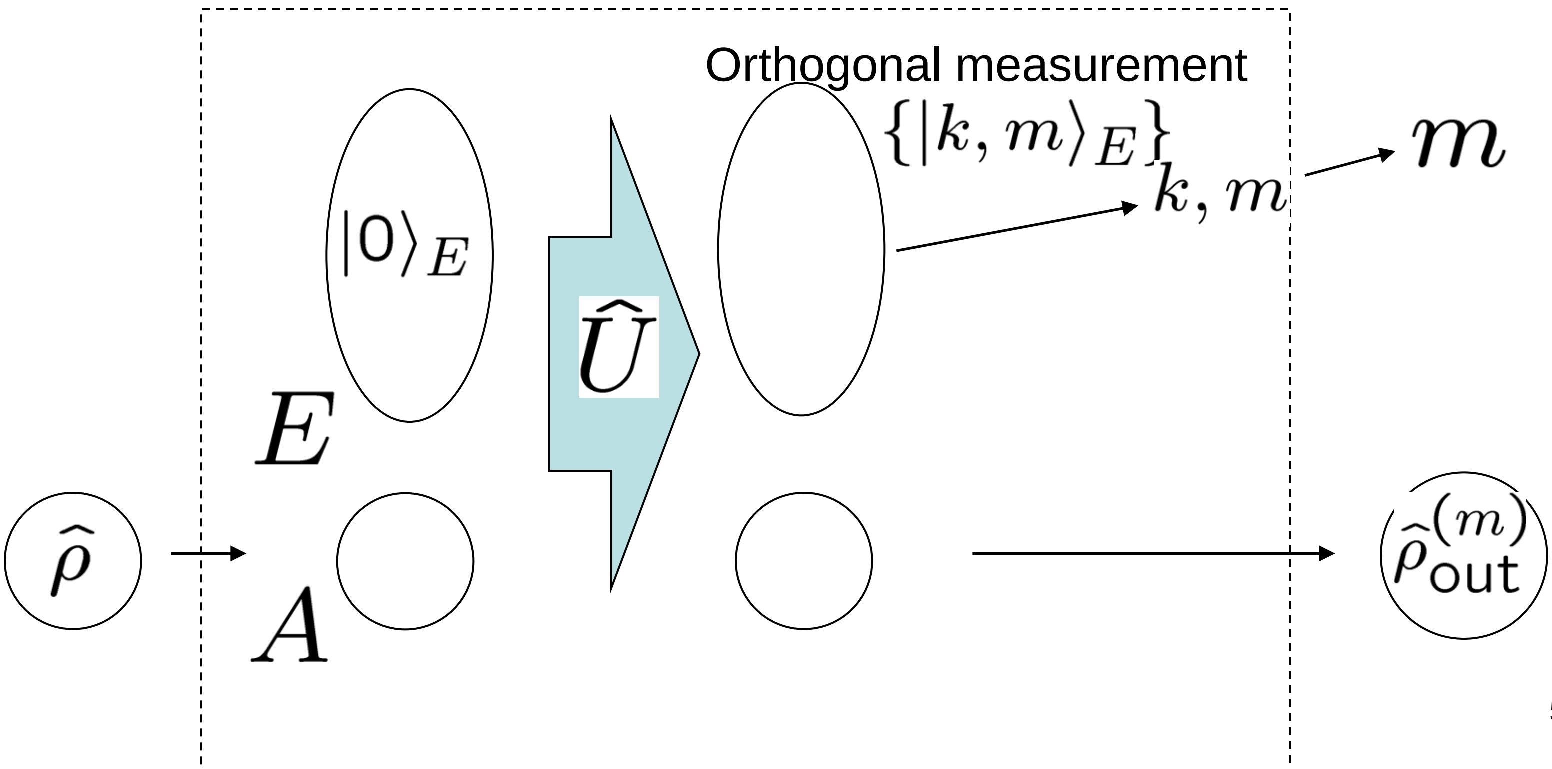
Black box with classical and quantum outputs

This is what we can do in principle

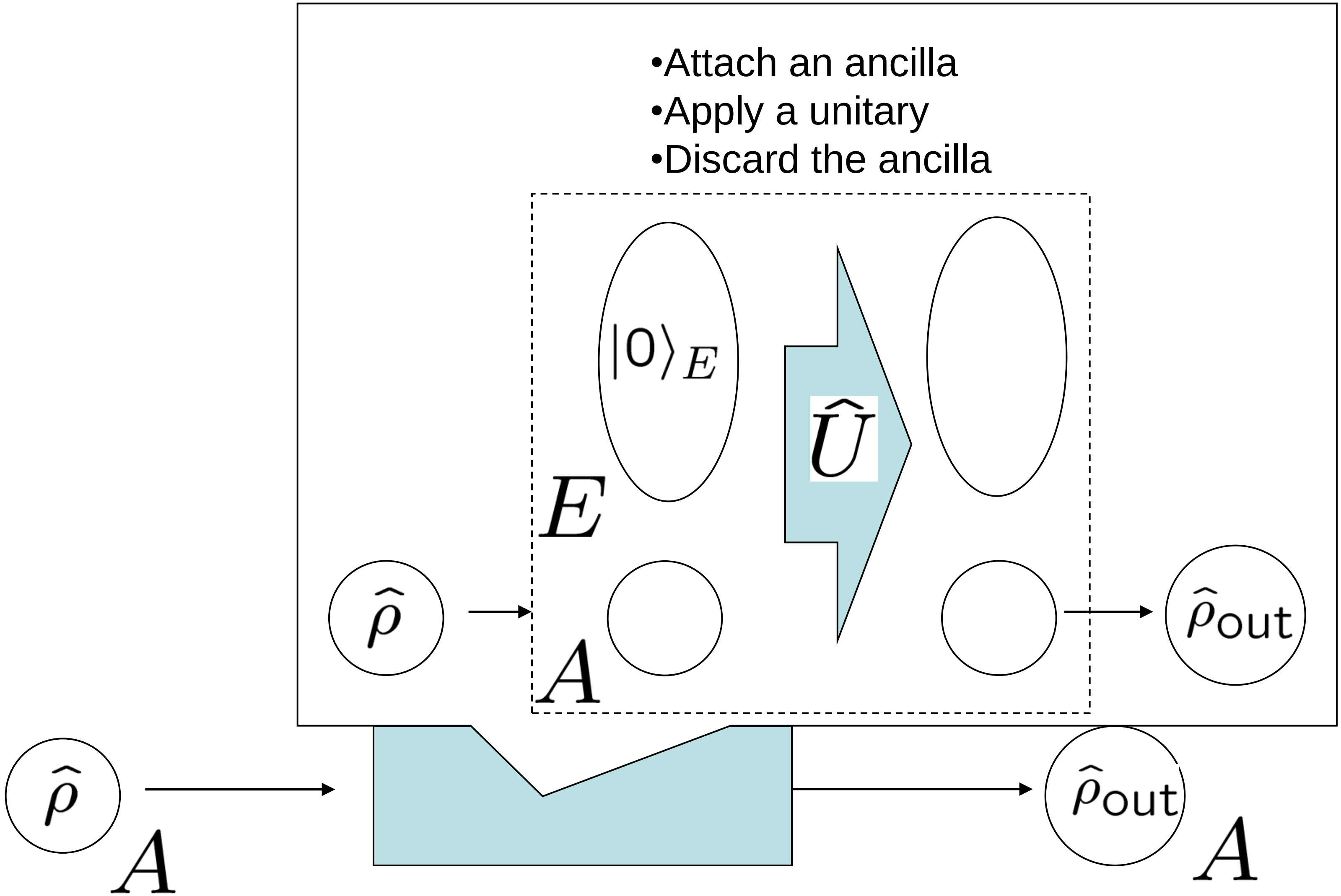


Any physical process should be represented in the following form:

$$p_m \hat{\rho}_{\text{out}}^{(m)} = \sum_k \hat{M}^{(k,m)} \hat{\rho} \hat{M}^{(k,m)\dagger} \qquad \sum_{m,k} \hat{M}^{(k,m)\dagger} \hat{M}^{(k,m)} = \hat{1}_A$$



What can we do in principle?



Black box with quantum output
(Quantum channel)

Maximally entangled states (MES)

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$$

Orthonormal bases $\{|k\rangle_A\}_{k=1,2,\ldots,d}$ $\{|k\rangle_B\}_{k=1,2,\ldots,d}$

$$\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

Maximally entangled state

Properties of MES (I): Relative states

Fix a maximally entangled state

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$$

$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A |k\rangle_B$$

A B

Relative states

$$|\phi\rangle_A = \sum_k \alpha_k |k\rangle_A \longleftrightarrow |\phi^*\rangle_B = \sum_k \overline{\alpha_k} |k\rangle_B$$

$$= \sqrt{d}\,{}_B\langle\phi^*||\Phi\rangle_{AB} \qquad = \sqrt{d}\,{}_A\langle\phi||\Phi\rangle_{AB}$$

$$|\Phi\rangle_{AB} \begin{cases} A \longrightarrow |\phi\rangle_A \\ B \xrightarrow{\text{Orthogonal measurement}} \text{outcome } j = 1 \end{cases}$$

Orthogonal measurement $\{|v_j\rangle_B\}_{j=1,2,\dots,d}$

outcome $j = 1$

$p_1 = \frac{1}{d}$

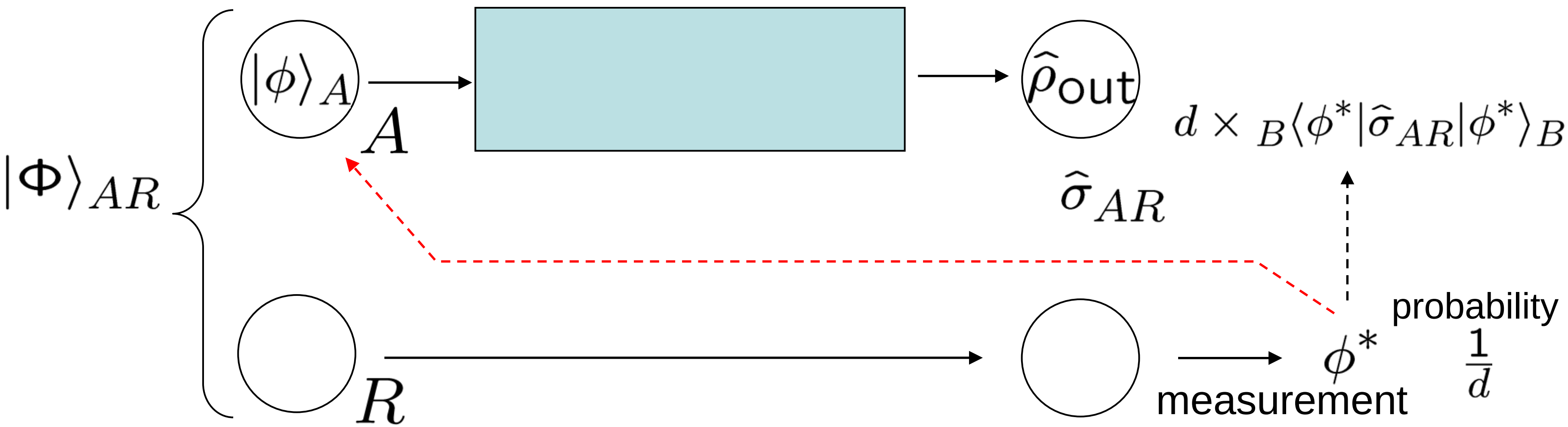
$|v_1\rangle_B = |\phi^*\rangle_B$

$$\boxed{\frac{1}{\sqrt{d}}|\phi\rangle_A = {}_B\langle\phi^*||\Phi\rangle_{AB}}$$

Quantum operation and bipartite state

We can remotely prepare system A in any state with a nonzero success probability.

At any time



$\hat{\sigma}_{AR}$:The state obtained when a half of an MES is fed to the channel.

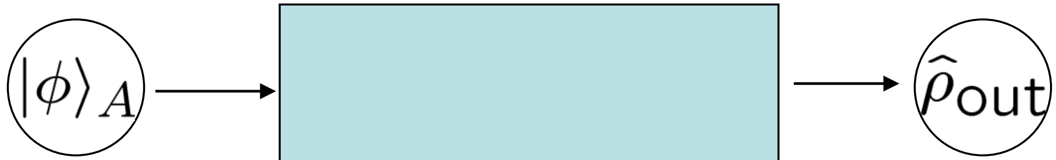
If this state is known,

$$\hat{\rho}_{\text{out}} = {}_B\langle\phi^*|\hat{\sigma}_{AR}|\phi^*\rangle_B d$$

Output for every input state is known!

Characterization of a process = Characterization of a state

Quantum operation and bipartite state



$$\hat{\rho}_{\text{out}} = \sqrt{d}\,{}_R\langle\phi^*|\hat{\sigma}_{AR}|\phi^*\rangle_R\sqrt{d}$$

$${}_R\langle\phi^*| = \sqrt{d}\ {}_{AR}\langle\Phi||\phi\rangle_A \qquad \hat{\sigma}_{AR} = \sum_j |\Psi_j\rangle_{AR}\ {}_{AR}\langle\Psi_j|$$

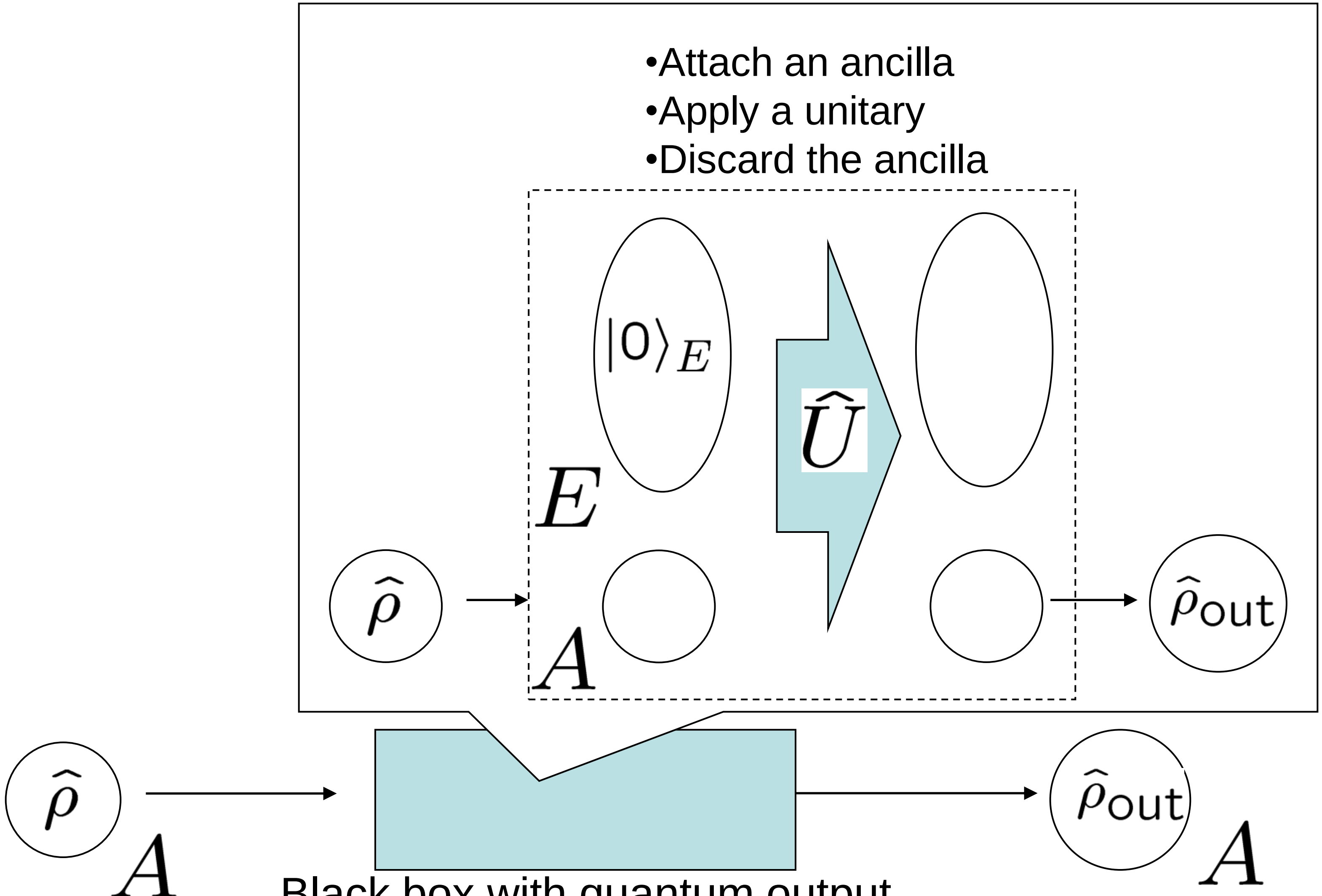
unnormalized

$$\sqrt{d}\,{}_R\langle\phi^*||\Psi_j\rangle_{AR} = \hat{M}^{(j)}|\phi\rangle_A$$ (A linear map)

$$\hat{\rho}_{\text{out}} = \sum_j \hat{M}^{(j)}|\phi\rangle_{A\,A}\langle\phi|\hat{M}^{(j)\dagger}$$

$${}_{AR}\langle\Phi|\begin{matrix}|\phi\rangle_A \\ |\Psi_j\rangle_{AR}\end{matrix}$$

What we can do in principle



Black box with quantum output
(Quantum channel)

$$\hat{\rho}_{\text{out}} = \sum_j \widehat{M}^{(j)} \hat{\rho} \widehat{M}^{(j)\dagger}$$

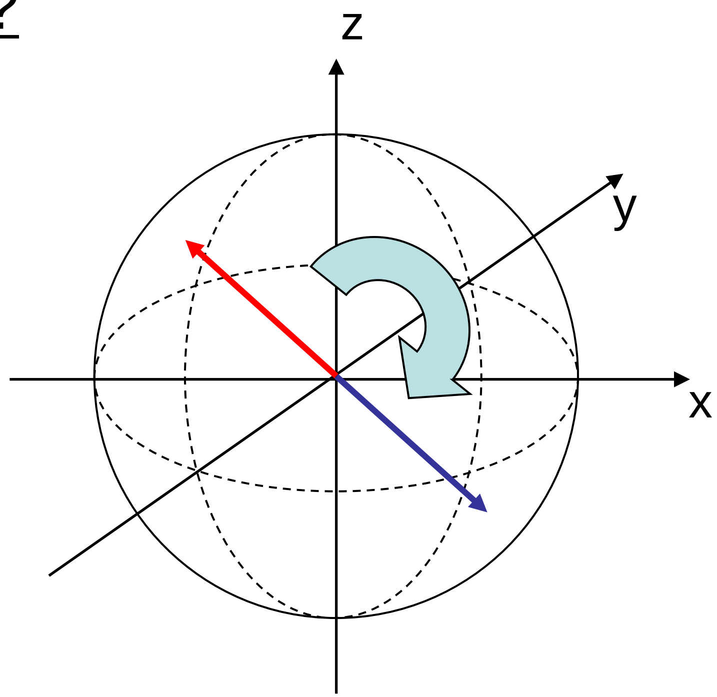
Universal NOT ? Spin reversal ?

Bloch vector

$$\boldsymbol{P} \to -\boldsymbol{P}$$

linear map $\hat{\rho} \to \mathcal{C}(\hat{\rho})$

$$\mathcal{C}(\hat{1}) = \hat{1} \qquad \mathcal{C}(\hat{\sigma}_x) = -\hat{\sigma}_x$$
$$\mathcal{C}(\hat{\sigma}_y) = -\hat{\sigma}_y \qquad \mathcal{C}(\hat{\sigma}_z) = -\hat{\sigma}_z$$

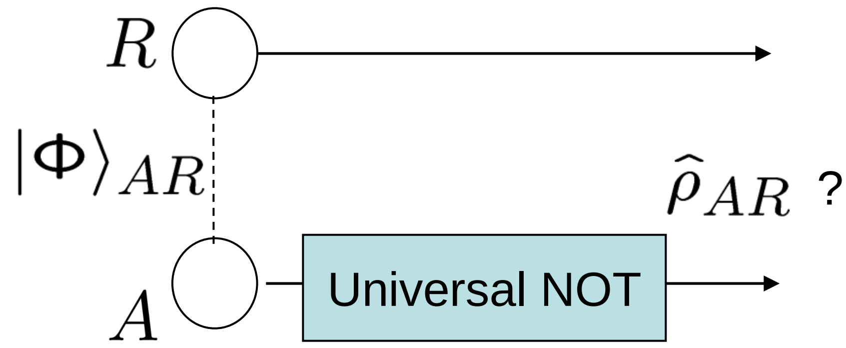


$$\mathcal{C}(|0\rangle\langle 0|) = |1\rangle\langle 1|$$
$$\mathcal{C}(|1\rangle\langle 1|) = |0\rangle\langle 0|$$
$$\mathcal{C}(|0\rangle\langle 1|) = -|0\rangle\langle 1|$$
$$\mathcal{C}(|1\rangle\langle 0|) = -|1\rangle\langle 0|$$

$$\hat{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$
$$\hat{\sigma}_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$
$$\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$
$$\hat{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

This map is positive, but...

Universal NOT ? Spin reversal ?



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$$\begin{aligned} 2|\Phi\rangle\langle\Phi| &= (|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \\ &= |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \end{aligned}$$

$$\begin{aligned} 2\widehat{\rho}_{AR} \equiv 2(\mathcal{C}\otimes\mathcal{I})|\Phi\rangle\langle\Phi| &= \\ &= |10\rangle\langle 10| - |00\rangle\langle 11| - |11\rangle\langle 00| + |01\rangle\langle 01| \end{aligned}$$

$$2\widehat{\rho}_{AR}(|00\rangle + |11\rangle) = -|11\rangle - |00\rangle = -(|00\rangle + |11\rangle)$$

$\widehat{\rho}_{AR}$ has a negative eigenvalue! (The map is not completely positive.)

$\longrightarrow$ Universal NOT is impossible.

Distinguishability

Examples

$$\frac{1}{2}||\hat{\rho}-\hat{\sigma}||$$

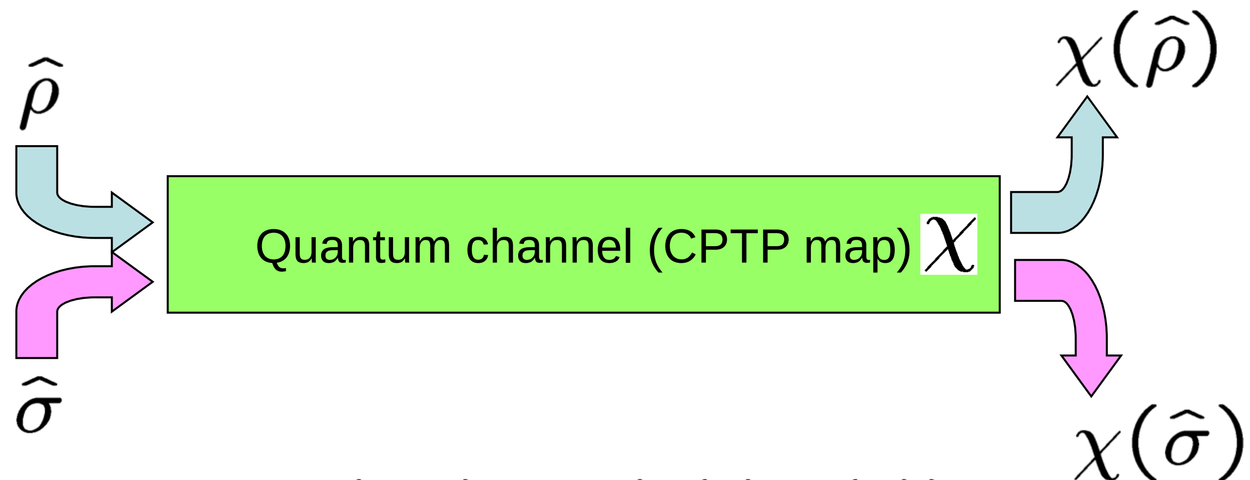
$$1-F(\hat{\rho},\hat{\sigma})$$

Measure of distinguishability between two states $D(\hat{\rho},\hat{\sigma})$

A quantity describing how we can distinguish between the two states in principle.

The distinguishability should never be improved by a quantum operation.

Monotonicity under quantum operations

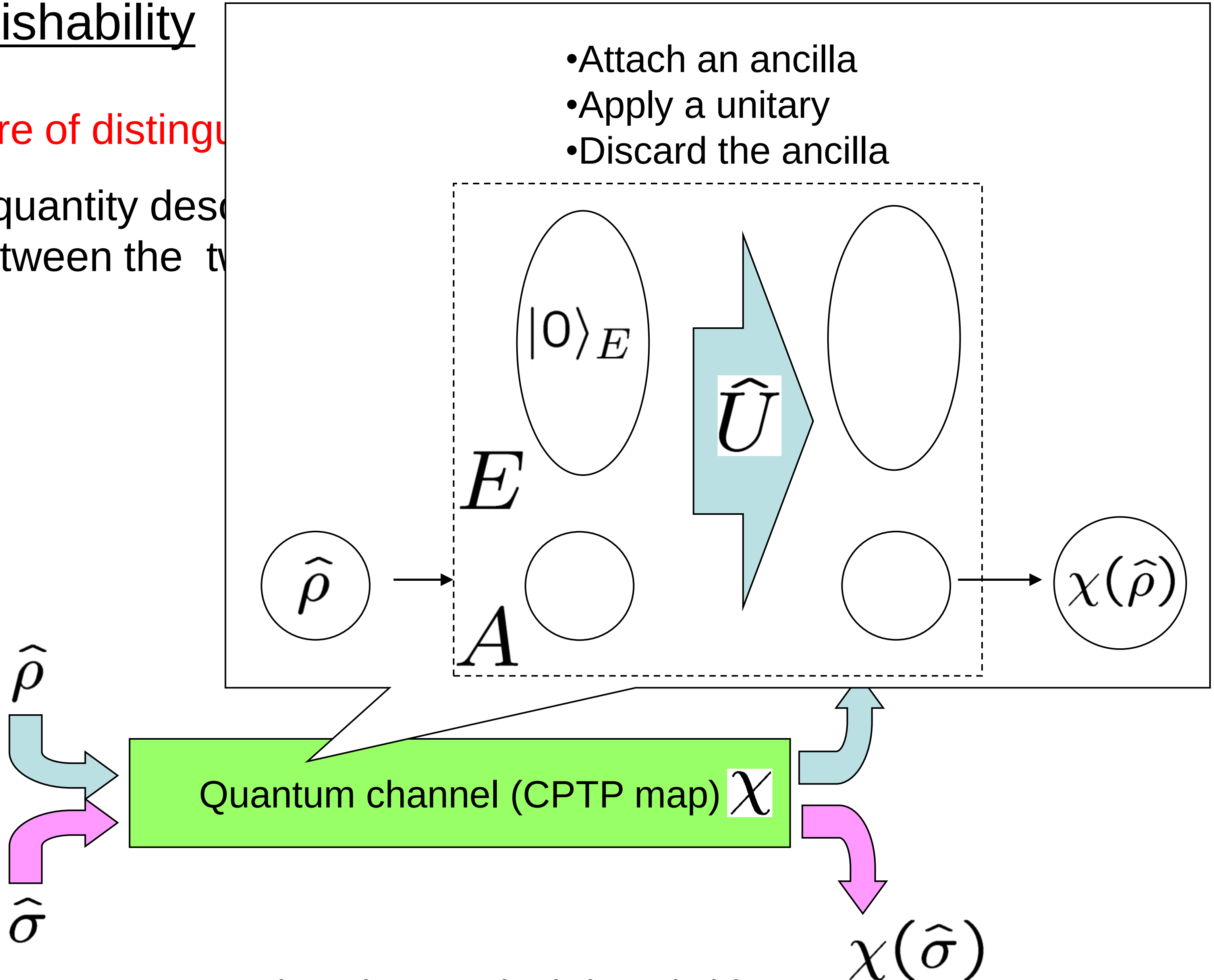


$$D(\hat{\rho},\hat{\sigma}) \geq D(\chi(\hat{\rho}),\chi(\hat{\sigma}))$$

Distinguishability

Measure of distingu

A quantity desc
between the tw



$$D(\hat{\rho},\hat{\sigma}) \geq D(\chi(\hat{\rho}),\chi(\hat{\sigma}))$$

Trace distance

$\|\cdot\|$: trace norm

$$\frac{1}{2}\|\hat\rho-\hat\sigma\|$$

Zero when $\hat\rho=\hat\sigma$ (the same state)

Unity when $\hat\rho\hat\sigma=0$ (perfectly distinguishable)

Monotonicity? $\boxed{\|\hat\rho-\hat\sigma\|\geq\|\chi(\hat\rho)-\chi(\hat\sigma)\|}$

- Attach an ancilla $\hat\rho\to\hat\rho\otimes\hat\tau$ $\hat\sigma\to\hat\sigma\otimes\hat\tau$

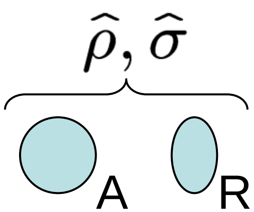
$$\mathrm{Tr}|\hat A\otimes\hat B|=\mathrm{Tr}(\sqrt{\hat A^\dagger\hat A}\otimes\sqrt{\hat B^\dagger\hat B})=\mathrm{Tr}|\hat A|\mathrm{Tr}|\hat B|$$

$$\|\hat\rho\otimes\hat\tau-\hat\sigma\otimes\hat\tau\|=\|(\hat\rho-\hat\sigma)\otimes\hat\tau\|=\|\hat\rho-\hat\sigma\|\times\|\hat\tau\|=\|\hat\rho-\hat\sigma\|$$

- Apply a unitary $\hat\rho\to\hat U\hat\rho\hat U^\dagger$ $\hat\sigma\to\hat U\hat\sigma\hat U^\dagger$

$$\max_{\hat V}|\mathrm{Tr}(\hat A\hat V)|=\max_{\hat V}|\mathrm{Tr}(\hat U\hat A\hat U^\dagger\hat V)|$$

$$\|\hat U\hat\rho\hat U^\dagger-\hat U\hat\sigma\hat U^\dagger\|=\|\hat U(\hat\rho-\hat\sigma)\hat U^\dagger\|=\|\hat\rho-\hat\sigma\|$$



- Discard the ancilla $\hat\rho\to\mathrm{Tr}_R(\hat\rho)$ $\hat\sigma\to\mathrm{Tr}_R(\hat\sigma)$

$$\max_{\hat V_A}|\mathrm{Tr}[(\mathrm{Tr}_R\hat\rho-\mathrm{Tr}_R\hat\sigma)\hat V_A]|=\max_{\hat V_A}|\mathrm{Tr}[(\hat\rho-\hat\sigma)(\hat V_A\otimes\hat 1_R)]|$$
$$\leq\max_{\hat U_{AR}}|\mathrm{Tr}[(\hat\rho-\hat\sigma)\hat U_{AR}]|$$

Fidelity

$$F(\hat{\rho},\hat{\sigma}) \equiv \max |\langle \phi_\rho | \phi_\sigma \rangle|^2 = \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\hat{\sigma}}\hat{\rho}\sqrt{\hat{\sigma}}}\right)^2$$

$$F(\hat{\rho},\hat{\sigma}) = 1 \text{ when } \hat{\rho} = \hat{\sigma} \qquad F(\hat{\rho},\hat{\sigma}) = 0 \text{ when } \hat{\rho}\hat{\sigma} = 0$$

$$F(\hat{\rho}, |\psi\rangle\langle\psi|) = \langle\psi|\hat{\rho}|\psi\rangle$$

$1 - F(\hat{\rho},\hat{\sigma})$ is a measure of distinguishability. (not a distance)

Monotonicity $\boxed{F(\hat{\rho},\hat{\sigma}) \le F(\chi(\hat{\rho}),\chi(\hat{\sigma}))}$

- Attach an ancilla

$$F(\hat{\rho}\otimes\hat{\tau}, \hat{\sigma}\otimes\hat{\tau}) = F(\hat{\rho},\hat{\sigma})F(\hat{\tau},\hat{\tau}) = F(\hat{\rho},\hat{\sigma})$$

- Apply a unitary

$$F(\hat{U}\hat{\rho}\hat{U}^\dagger, \hat{U}\hat{\sigma}\hat{U}^\dagger) = \|\hat{U}\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\hat{U}^\dagger\|^2 = \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|^2 = F(\hat{\rho},\hat{\sigma})$$

- Discard the ancilla

$$F(\hat{\rho},\hat{\sigma}) = \max |\langle \phi_\rho | \phi_\sigma \rangle|^2$$

$$F(\mathrm{Tr}_R\hat{\rho}, \mathrm{Tr}_R\hat{\sigma}) = \max |\langle \phi'_\rho | \phi'_\sigma \rangle|^2$$

$$\max |\langle \phi_\rho | \phi_\sigma \rangle|^2 \le \max |\langle \phi'_\rho | \phi'_\sigma \rangle|^2$$