1. Basic rules of quantum mechanics

How to describe the states of an ideally controlled system?

How to describe changes in an ideally controlled system?

How to describe measurements on an ideally controlled system?

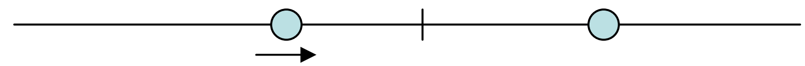
How to treat composite systems?

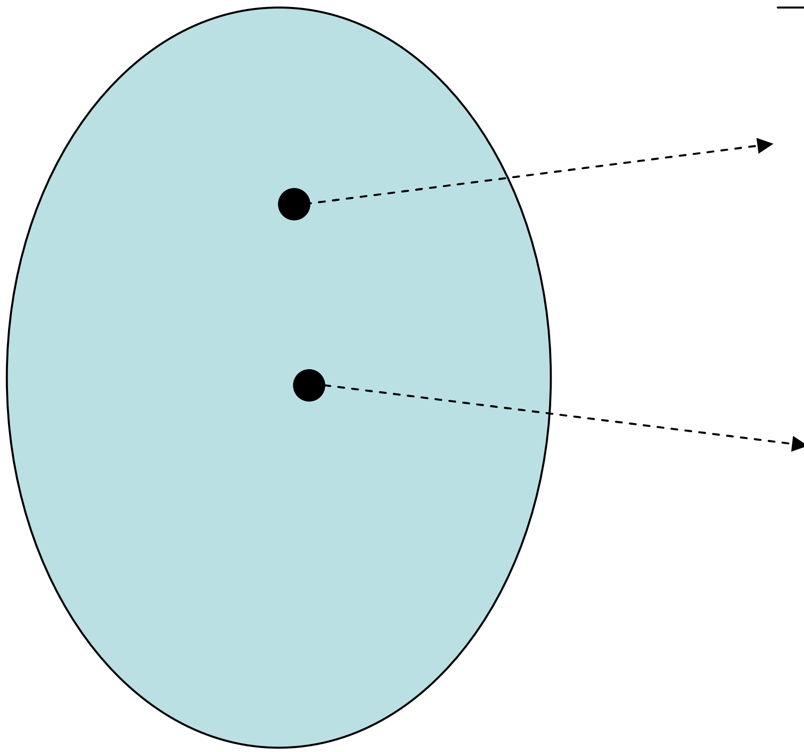
How to describe the states of an ideally controlled system?

(Basic rule I)

Example of a classical system

A particle on a 1D line





It is at 3.4 cm to the right of the origin and stands still.

It is at 2.3 cm to the left of the origin and moves to the right 0.3 cm/sec.

Set of all the states

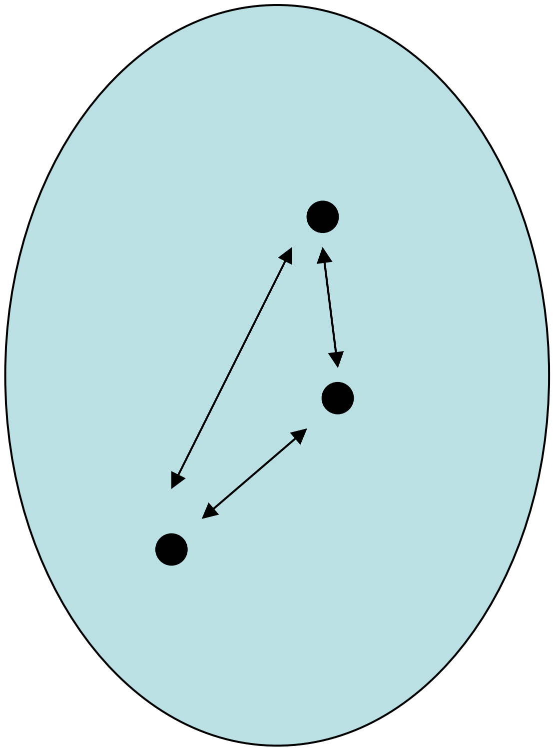
Is there any common structure in the set?

Relation between a pair of states?

Closeness?

How to describe the states of an ideally controlled system?

(Basic rule I)



Set of all the states

Quantum system

State A and State B may not be perfectly distinguishable.

Distinguishablity: Can be operationally defined.

Applicable to *any* system

Common structure

A quantity representing the distiguishablity is assigned to every pair of states.

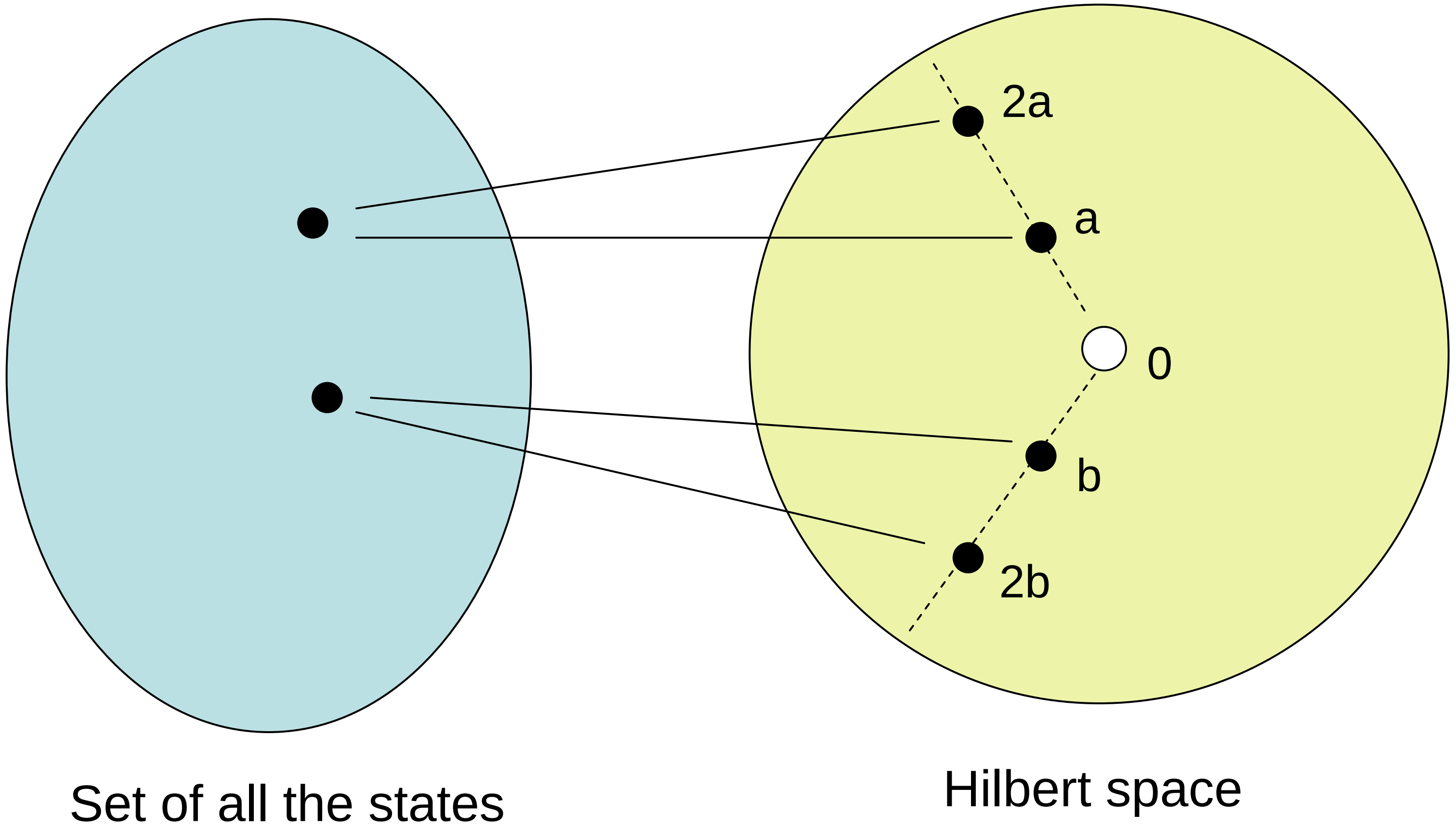
Hilbert space

- Linear space over $\mathbb{C}$
- Inner product (a, b)
- Complete in the norm $\|a\| \equiv \sqrt{(a, a)}$

$\mathbb{C}$: set of complex numbers

How to describe the states of an ideally controlled system?

(Basic rule I)



A state $\leftrightarrow$ a **ray** in the Hilbert space

ray including vector $a \neq 0$ is

$\{\alpha a | \alpha \in \mathbb{C}, \alpha \neq 0\}$.

How to describe the states of an ideally controlled system?

(Basic rule I)

A physical system $\leftrightarrow$ a **Hilbert space** $\mathcal{H}$

A state $\leftrightarrow$ a **ray** in the Hilbert space

Usually, we use a normalized vector ϕ satisfying $(\phi, \phi) = 1$ as a representative of the ray.

Distinguishability — inner product

For normalized vectors ϕ and ψ,
$|(\phi, \psi)| = 0$ — perfectly distinguishable
$|(\phi, \psi)| = 1$ — completely indistinguishable (the same state)

Dirac notation

'ket' $|\phi\rangle$ — vector $\phi \in \mathcal{H}$.
'bra' $\langle\phi|$ — linear functional $(\phi, \cdot) : \mathcal{H} \to \mathbb{C}$.

$$\langle\phi|\psi\rangle \text{ — } (\phi, \psi)$$

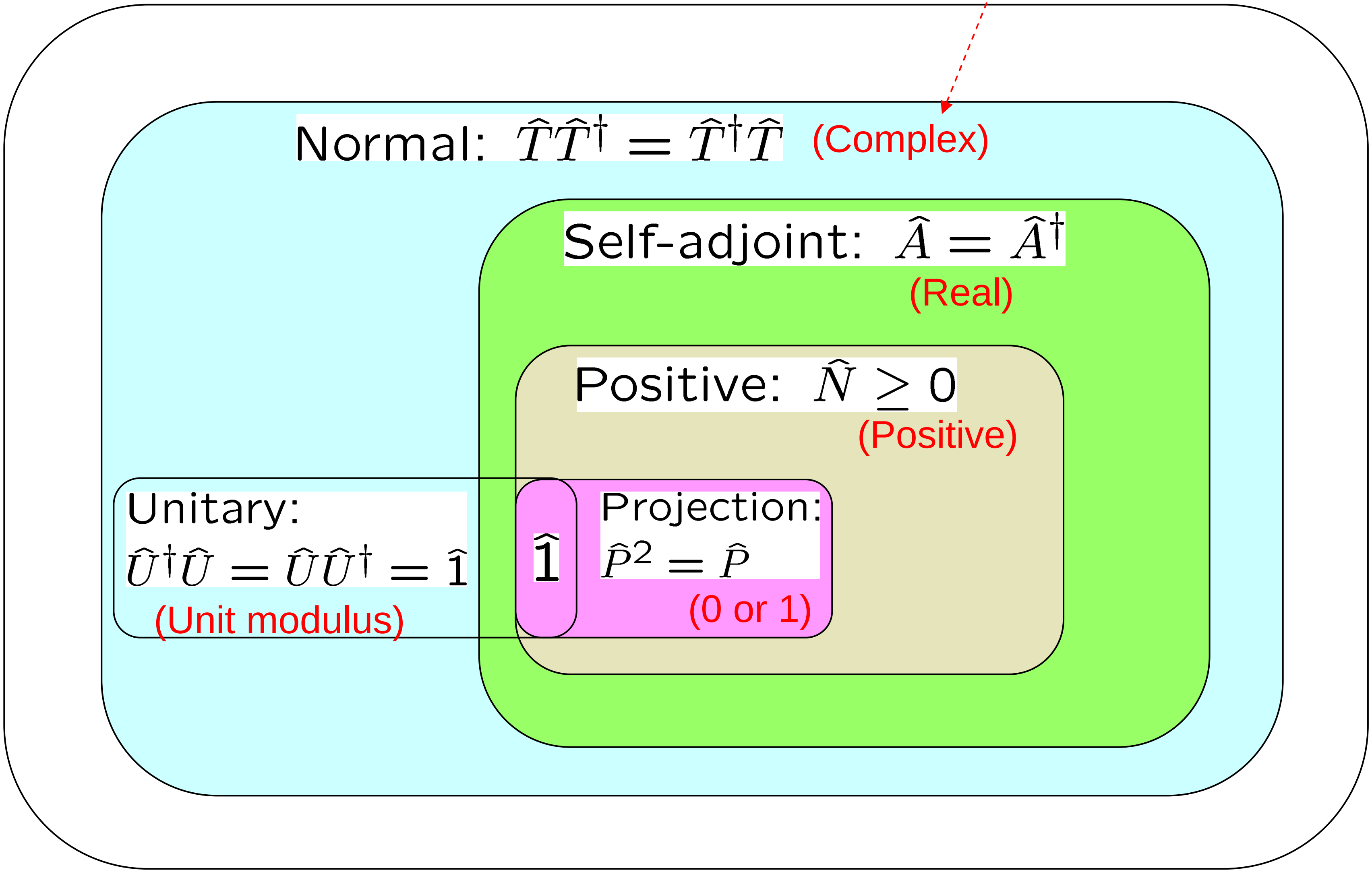
Linear operators: $\mathcal{H} \to \mathcal{H}$.

$\hat{T}$ is normal $\leftrightarrow$ $\hat{T}$ is diagonalizable.

$$\hat{T} = \sum_j \lambda_j |u_j\rangle\langle u_j|$$

Eigenvalues

An orthonormal basis



How to describe changes in an ideally controlled system?

(Basic rule II)

Reversible evolution

A unitary operator $\widehat{U}$:

$$|\phi_{\text{out}}\rangle = \widehat{U}|\phi_{\text{in}}\rangle$$

Inner products are preserved by unitary operations.

Distinguishability should never be improved by any operation.

↓

Distinguishability should be unchanged by any reversible operation.

↓

Inner products will be preserved in any reversible operation.

Infinitesimal change

$$|\phi(t_2)\rangle = \widehat{U}(t_2, t_1)|\phi(t_1)\rangle$$

$$|\phi(t+dt)\rangle = \widehat{U}(t+dt, t)|\phi(t)\rangle$$

$$\widehat{U}(t+dt, t) \cong \hat{1} - (i/\hbar)\widehat{H}(t)dt$$

Self-adjoint operator $\widehat{H}(t)$: Hamiltonian of the system

Schrödinger equation:

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = \widehat{H}(t)|\phi(t)\rangle$$

How to describe measurements on an ideally controlled system?

(Basic rule III)

An ideal measurement with outcome $j = 1, \ldots, d$

For every j,

(1) There exists an input state $|a_j\rangle$ that produces outcome j with probability 1.

The states $\{|a_k\rangle\}(k \neq j)$ produce

(2) ~~Any other state produces~~ outcome j with probability 0.

(3) The number of outcomes d is maximal.

$\downarrow$

$\{|a_j\rangle\}_{j=1,\cdots,d}$ is an orthonormal basis of $\mathcal{H}$.

$d = \dim \mathcal{H}$.

Note: This is not the unique way of defining the 'best' measurement. We'll see later.

How to describe measurements on an ideally controlled system?

(Basic rule III)

Orthogonal measurement on an orthonormal basis $\{|a_j\rangle\}_{j=1,\cdots,d}$
(von Neumann measurement, projection measurement)

Input state $|\phi\rangle = \sum_j |a_j\rangle\langle a_j|\phi\rangle$

Probability of outcome j $\quad P(j) = |\langle a_j|\phi\rangle|^2$

Closure relation

$\sum_j |a_j\rangle\langle a_j| = \hat{1}$

Measurement of an observable

Self-adjoint operator $\hat{A}$

$$\hat{A} = \sum_j \lambda_j |a_j\rangle\langle a_j|$$

Measurement on $\{|a_j\rangle\}_{j=1,\cdots,d}$ $\quad$ Assign $j \to \lambda_j$

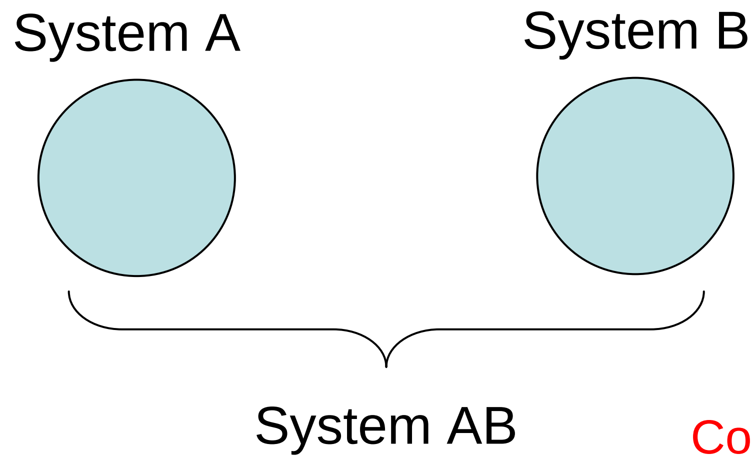
$$\langle \hat{A} \rangle \equiv \sum_j P(j)\lambda_j = \sum_j \langle\phi|a_j\rangle\langle a_j|\phi\rangle\lambda_j = \langle\phi|\hat{A}|\phi\rangle$$

How to treat composite systems?

(Basic rule IV)

We know how to describe each of the systems A and B.

How to describe AB as a single system?



Subsystems

Composite system

System A: Hilbert space $\mathcal{H}_A$ Basis $\{|a_i\rangle\}_{i=1,\cdots,d_A}$

System B: Hilbert space $\mathcal{H}_B$ Basis $\{|b_j\rangle\}_{j=1,\cdots,d_B}$

Composite system AB:

Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ $\{|a_i\rangle \otimes |b_j\rangle\}_{i=1,\cdots,d_A;j=1,\cdots,d_B}$

Tensor product

$$\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = \dim \mathcal{H}_A \dim \mathcal{H}_B$$

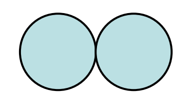
How to treat composite systems?

(Basic rule IV)

When system A and system B are independently accessed …

	State preparation	Unitary evolution	Orthogonal measurement
System A	$\vert\phi\rangle_A$	$\hat{U}_A$	$\{\vert a_i\rangle_A\}_{i=1,\cdots,d_A}$
System B	$\vert\psi\rangle_B$	$\hat{V}_B$	$\{\vert b_j\rangle_B\}_{j=1,\cdots,d_B}$
System AB	$\vert\phi\rangle_A \otimes \vert\psi\rangle_B$	$\hat{U}_A \otimes \hat{V}_B$	$\{\vert a_i\rangle_A \otimes \vert b_j\rangle_B\}_{i=1,\cdots,d_A}^{j=1,\cdots,d_B}$
	Separable states	Local unitary operations	Local measurements

When system A and system B are directly interacted …



State preparation	Unitary evolution	Orthogonal measurement
$\vert\Psi\rangle_{AB} \in \mathcal{H}_{AB}$ $\sum_k \alpha_k \vert\phi_k\rangle_A \otimes \vert\psi_k\rangle_B$	$\hat{U}_{AB} : \mathcal{H}_{AB} \to \mathcal{H}_{AB}$	$\{\vert\Psi_k\rangle_{AB}\}_{k=1,2,\ldots,d_A d_B}$
Entangled states	Global unitary operations	Global measurements

2. State of a subsystem

Rule for a local measurement

State after discarding a subsystem (marginal state)

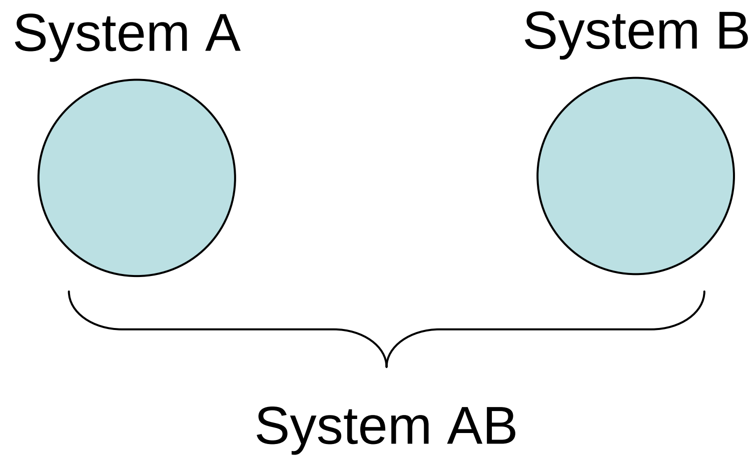
- Alternative description: density operator
- Properties of density operators
- Rules in terms of density operators

Which is the better description?

- Schmidt decomposition
- Pure states with the same marginal state
- Ensembles with the same density operator

Entanglement

Suppose that the whole system (AB) is ideally controlled (prepared in a definite state).



state: $|\Phi\rangle_{AB}$

Intuition in a ‘classical’ world:

If the whole is under a good control, so are the parts.

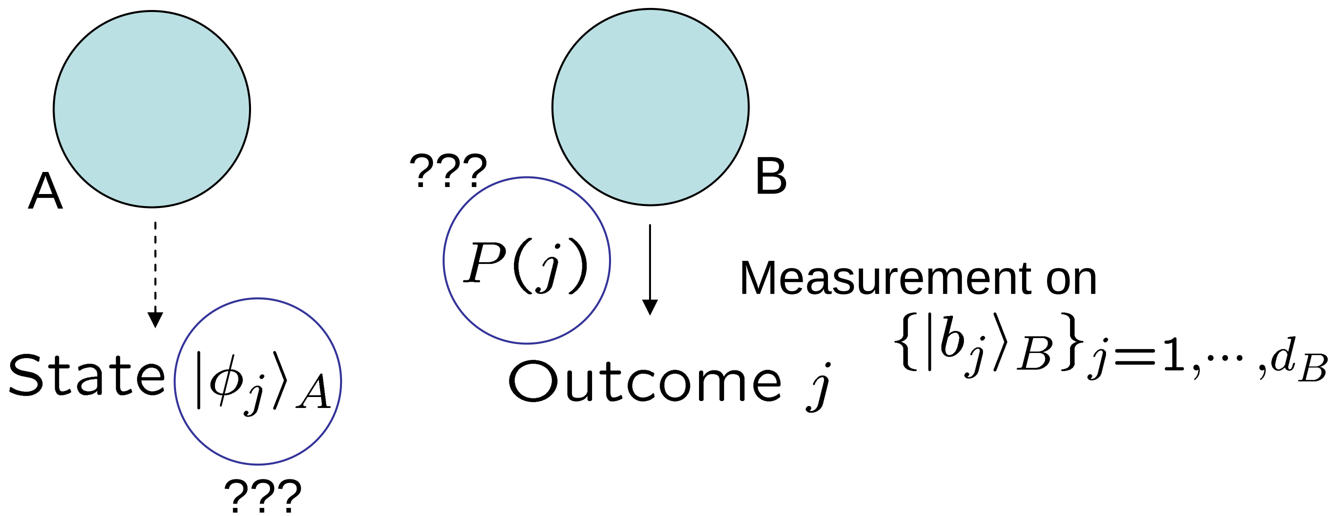
But ….

It is not always possible to assign a state vector to subsystem A.

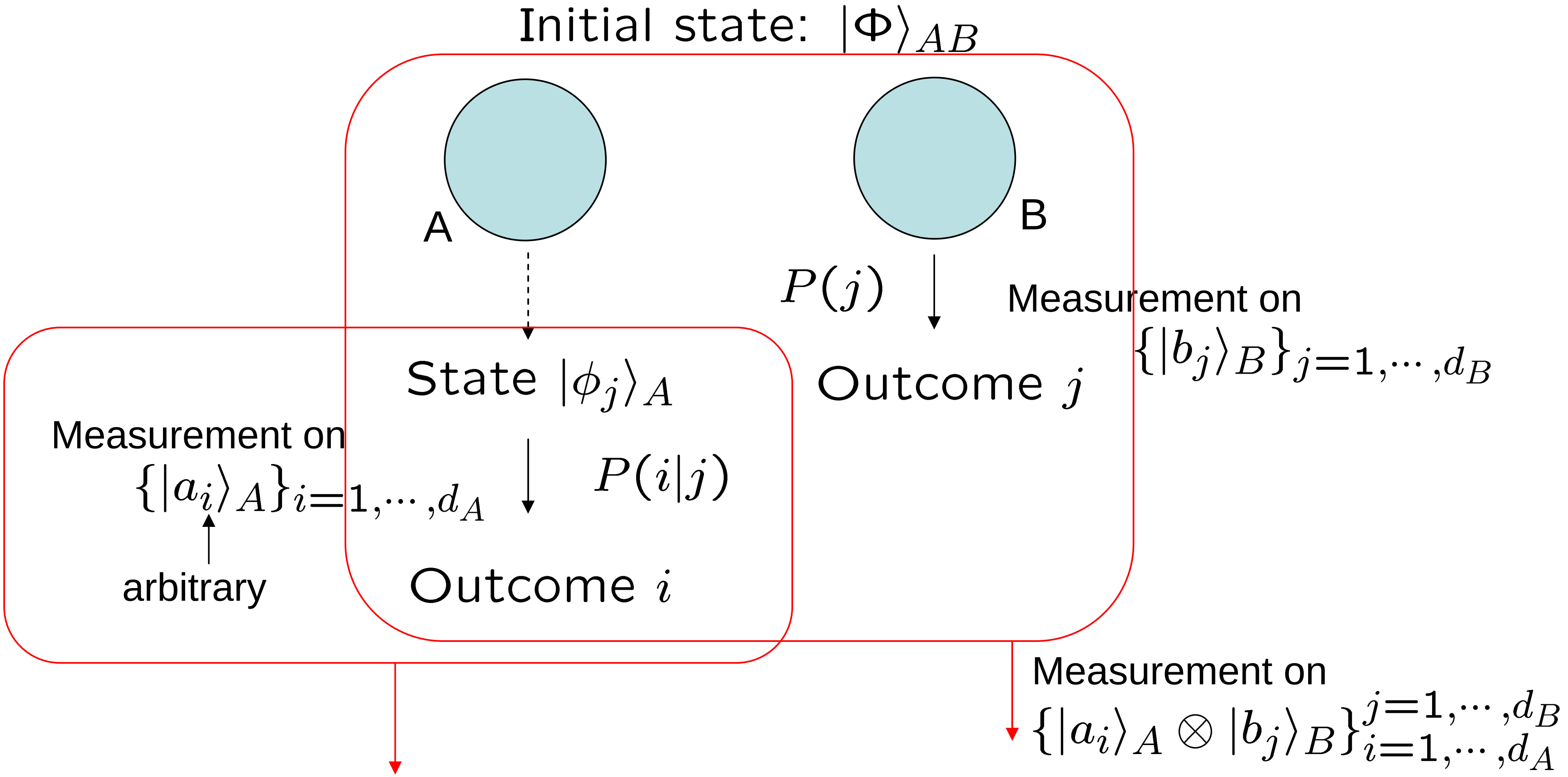
What is the state of subsystem A?

Rule for a local measurement

Initial state: $|\Phi\rangle_{AB}$



Rule for a local measurement



$$P(i|j) = |{}_A\langle a_i|\phi_j\rangle_A|^2$$

$$P(i,j) = \left|{}_A\langle a_i|{}_B\langle b_j||\Phi\rangle_{AB}\right|^2$$

$$P(i,j) = P(i|j)P(j) = |{}_A\langle a_i|\sqrt{P(j)}|\phi_j\rangle_A|^2$$

A remark on notations

$$
{}_A\langle a_i| \otimes {}_B\langle b_j||\Phi\rangle_{AB}
$$

$$
= {}_A\langle a_i|(\hat{1}_A \otimes {}_B\langle b_j|)|\Phi\rangle_{AB}
$$

abbreviation

$$
= {}_A\langle a_i|\ {}_B\langle b_j|\ |\Phi\rangle_{AB}
$$

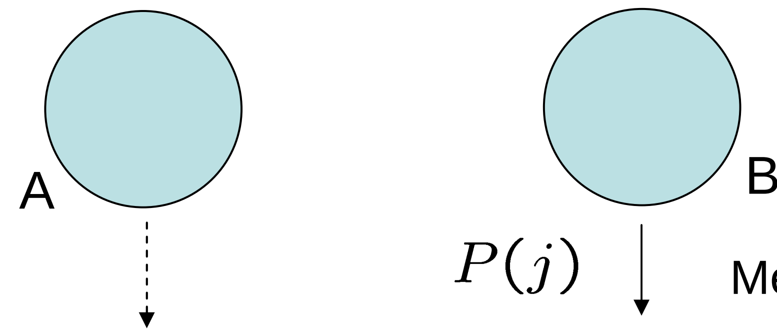
$$
\begin{matrix} {}_A\langle a_i| \\ {}_B\langle b_j| \end{matrix} \Big|\Phi\Big\rangle_{AB}
$$

$$
\begin{matrix} {}_A\langle a_i| & \\ & {}_B\langle b_j| \end{matrix} \Big|\Phi\Big\rangle_{AB}
$$

$$
\begin{aligned}
{}_B\langle b_j| &: \mathcal{H}_B \to \mathbb{C} \\
\hat{1}_A &: \mathcal{H}_A \to \mathcal{H}_A \\
\hat{1}_A \otimes {}_B\langle b_j| &: \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A
\end{aligned}
$$

Rule for a local measurement

Initial state: $|\Phi\rangle_{AB}$



Measurement on $\{|b_j\rangle_B\}_{j=1,\cdots,d_B}$

State $|\phi_j\rangle_A$ Outcome j

For arbitrary $\{|a_i\rangle_A\}_{i=1,\cdots,d_A}$

$$P(i,j) = \left|{}_A\langle a_i|{}_B\langle b_j||\Phi\rangle_{AB}\right|^2$$

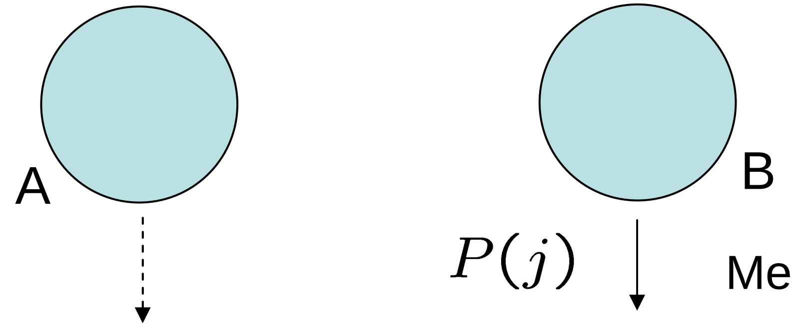
$$P(i,j) = P(i|j)P(j) = |{}_A\langle a_i|\sqrt{P(j)}|\phi_j\rangle_A|^2$$

$$\downarrow$$

$$\sqrt{P(j)}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

Rule for a local measurement

Initial state: $|\Phi\rangle_{AB}$



Measurement on $\{|b_j\rangle_B\}_{j=1,\cdots,d_B}$

State $|\phi_j\rangle_A$

Outcome j

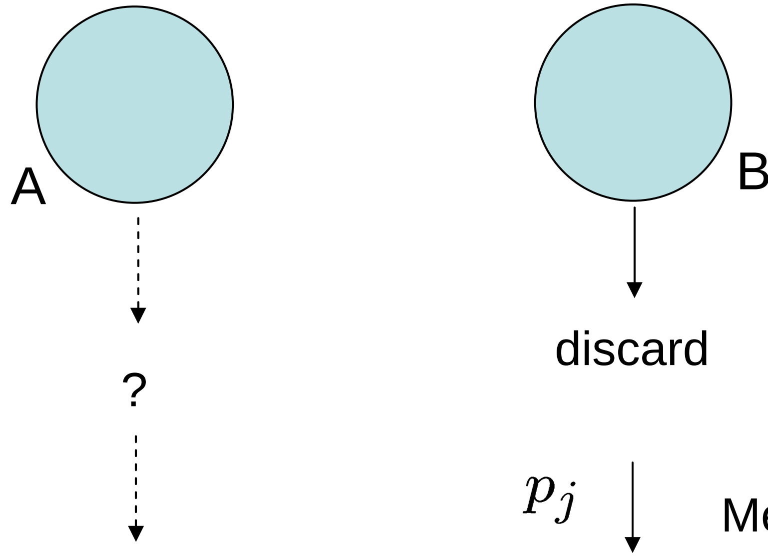
$$\boxed{\sqrt{P(j)}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}}$$

$$P(j) = \|{}_B\langle b_j||\Phi\rangle_{AB}\|^2$$

$$|\phi_j\rangle_A = \frac{{}_B\langle b_j||\Phi\rangle_{AB}}{\|{}_B\langle b_j||\Phi\rangle_{AB}\|}$$

State after discarding a subsystem (marginal state)

Initial state: $|\Phi\rangle_{AB}$



Measurement on $\{|b_j\rangle_B\}_{j=1,\cdots,d_B}$

State $|\phi_j\rangle_A$ Outcome j

State of system A: $|\phi_j\rangle_A$ with probability p_j $\longrightarrow$ $\{p_j, |\phi_j\rangle_A\}$

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

This description is correct, but dependence on the fictitious measurement is weird…

Alternative description: density operator

$\{p_j, |\phi_j\rangle_A\}$ $\quad |\phi_j\rangle_A$ with probability p_j

$$\hat{\rho}_A \equiv \sum_j p_j |\phi_j\rangle_{AA}\langle\phi_j|$$

Cons

$\{q_k, |\psi_k\rangle_A\}$

$\{p_j, |\phi_j\rangle_A\}$

Same $\hat{\rho}_A$

Two different physical states could have the same density operator.
(The description could be insufficient.)

Pros

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

$$\hat{\rho}_A = \sum_j p_j |\phi_j\rangle_{AA}\langle\phi_j| = \sum_j \sqrt{p_j}|\phi_j\rangle_{AA}\langle\phi_j|\sqrt{p_j}$$

$$= \sum_j {}_B\langle b_j||\Phi\rangle\langle\Phi||b_j\rangle_B = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

Independent of the choice of the fictitious measurement

Properties of density operators

$$\hat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

For any $|\psi\rangle$, $\langle\psi|\hat{\rho}|\psi\rangle = \sum_j p_j |\langle\psi|\phi_j\rangle|^2 \geq 0$ Positive

$$\mathrm{Tr}(\hat{\rho}) = \sum_j p_j \mathrm{Tr}(|\phi_j\rangle\langle\phi_j|)$$
$$= \sum_j p_j \langle\phi_j|\phi_j\rangle = \sum_j p_j = 1$$ Unit trace

Positive & Unit trace $\longrightarrow$ $\hat{\rho} = \sum_j p_j |\phi_j\rangle\langle\phi_j|$

probability

This decomposition is by no means unique!

Mixed state $\hat{\rho} = \sum_j p_j |\phi_j\rangle\langle\phi_j|$

Pure state $\hat{\rho} = |\phi\rangle\langle\phi|$ (One eigenvalue is 1)

Rules in terms of density operators

Prepare $|\phi_j\rangle$ with probability p_j

$$\hat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

Prepare $\hat{\rho}_j$ with probability p_j

$$\hat{\rho} = \sum_j p_j \hat{\rho}_j$$

Unitary evolution

$$|\phi_{\text{out}}\rangle = \hat{U}|\phi_{\text{in}}\rangle$$

Hint: $|\phi_{\text{out}}\rangle\langle\phi_{\text{out}}| = \hat{U}|\phi_{\text{in}}\rangle\langle\phi_{\text{in}}|\hat{U}^\dagger$

$$\hat{\rho}_{\text{out}} = \hat{U}\hat{\rho}_{\text{in}}\hat{U}^\dagger$$

Orthogonal measurement on basis $\{|a_j\rangle\}$

$$P(j) = |\langle a_j|\phi\rangle|^2$$

Hint: $P(j) = \langle a_j|\phi\rangle\langle\phi|a_j\rangle$

$$P(j) = \langle a_j|\hat{\rho}|a_j\rangle$$

Expectation value of an observable $\hat{A}$

$$\langle\hat{A}\rangle = \langle\phi|\hat{A}|\phi\rangle$$

Hint: $\langle\hat{A}\rangle = \text{Tr}(\hat{A}|\phi\rangle\langle\phi|)$

$$\langle\hat{A}\rangle = \text{Tr}(\hat{A}\hat{\rho})$$

Rules in terms of density operators

Independently prepared systems A and B

$$|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\psi\rangle_B \qquad \widehat{\rho}_{AB} = \widehat{\rho}_A \otimes \widehat{\rho}_B$$

Local measurement on system B on basis $\{|b_j\rangle_B\}$

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB} \qquad p_j\widehat{\rho}_A^{(j)} = {}_B\langle b_j|\widehat{\rho}_{AB}|b_j\rangle_B$$

Discarding system B

$$\widehat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|) \qquad \widehat{\rho}_A = \mathrm{Tr}_B[\widehat{\rho}_{AB}]$$

All the rules so far can be written in terms of density operators.

Which is the better description?

$$\{p_j, |\phi_j\rangle\}$$

This looks natural. The system is in one of the pure states, but we just don't know. Quantum mechanics may treat just the pure states, and leave mixed states to statistical mechanics or probability theory.

$$\widehat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

Best description

All the rules so far can be written in terms of density operators.

Which description has one-to-one correspondence to physical states?

Theorem: Two states $\{p_j, |\phi_j\rangle\}$ and $\{q_k, |\psi_k\rangle\}$ with the same density operator are physically indistinguishable (hence are the same state).

Schmidt decomposition

Bipartite pure states have a very nice standard form.

Any orthonormal bases $\{|a_i\rangle_A\}$ $\{|b_j\rangle_B\}$

$$|\Phi\rangle_{AB} = \sum_{ij} \alpha_{ij} |a_i\rangle_A |b_j\rangle_B$$

We can always choose the two bases such that

$$|\Phi\rangle_{AB} = \sum_{i} \sqrt{p_i} |a_i\rangle_A |b_i\rangle_B$$

Schmidt decomposition

$\{|a_i\rangle_A\}$: Diagonalizes $\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$

Proof: $|\Phi\rangle_{AB} = \sum_i |a_i\rangle_A |\tilde{b}_i\rangle_B$ $\quad$ $|\tilde{b}_i\rangle_B \equiv {}_A\langle a_i||\Phi\rangle_{AB}$

unnormalized

$$\begin{aligned} {}_B\langle \tilde{b}_j|\tilde{b}_i\rangle_B &= \mathrm{Tr}[{}_A\langle a_i||\Phi\rangle_{AB\,AB}\langle\Phi||a_j\rangle_A] \\ &= {}_A\langle a_i|\mathrm{Tr}_B[|\Phi\rangle_{AB\,AB}\langle\Phi|]|a_j\rangle_A \\ &= {}_A\langle a_i|\hat{\rho}_A|a_j\rangle_A = p_j\delta_{ij}. \end{aligned}$$

$$\sqrt{p_j}|b_j\rangle \equiv |\tilde{b}_j\rangle_B$$

Entangled states and separable states

$|\phi\rangle_A \otimes |\psi\rangle_B$ — Separable states

$\sum_k \alpha_k |\phi_k\rangle_A \otimes |\psi_k\rangle_B$ — Entangled states

Are there any procedure to distinguish between the two classes?

→ Schmidt decomposition $$|\Phi\rangle_{AB} = \sum_{i=1}^{s} \sqrt{p_i}|a_i\rangle_A |b_i\rangle_B$$

$$p_1 \geq p_2 \geq \cdots \geq p_s > 0$$

$\{p_j\}$: The eigenvalues of the marginal density operators (the same for A and B)

Schmidt number

Number of nonzero coefficients in Schmidt decomposition

= The rank of the marginal density operators

'Symmetry' between A and B

$\hat{\rho}_A, \hat{\rho}_B$ The same set of eigenvalues

$$\mathrm{Rank}(\hat{\rho}_A) = \mathrm{Rank}(\hat{\rho}_B) = s$$

Separable states — Schmidt number = 1

$$p_1 = 1$$

Entangled states — Schmidt number > 1

$$p_1 \geq p_2 > 0$$

Range and Kernel of $\hat{\rho}$

$$\mathrm{Ran}\ \hat{\rho} \equiv \{\hat{\rho}|x\rangle \mid |x\rangle \in \mathcal{H}\}$$

Subspace in which $\hat{\rho} > 0$

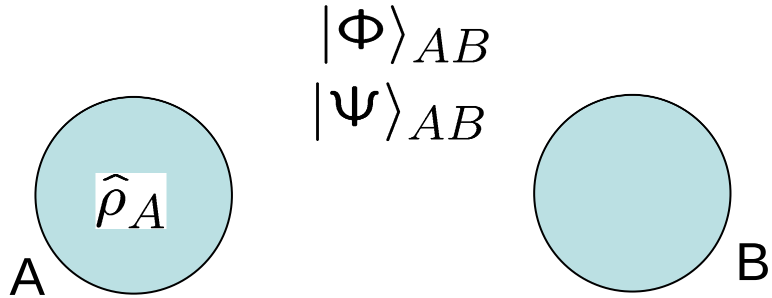
$$\mathrm{Ker}\ \hat{\rho} \equiv \{|y\rangle \mid \hat{\rho}|y\rangle = 0\}$$

Subspace in which $\hat{\rho} = 0$

$$\mathcal{H} = (\mathrm{Ran}\ \hat{\rho}) \oplus (\mathrm{Ker}\ \hat{\rho})$$

$$\mathrm{Rank}(\hat{\rho}) \equiv \dim \mathrm{Ran}\ \hat{\rho}$$

Pure states with the same marginal state

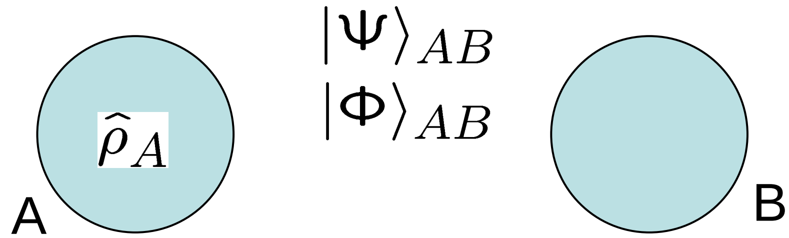


$$\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$
$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$$

$|\Phi\rangle_{AB} \longrightarrow \hat{\rho}_A$ Marginal state (unique)

$\hat{\rho}_A \longrightarrow |\Phi\rangle_{AB}$, $\hat{\rho}_A \longrightarrow |\Psi\rangle_{AB}$ Purification / Pure Extension (not unique)

Pure states with the same marginal state



$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

Schmidt decomposition

Orthonormal basis $\{|a_i\rangle_A\}$ that diagonalizes $\hat{\rho}_A$

$$|\Psi\rangle_{AB} = \sum_i \sqrt{p_i}|a_i\rangle_A|\mu_i\rangle_B$$

$$|\Phi\rangle_{AB} = \sum_i \sqrt{p_i}|a_i\rangle_A|\nu_i\rangle_B$$

$\{|\mu_i\rangle_B\}$ Orthonormal basis

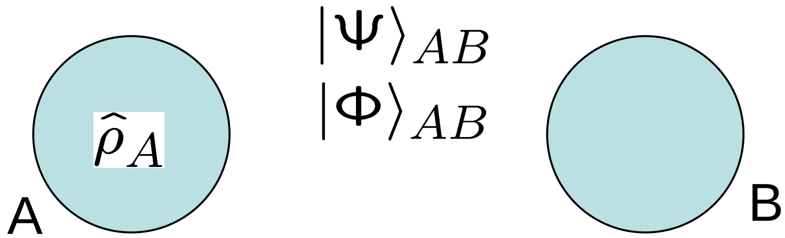
$\{|\nu_i\rangle_B\}$ Orthonormal basis

$$|\nu_i\rangle_B = \hat{U}_B|\mu_i\rangle_B$$

unitary

$$|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}$$

Pure states with the same marginal state

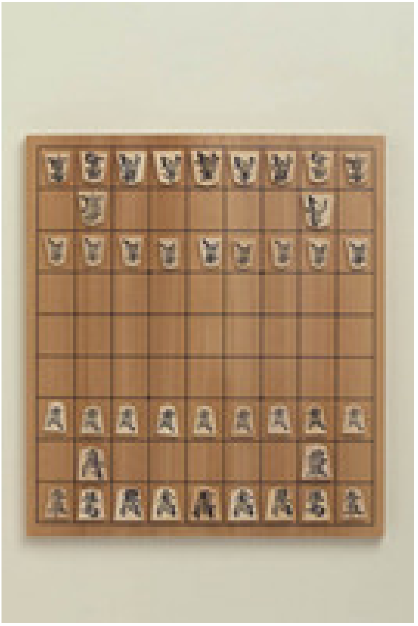


$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

$$\boxed{|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}}$$

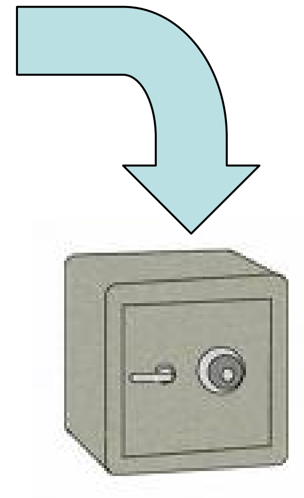
Theorem: If $|\Psi\rangle_{AB}$ and $|\Phi\rangle_{AB}$ are purifications of the same state $\hat{\rho}_A$, state $|\Psi\rangle_{AB}$ can be physically converted to state $|\Phi\rangle_{AB}$ without touching system A.

<u>Sealed move</u> (封じ手)



Chess, Go, Shogi …

Bb5
4六銀



Let us call it a day and shall we start over tomorrow, with Bob’s move.

While they are (suppose to be) sleeping...

- Alice should not learn the sealed move.
- Bob should not alter the sealed move.

Sealed move

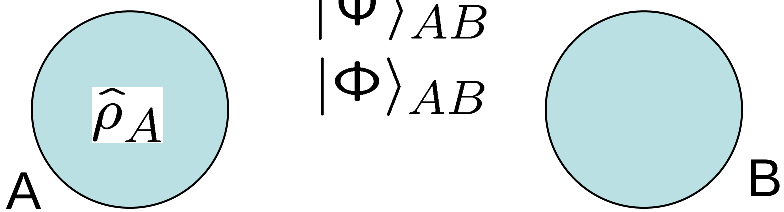
- Alice should not learn the sealed move.
- Bob should not alter the sealed move.

If there is no reliable safe available …

(If there is no system out of both Alice's and Bob's reach …)

$|\Psi\rangle_{AB}$

$|\Phi\rangle_{AB}$



Alice has no knowledge

↓

Bob can alter the states

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

$$\boxed{|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}}$$

Function of the "safe" cannot be realized.

Impossibility of unconditionally secure quantum bit commitment

(Lo, Mayers)

Ensembles with the same density operator

$\{p_j, |\phi_j\rangle_A\}$ $\qquad |\phi_j\rangle_A$ with probability p_j

$\{q_k, |\psi_k\rangle_A\}$ $\qquad |\psi_k\rangle_A$ with probability q_k

$$\hat{\rho}_A \equiv \sum_j p_j |\phi_j\rangle_{AA}\langle\phi_j| = \sum_k q_k |\psi_k\rangle_{AA}\langle\psi_k|$$

A scheme to realize the ensemble $\{p_j, |\phi_j\rangle_A\}$

Prepare system AB in state

$\{|b_j\rangle_B\}$ Orthonormal basis

$$|\Phi\rangle_{AB} \equiv \sum_j \sqrt{p_j} |\phi_j\rangle_A |b_j\rangle_B$$

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

Measure system B on basis $\{|b_j\rangle_B\}$

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

$|\phi_j\rangle_A$ with probability p_j

Ensembles with the same density operator

Prepare system AB in state

$$|\Psi\rangle_{AB} \equiv \sum_k \sqrt{q_k}|\psi_k\rangle_A|b_k\rangle_B$$

Apply unitary operation $\hat{U}_B$ to system B

$$|\Phi\rangle_{AB} \equiv \sum_j \sqrt{p_j}|\phi_j\rangle_A|b_j\rangle_B$$

Measure system B on basis $\{|b_j\rangle_B\}$

$|\phi_j\rangle_A$ with probability p_j

$$\{p_j, |\phi_j\rangle_A\}$$

$$|\Psi\rangle_{AB} \equiv \sum_k \sqrt{q_k}|\psi_k\rangle_A|b_k\rangle_B$$

Measure system B on basis $\{|b_k\rangle_B\}$

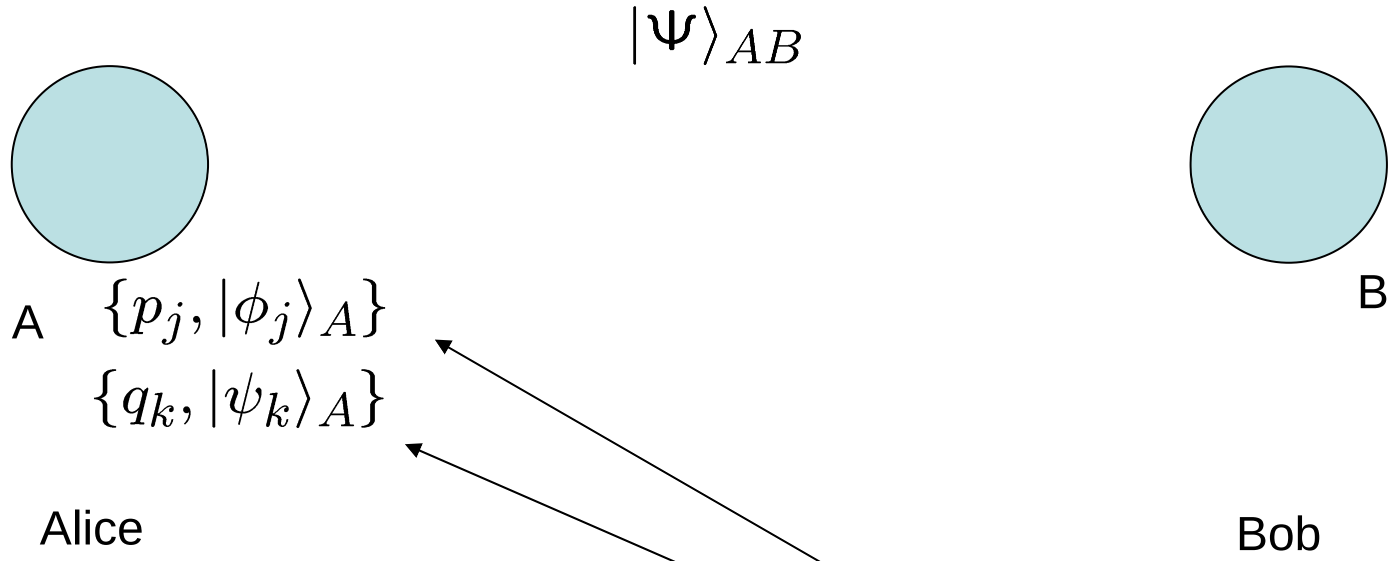
$|\psi_k\rangle_A$ with probability q_k

$$\{q_k, |\psi_k\rangle_A\}$$

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

$$|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}$$

Ensembles with the same density operator



Can Alice distinguish the two states even partially?

NO!

Bob can remotely decide which of the states the system A is in.

Bob can postpone his decision indefinitely.

Theorem: Two states $\{p_j, |\phi_j\rangle\}$ and $\{q_k, |\psi_k\rangle\}$ with the same density operator are physically indistinguishable (hence are the same state).

Density operator

One-to-one

Physical state

Ensembles with the same density operator: an alternative condition

$$\{p_j, |\phi_j\rangle_A\} \qquad \{q_k, |\psi_k\rangle_A\}$$

A necessary and sufficient condition for

$$\hat{\rho}_A \equiv \sum_j p_j |\phi_j\rangle_{AA}\langle\phi_j| = \sum_k q_k |\psi_k\rangle_{AA}\langle\psi_k|$$

$$\boxed{\sqrt{p_j}|\phi_j\rangle_A = \sum_k \underbrace{u_{jk}}_{\text{Unitary matrix}} \sqrt{q_k}|\psi_k\rangle_A}$$

Proof:

$$|\Phi\rangle_{AB} \equiv \sum_j \sqrt{p_j}|\phi_j\rangle_A |b_j\rangle_B \qquad |\Psi\rangle_{AB} \equiv \sum_k \sqrt{q_k}|\psi_k\rangle_A |b_k\rangle_B$$

$$|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}$$

$$\sum_j \sqrt{p_j}|\phi_j\rangle_A |b_j\rangle_B = \sum_k \sqrt{q_k}|\psi_k\rangle_A \hat{U}_B |b_k\rangle_B$$

$$\sqrt{p_j}|\phi_j\rangle_A = \sum_k \underbrace{\langle b_j|\hat{U}_B|b_k\rangle}_{u_{jk}} \sqrt{q_k}|\psi_k\rangle_A$$

3. Qubits

Pauli operators (Pauli matrices)

Bloch representation (Bloch sphere)

Orthogonal measurement

Unitary operation

Qubit

$$\dim \mathcal{H} = 2$$

Take a standard basis $\{|0\rangle, |1\rangle\}$

Linear operator $\hat{A}$

Matrix representation (for $\{|0\rangle, |1\rangle\}$)

$$\hat{A} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

$$A_{ij} = \langle i|\hat{A}|j\rangle$$

$$\hat{A} = \sum_{ij} A_{ij}|i\rangle\langle j|$$

4 complex parameters

$$\hat{A} = \alpha_0\hat{\sigma}_0 + \alpha_1\hat{\sigma}_1 + \alpha_2\hat{\sigma}_2 + \alpha_3\hat{\sigma}_3$$

Pauli operators (Pauli matrices)

Take a standard basis $\{|0\rangle, |1\rangle\}$

$$\hat{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_x = \hat{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{\sigma}_y = \hat{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \hat{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Unitary and self-adjoint

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k$$

$$\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{i,j}\hat{1}$$

$$\mathrm{Tr}(\hat{\sigma}_i) = 0, \ \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j) = 2\delta_{i,j}.$$

$i, j = 1, 2, 3$

Levi-Civita symbol

$$\begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \\ \text{Otherwise } \epsilon_{ijk} = 0 \end{cases}$$

Einstein notation

$\sum_k$ is omitted.

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z$$

$$\hat{\sigma}_x^2 = \hat{1}$$

$$\{\hat{\sigma}_x, \hat{\sigma}_z\} \equiv \hat{\sigma}_x\hat{\sigma}_z + \hat{\sigma}_z\hat{\sigma}_x = 0$$

$$\mathrm{Tr}(\hat{\sigma}_\mu\hat{\sigma}_\nu) = 2\delta_{\mu,\nu}$$

$$(\mu, \nu = 0, 1, 2, 3; \ \sigma_0 \equiv \hat{1})$$

'Orthogonality' with respect to

$$(\hat{A}, \hat{B}) \equiv \mathrm{Tr}(\hat{A}^\dagger\hat{B})$$

Pauli operators (Pauli matrices)

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k$$
$$\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{i,j}\hat{1}$$
$$\mathrm{Tr}(\hat{\sigma}_i) = 0,\ \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j) = 2\delta_{i,j}.$$

Linear operator $\hat{A}$ 4 complex parameters (P_0, P_x, P_y, P_z)

$$\hat{A} = \frac{1}{2}\left(P_0\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) = \frac{1}{2}\begin{pmatrix} P_0 + P_z & P_x - iP_y \\ P_x + iP_y & P_0 - P_z \end{pmatrix}$$

$$\boldsymbol{P} = (P_x, P_y, P_z)$$
$$\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$$

$$P_0 = \mathrm{Tr}(\hat{A}) \qquad \boldsymbol{P} = \mathrm{Tr}(\hat{\boldsymbol{\sigma}}\hat{A})$$

Pauli operators (Pauli matrices)

$$\hat{A} = \frac{1}{2}\left(P_0 \hat{1} + \boldsymbol{P} \cdot \hat{\boldsymbol{\sigma}}\right) = \frac{1}{2}\begin{pmatrix} P_0 + P_z & P_x - iP_y \\ P_x + iP_y & P_0 - P_z \end{pmatrix}$$

$\hat{A}$ is self-adjoint. $\longleftrightarrow$ P_0 and $\boldsymbol{P}$ are real.

Eigenvalues λ_+, λ_-

$$\det(\hat{A}) = \lambda_+ \lambda_- = \tfrac{1}{4}(P_0^2 - |\boldsymbol{P}|^2)$$
$$\mathrm{Tr}(\hat{A}) = \lambda_+ + \lambda_- = P_0$$
$$\downarrow$$
$$\lambda_\pm = (P_0 \pm |\boldsymbol{P}|)/2$$

$\hat{A}$ is positive. $\longleftrightarrow$ P_0 and $\boldsymbol{P}$ are real, $P_0 \geq |\boldsymbol{P}|$

Bloch representation (Bloch sphere)

Density operator

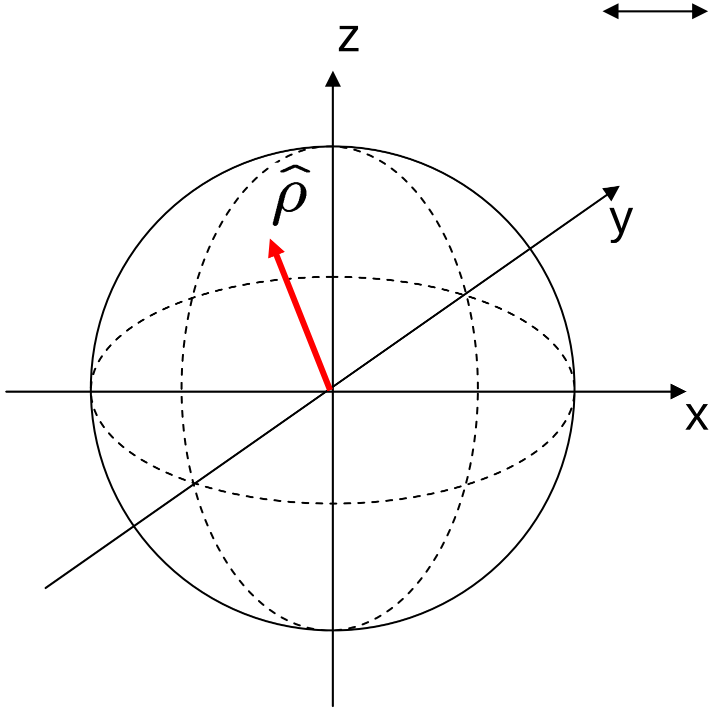
Positive & Unit trace

$P_0 \geq |\boldsymbol{P}| \qquad P_0 = 1$

$$\hat{\rho} = \frac{1}{2}\left(\hat{1} + \boldsymbol{P} \cdot \hat{\boldsymbol{\sigma}}\right) \qquad |\boldsymbol{P}| \leq 1$$

Density operator for a qubit system

$\longleftrightarrow$ A 3D real vector of length no greater than 1



A point inside or on the sphere of radius 1

$$\boldsymbol{P} = (P_x, P_y, P_z)$$

Bloch vector

$$\lambda_{\pm} = (P_0 \pm |\boldsymbol{P}|)/2$$

Pure states $\longleftrightarrow$ $\lambda_+ = 1, \lambda_- = 0$

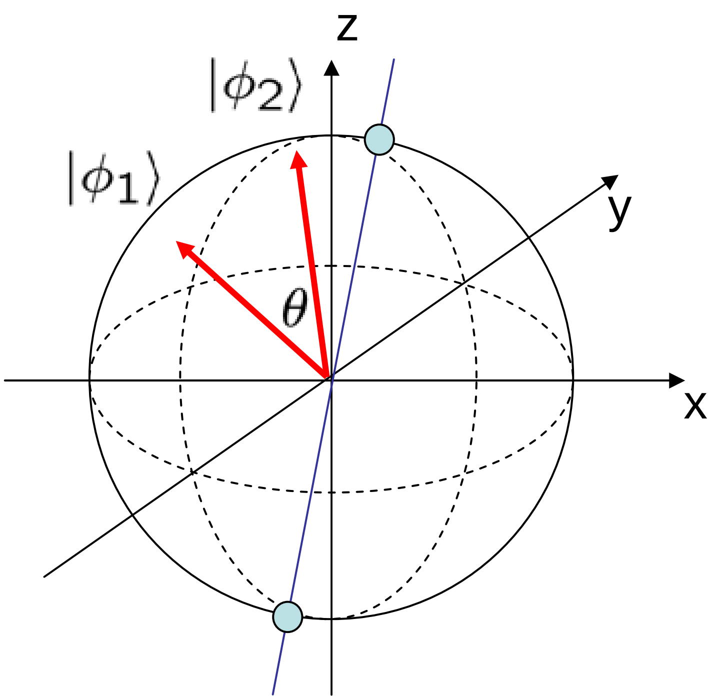
$\longleftrightarrow$ $|\boldsymbol{P}| = 1$

$\longleftrightarrow$ On the sphere

Pure states

$$\hat{\rho}_j = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}_j \cdot \hat{\boldsymbol{\sigma}}\right)$$

$$|\langle\phi_1|\phi_2\rangle|^2 = \mathrm{Tr}[\hat{\rho}_1\hat{\rho}_2]$$
$$= \frac{1 + \boldsymbol{P}_1 \cdot \boldsymbol{P}_2}{2} = \cos^2\frac{\theta}{2}$$

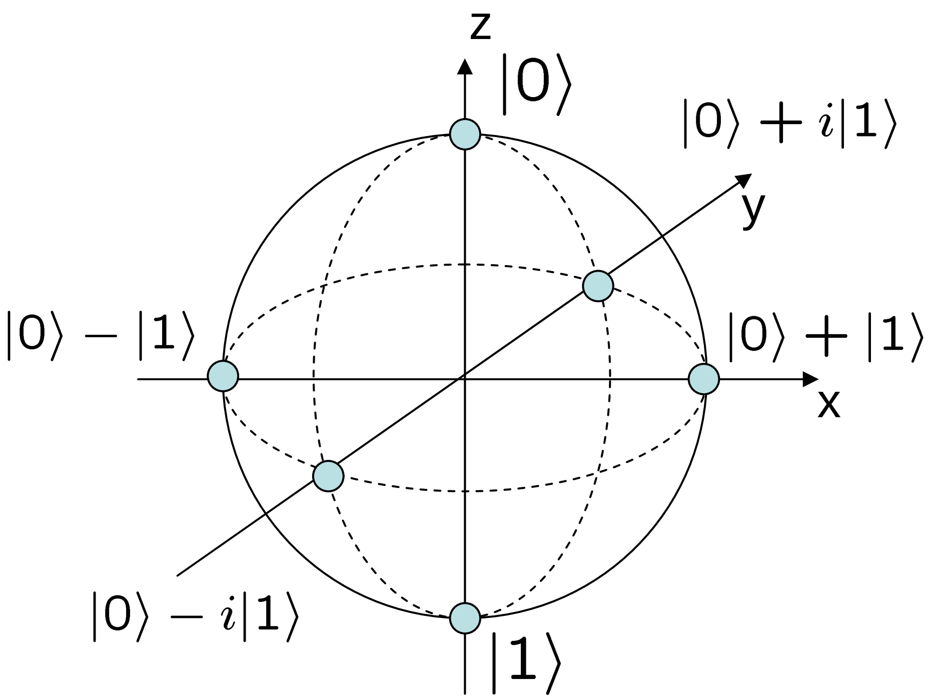


$$\boldsymbol{P}_1 \cdot \boldsymbol{P}_2 = \cos\theta$$

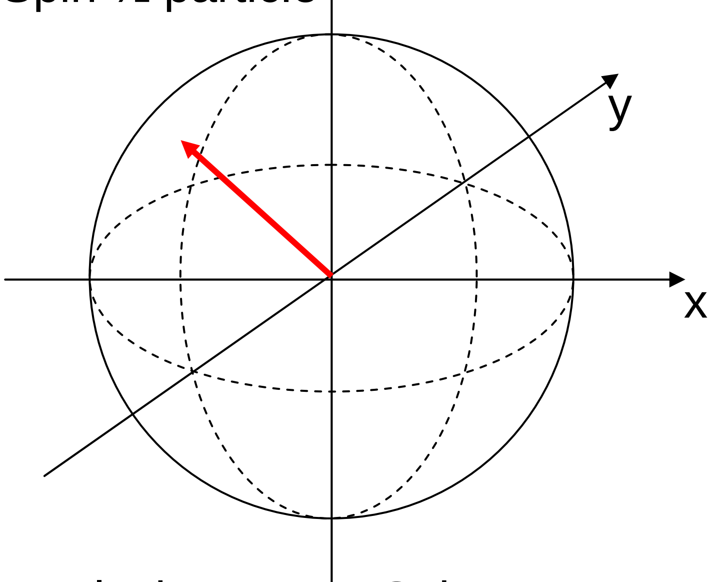
Orthogonal states $\longleftrightarrow$ $\theta = \pi$

Orthonormal basis $\longleftrightarrow$ A line through the origin

Examples

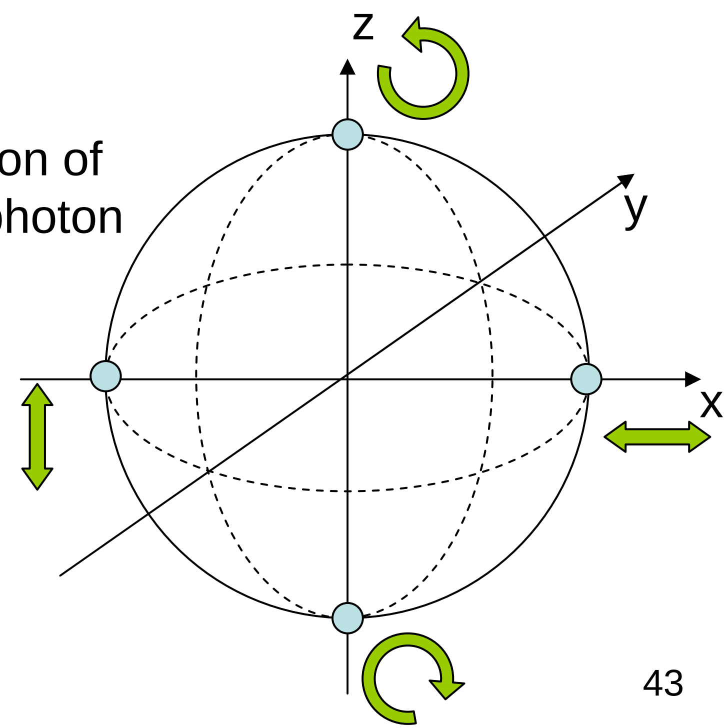


Spin ½ particle



Bloch vector = Spin vector

Polarization of a single photon

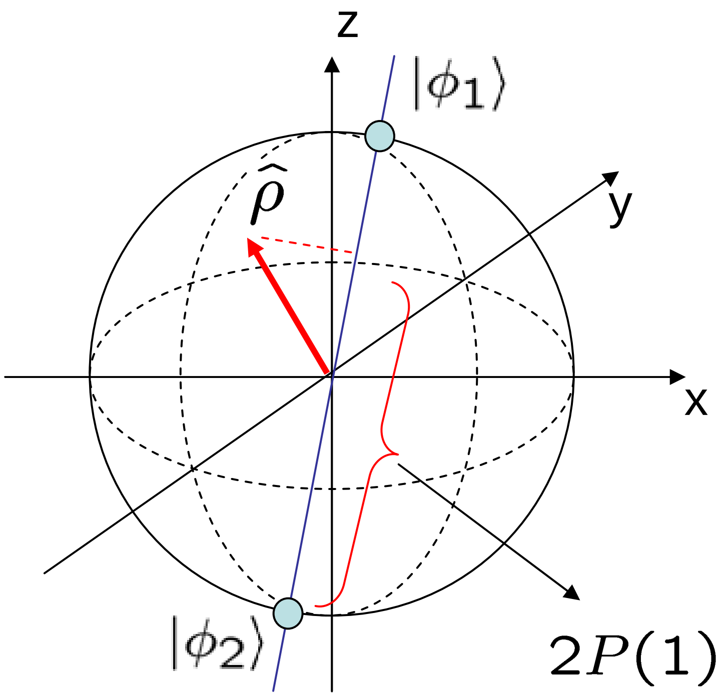


Orthogonal measurement

Orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle\}$ $\longleftrightarrow$ A line through the origin

$$P(1) = \langle\phi_1|\hat{\rho}|\phi_1\rangle = \mathrm{Tr}(\hat{\rho}_1\hat{\rho}) = \frac{1+\boldsymbol{P}_1\cdot\boldsymbol{P}}{2}$$

$$P(2) = \frac{1-\boldsymbol{P}_1\cdot\boldsymbol{P}}{2}$$



Example

Measurement of observable $\hat{\sigma}_z$

↓

Z axis

Unitary operation

$|\psi\rangle, e^{i\theta}|\psi\rangle$ The same physical state

$\widehat{U}, \quad e^{i\theta}\widehat{U}$ The same physical operation

$$\det(e^{i\theta}\widehat{U}) = e^{2i\theta}\det\widehat{U}$$

group $SU(2)$: Set of $\widehat{U}$ with $\det\widehat{U} = 1$ $\widehat{U} \in SU(2) \leftrightarrow -\widehat{U} \in SU(2)$

(2 to 1 correspondence to the physical unitary operations)

$$\widehat{U} = \exp[i\widehat{S}] \qquad \widehat{U} = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$$

Self-adjoint, traceless

$$\widehat{S} = \frac{1}{2}(\boldsymbol{P}\cdot\widehat{\boldsymbol{\sigma}}) \qquad \widehat{S} = \begin{pmatrix} \varphi/2 & 0 \\ 0 & -\varphi/2 \end{pmatrix}$$

We can parameterize the elements of SU(2) as

$$\widehat{U}(\boldsymbol{n},\varphi) \equiv \exp[-i(\varphi/2)\boldsymbol{n}\cdot\widehat{\boldsymbol{\sigma}}]$$

Unit vector

Unitary operation

$$\hat{\rho} = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}\cdot\hat{\boldsymbol{\sigma}}\right) \xrightarrow{\hat{U}(\boldsymbol{n},\varphi)} \hat{\rho}' = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}'\cdot\hat{\boldsymbol{\sigma}}\right)$$

How does the Bloch vector changes?

Infinitesimal change $\hat{U}(\boldsymbol{n},\delta\varphi) \sim \hat{1} - i(\delta\varphi/2)\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}$

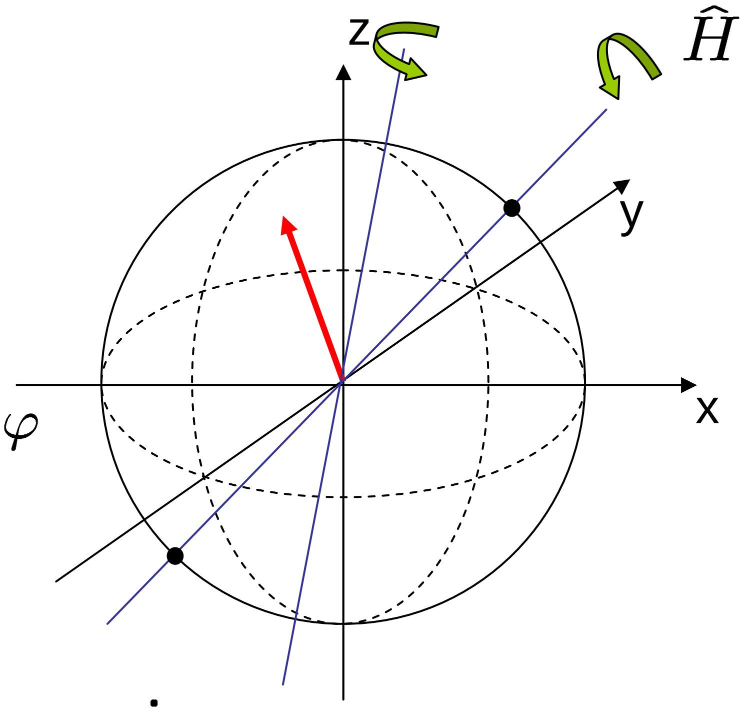
$$\delta\boldsymbol{P} \equiv \boldsymbol{P}' - \boldsymbol{P} = \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}'] - \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}]$$

$$\begin{aligned}
&= \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{U}(\boldsymbol{n},\delta\varphi)\hat{\rho}\hat{U}^\dagger(\boldsymbol{n},\delta\varphi)] - \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}] \\
&= \mathrm{Tr}[\hat{U}^\dagger(\boldsymbol{n},\delta\varphi)\hat{\boldsymbol{\sigma}}\hat{U}(\boldsymbol{n},\delta\varphi)\hat{\rho}] - \mathrm{Tr}[\hat{\boldsymbol{\sigma}}\hat{\rho}] \\
&\sim \mathrm{Tr}\{(i\delta\varphi/2)[(\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}),\hat{\boldsymbol{\sigma}}]\hat{\rho}\} = -\delta\varphi\mathrm{Tr}[n_i\epsilon_{ijk}\hat{\sigma}_k\hat{\rho}] \\
&= \delta\varphi\mathrm{Tr}[(\boldsymbol{n}\times\hat{\boldsymbol{\sigma}})\hat{\rho}] = \delta\varphi\boldsymbol{n}\times\boldsymbol{P}.
\end{aligned}$$

Rotation around axis $\boldsymbol{n}$ by angle $\delta\varphi$

Unitary operation

$$\hat{U} \in SU(2)$$



$$\hat{U} = \exp[-i(\varphi/2)\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}]$$

Rotation around axis $\boldsymbol{n}$ by angle φ

Examples

$\hat{\sigma}_z$: π rotation around z axis

$\hat{\sigma}_x$: π rotation around x axis

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hadamard transform

π rotation (interchanges z and x axes)

4. Power of an ancillary system

Kraus representation (Operator-sum rep.)

Generalized measurement

Unambiguous state discrimination

Quantum operation (Quantum channel, CPTP map)

Relation between quantum operations and bipartite states

A maximally entangled state and relative states

Size of the auxiliary system

Kraus operators for the same CPTP map

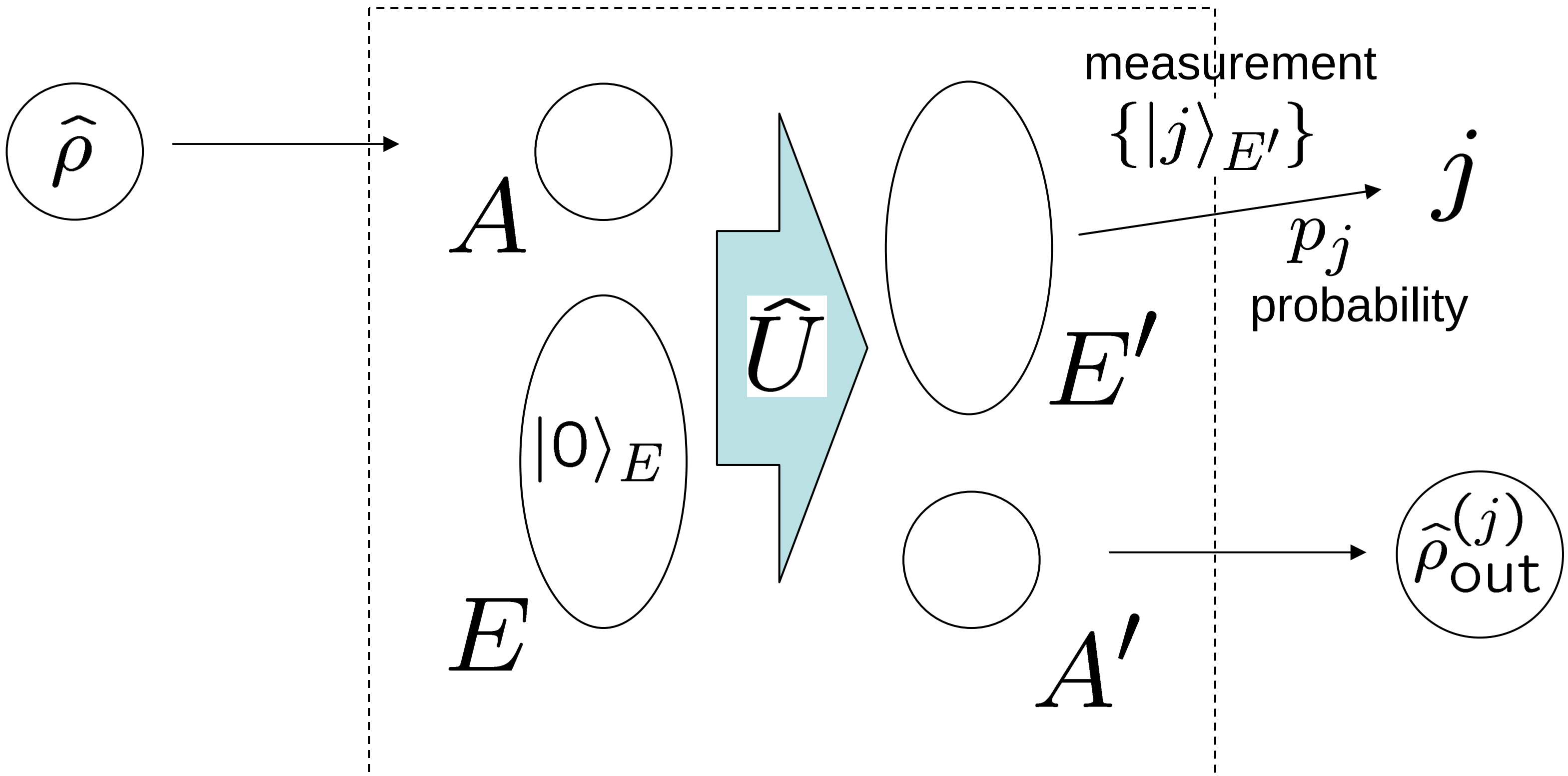
What can we do in principle?

Power of an ancilla system

Basic operations
- Unitary operations
- Orthogonal measurements

\+

An auxiliary system (ancilla)

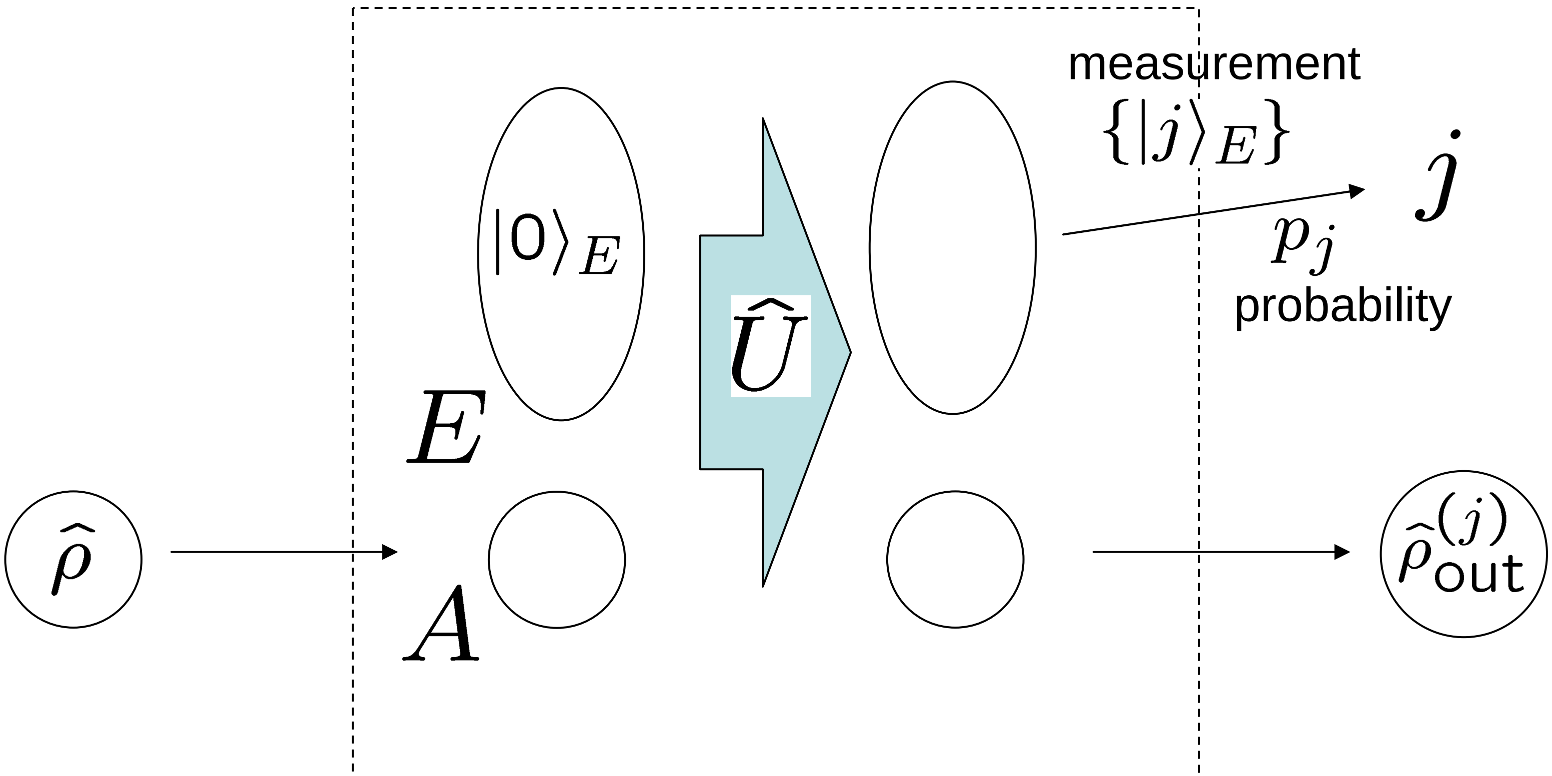


Power of an ancilla system

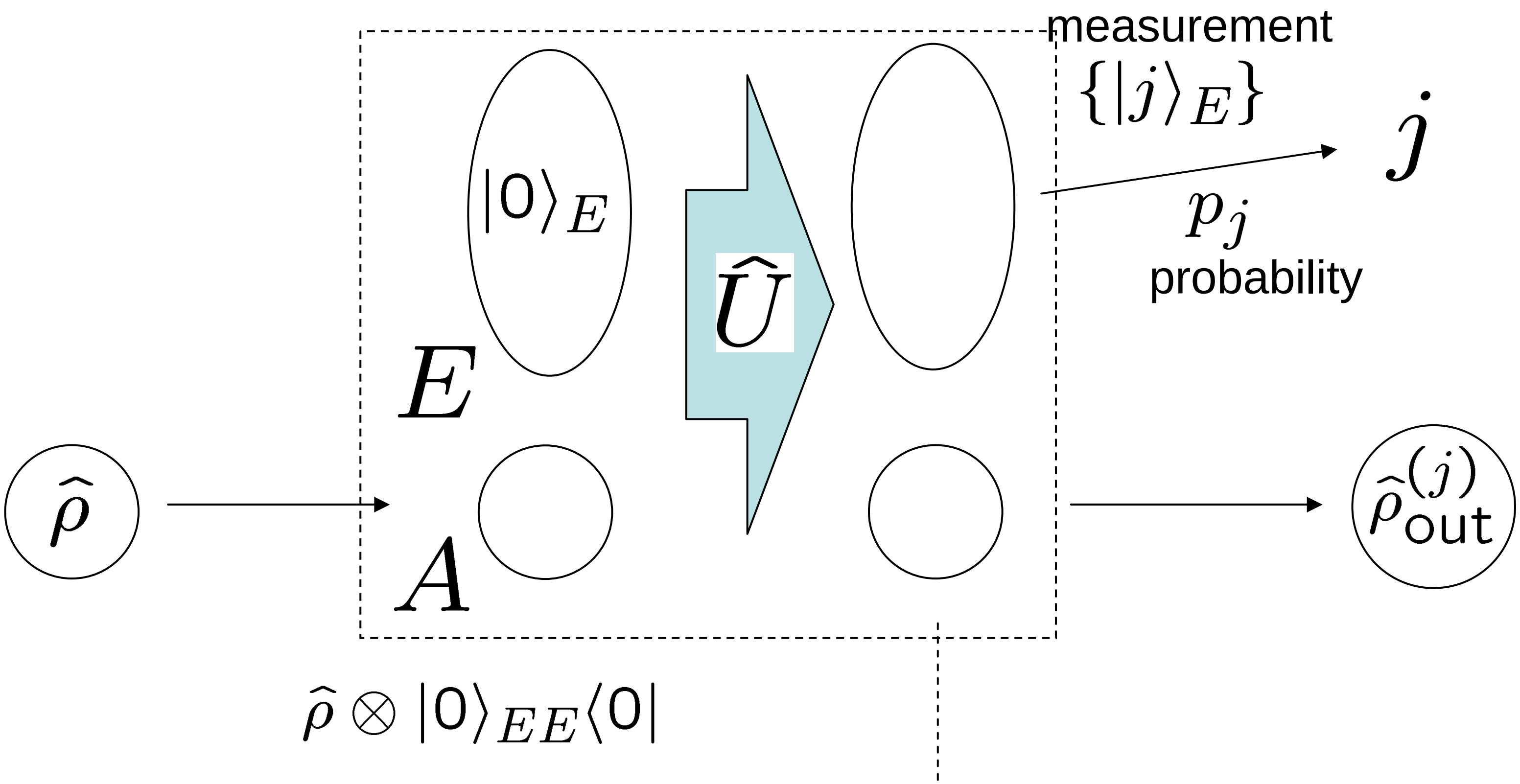
Basic operations
Unitary operations
Orthogonal measurements

+

An auxiliary system
(ancilla)



Power of an ancilla system



$$p_j\widehat{\rho}_{\mathrm{out}}^{(j)} = {}_E\langle j|\widehat{U}(\widehat{\rho}\otimes|0\rangle_{EE}\langle 0|)\widehat{U}^\dagger|j\rangle_E$$
$$= \widehat{M}^{(j)}\widehat{\rho}\widehat{M}^{(j)\dagger}$$
$$\widehat{M}^{(j)} \equiv {}_E\langle j|\widehat{U}|0\rangle_E$$

$${}_E\langle j|\ \widehat{U}\ |0\rangle_E$$
$$\widehat{M}^{(j)}:\mathcal{H}_A\to\mathcal{H}_A$$

Kraus representation (Operator-sum rep.)

$$p_j \widehat{\rho}_{\text{out}}^{(j)} = {}_E\langle j|\widehat{U}(\widehat{\rho}\otimes|0\rangle_{EE}\langle 0|)\widehat{U}^{\dagger}|j\rangle_E$$

$$\Big\downarrow \quad \widehat{M}^{(j)} \equiv {}_E\langle j|\widehat{U}|0\rangle_E$$ Kraus operators

$$p_j \widehat{\rho}_{\text{out}}^{(j)} = \widehat{M}^{(j)}\widehat{\rho}\widehat{M}^{(j)\dagger} \text{ with } \textstyle\sum_j \widehat{M}^{(j)\dagger}\widehat{M}^{(j)} = \widehat{1}$$

Representation with no reference to the ancilla system

$$\begin{aligned}\sum_j \widehat{M}^{(j)\dagger}\widehat{M}^{(j)} &= \sum_j {}_E\langle 0|\widehat{U}^{\dagger}|j\rangle_{EE}\langle j|\widehat{U}|0\rangle_E \\ &= {}_E\langle 0|\widehat{U}^{\dagger}\widehat{U}|0\rangle_E \\ &= {}_E\langle 0|\widehat{1}_A\otimes\widehat{1}_E|0\rangle_E \\ &= \widehat{1}_A\end{aligned}$$

Kraus operators $\longrightarrow$ Physical realization

$$p_j \hat{\rho}^{(j)}_{\text{out}} = {}_E\langle j|\hat{U}(\hat{\rho}\otimes|0\rangle_{EE}\langle 0|)\hat{U}^\dagger|j\rangle_E$$

$$\uparrow \downarrow \quad \hat{M}^{(j)} \equiv {}_E\langle j|\hat{U}|0\rangle_E$$ Kraus operators

$$p_j \hat{\rho}^{(j)}_{\text{out}} = \hat{M}^{(j)}\hat{\rho}\hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$$

Arbitrary set $\{\hat{M}^{(j)}\}$ satisfying $\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$

$|\phi\rangle_A \otimes |0\rangle_E \mapsto \sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E$ is linear.

preserves inner products.

For any two states $|\phi\rangle_A$ and $|\psi\rangle_A$,

$$\left(\sum_{j'} \hat{M}^{(j')}|\psi\rangle_A \otimes |j'\rangle_E\right)^\dagger \left(\sum_{j} \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E\right)$$

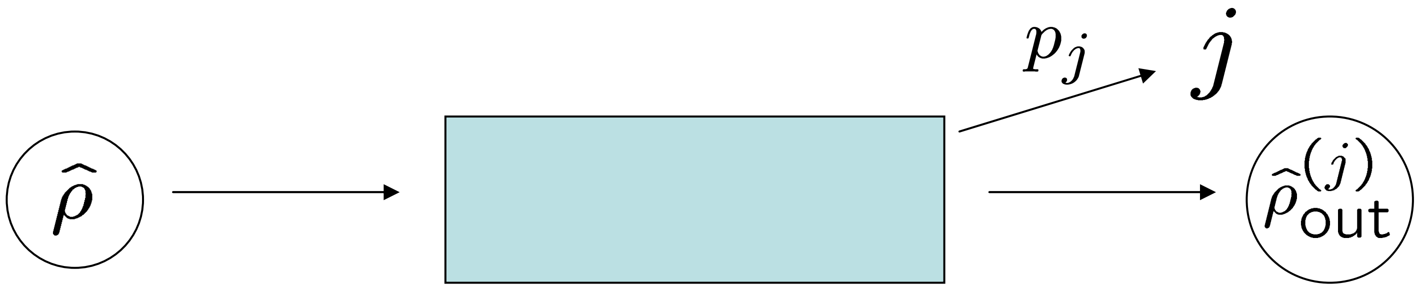
$$= {}_A\langle\psi|\phi\rangle_A = (|\psi\rangle_A \otimes |0\rangle_E)^\dagger(|\phi\rangle_A \otimes |0\rangle_E)\,.$$

$\downarrow$

There exists a unitary satisfying
$\hat{U}(|\phi\rangle_A \otimes |0\rangle_E) = \sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E$

Generalized measurement

$$p_j \hat{\rho}^{(j)}_{\mathrm{out}} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \quad \text{with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$



$$p_j = \mathrm{Tr}[\hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger}] = \mathrm{Tr}[\hat{F}^{(j)} \hat{\rho}]$$

$$\hat{F}^{(j)} \equiv \hat{M}^{(j)\dagger} \hat{M}^{(j)} \geq 0$$

positive

$$\boxed{p_j = \mathrm{Tr}[\hat{F}^{(j)} \hat{\rho}] \quad \text{with } \sum_j \hat{F}^{(j)} = \hat{1}}$$

$\{\hat{F}^{(j)}\}$ POVM

Positive operator valued measure

Generalized measurement

$$p_j = \mathrm{Tr}[\widehat{F}^{(j)}\hat{\rho}] \ \text{ with } \sum_j \widehat{F}^{(j)} = \hat{1}$$

Examples

Orthogonal measurement on basis $\{|a_j\rangle\}$

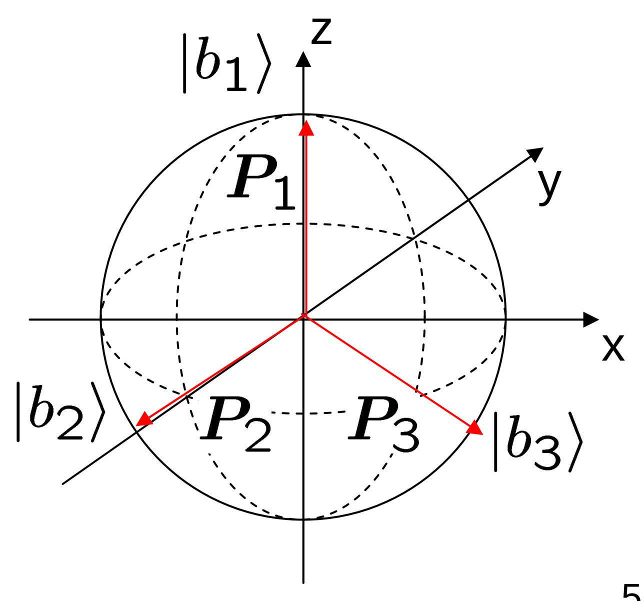
$$\widehat{F}^{(j)} = |a_j\rangle\langle a_j|$$

Trine measurement on a qubit

$$\widehat{F}^{(j)} = \tfrac{2}{3}|b_j\rangle\langle b_j|$$

$$|b_j\rangle\langle b_j| = \tfrac{1}{2}\left(\hat{1} + \boldsymbol{P}_j \cdot \hat{\boldsymbol{\sigma}}\right)$$

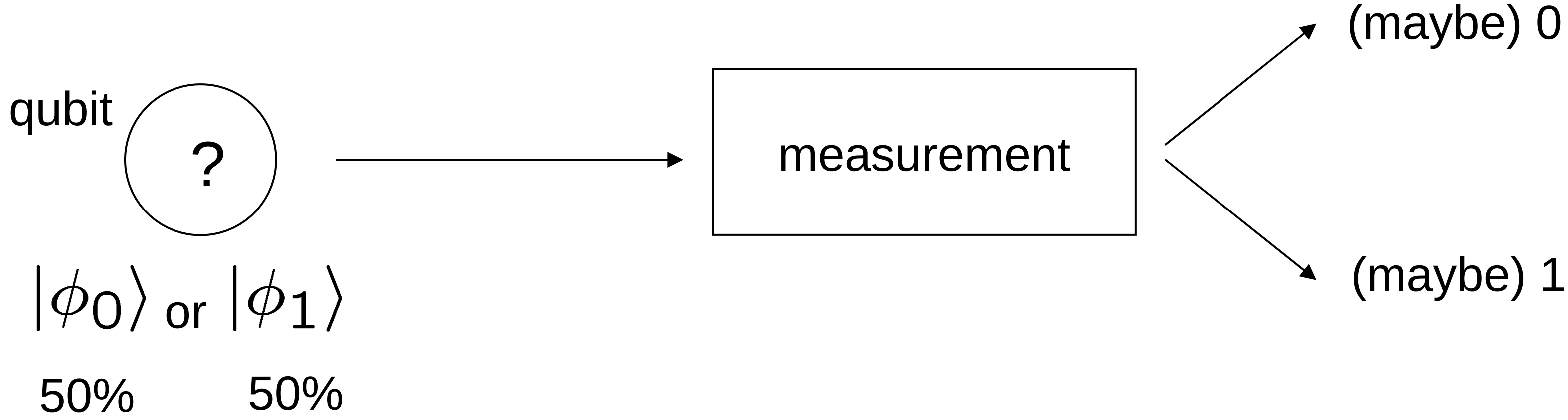
$$\sum_j \boldsymbol{P}_j = 0 \longrightarrow \sum_j \widehat{F}^{(j)} = \hat{1}$$



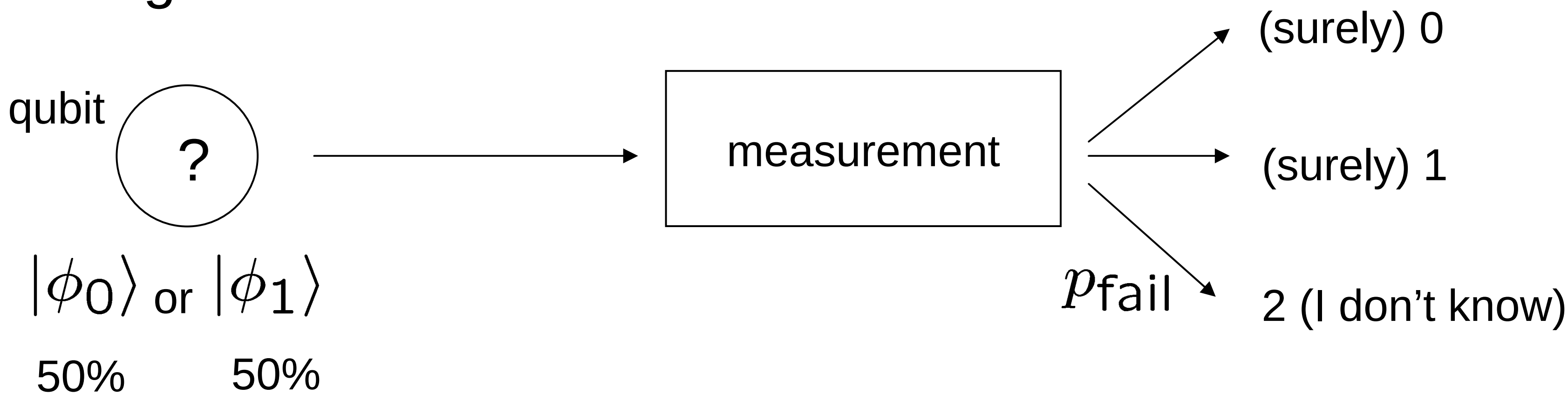
Distinguishing two nonorthogonal states

$$\langle \phi_0 | \phi_1 \rangle = s > 0$$

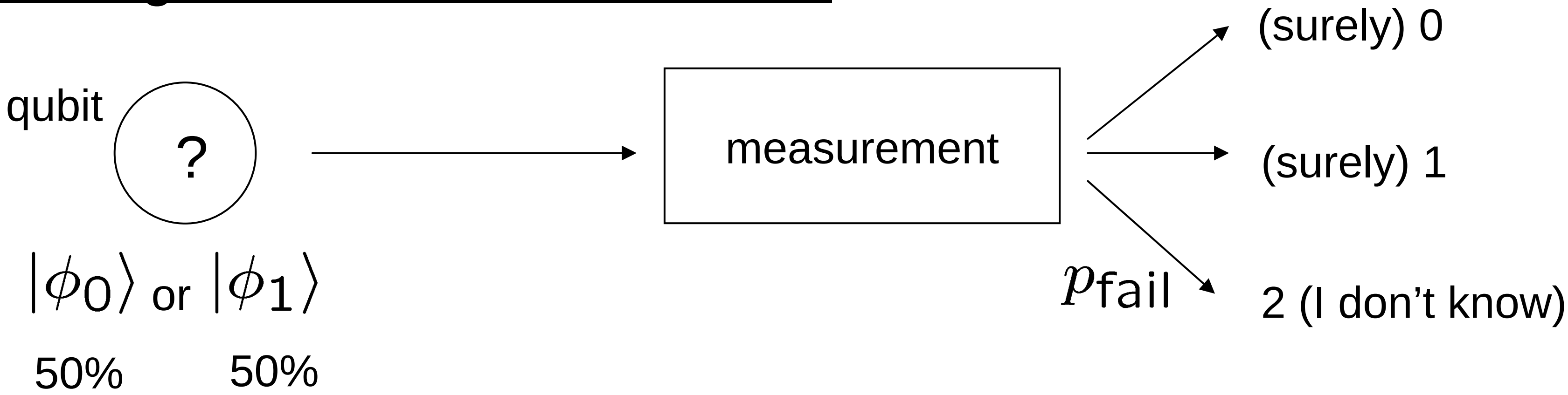
Minimum-error discrimination



Unambiguous state discrimination



Unambiguous state discrimination



$$\langle \phi_0 | \phi_1 \rangle = s > 0$$

Orthogonal measurement

$$\{|\phi_0\rangle, |\phi_0^{\perp}\rangle\}$$

2 (I don't know)   (surely) 1

If the initial state is $|\phi_0\rangle$

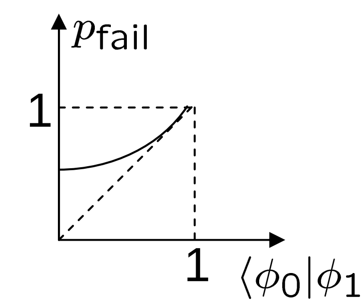
it always fails.

If the initial state is $|\phi_1\rangle$

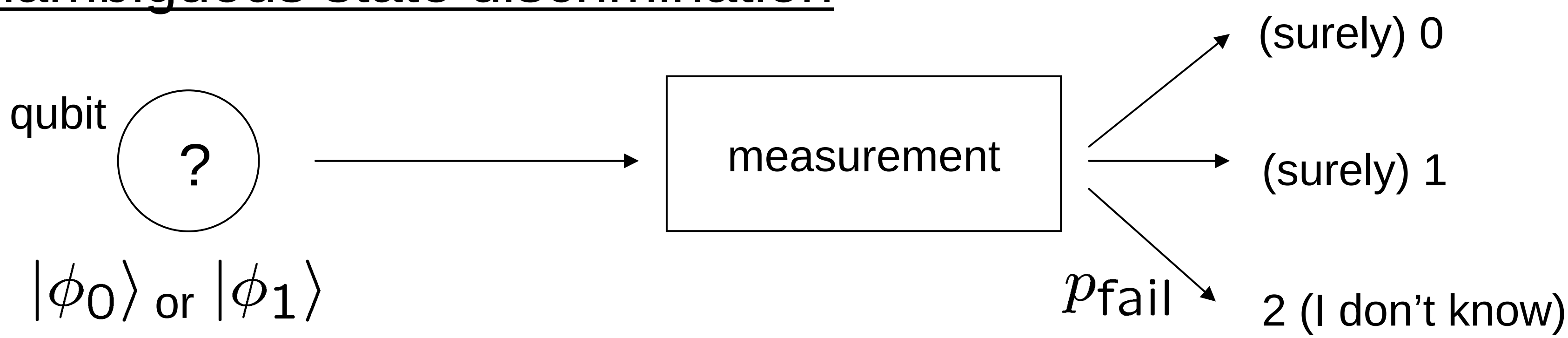
it fails with prob. $|\langle \phi_0 | \phi_1 \rangle|^2 = s^2$

$$\{|\phi_1\rangle, |\phi_1^{\perp}\rangle\}$$

$$p_{\text{fail}} = \frac{1 + s^2}{2}$$



Unambiguous state discrimination



$|\phi_0\rangle$ or $|\phi_1\rangle$

50% 50%

$$\langle\phi_0|\phi_1\rangle = s > 0$$

Generalized measurement

$$\hat{F}_0 := \mu|\phi_1^\perp\rangle\langle\phi_1^\perp|$$

$$\hat{F}_1 := \mu|\phi_0^\perp\rangle\langle\phi_0^\perp|$$

$$\hat{F}_2 := \hat{1} - \hat{F}_0 - \hat{F}_1$$

The only constraint on μ comes from $\hat{F}_2 \geq 0$

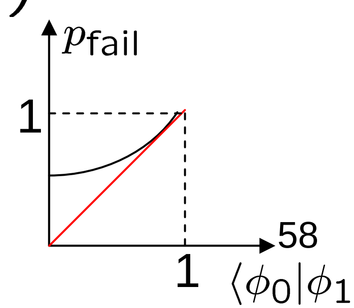
$$\langle\phi_0^\perp|\phi_1^\perp\rangle = s \qquad (\hat{F}_0 + \hat{F}_1 \leq \hat{1})$$

$$(\hat{F}_0 + \hat{F}_1)(|\phi_0^\perp\rangle \pm |\phi_1^\perp\rangle) = \mu(1 \pm s)(|\phi_0^\perp\rangle \pm |\phi_1^\perp\rangle)$$

The optimum: $\mu = (1+s)^{-1}$

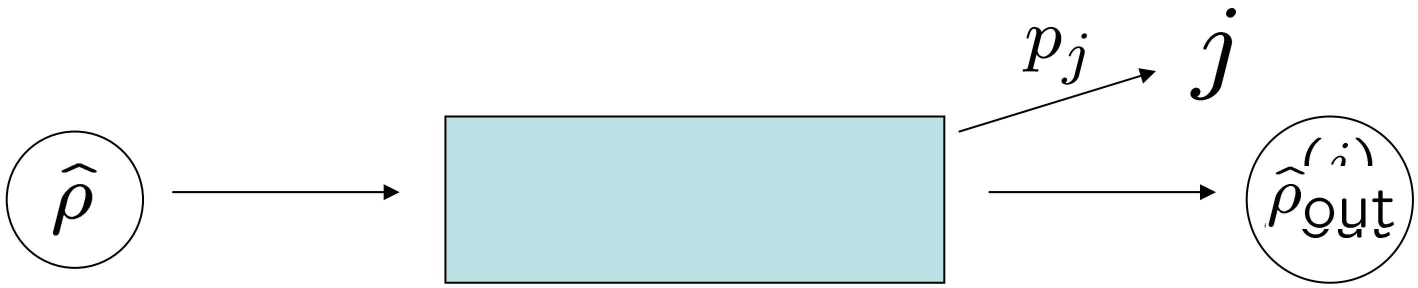
$$p_{\mathrm{fail}} = 1 - \frac{\mu}{2}|\langle\phi_0|\phi_1^\perp\rangle|^2 - \frac{\mu}{2}|\langle\phi_1|\phi_0^\perp\rangle|^2 = 1 - \mu(1 - s^2)$$

$$p_{\mathrm{fail}} = s$$



Quantum operation (Quantum channel, CPTP map)

$$p_j \hat{\rho}_{\text{out}}^{(j)} = \widehat{M}^{(j)} \hat{\rho} \widehat{M}^{(j)\dagger} \text{ with } \sum_j \widehat{M}^{(j)\dagger} \widehat{M}^{(j)} = \hat{1}$$



$$\begin{aligned}
\hat{\rho}_{\text{out}} = \sum_j p_j \hat{\rho}_{\text{out}}^{(j)} &= \sum_j \widehat{M}^{(j)} \hat{\rho} \widehat{M}^{(j)\dagger} \\
&= \sum_j {}_E\langle j | \widehat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \widehat{U}^\dagger | j \rangle_E \\
&= \text{Tr}_E[\widehat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \widehat{U}^\dagger]
\end{aligned}$$

$$\boxed{\begin{aligned}
\hat{\rho}_{\text{out}} &= \sum_j \widehat{M}^{(j)} \hat{\rho} \widehat{M}^{(j)\dagger} \\
&= \text{Tr}_E[\widehat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \widehat{U}^\dagger]
\end{aligned}}$$

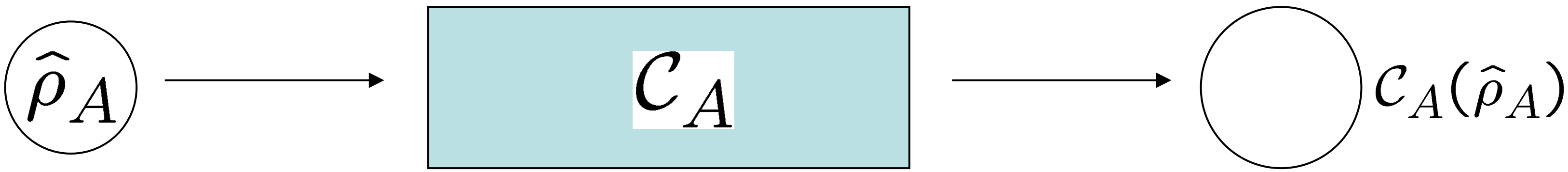
$\hat{\rho}_{\text{out}} = \mathcal{C}(\hat{\rho})$ completely-positive trace-preserving map
CPTP map

Positive maps and completely-positive maps

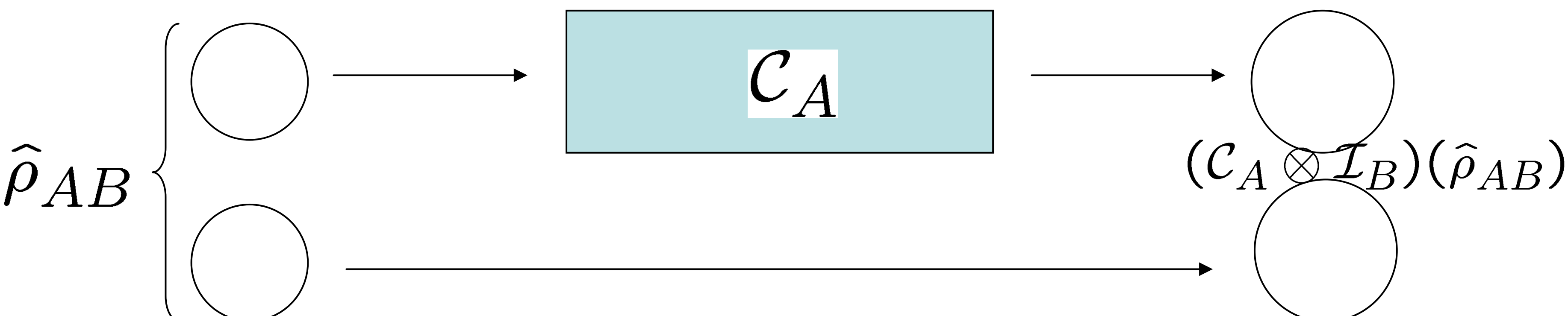
Linear map

$$\hat{\rho}_A \mapsto \mathcal{C}_A(\hat{\rho}_A)$$

"positive": $\mathcal{C}_A(\hat{\rho}_A)$ is positive whenever $\hat{\rho}_A$ is positive



"completely-positive": $(\mathcal{C}_A \otimes \mathcal{I}_B)(\hat{\rho}_{AB})$ is positive whenever $\hat{\rho}_{AB}$ is positive



$$(\mathcal{C}_A \otimes \mathcal{I}_B)(\hat{\rho}_{AB}) = \sum_j \hat{M}_A^{(j)} \hat{\rho}_{AB} \hat{M}_A^{(j)\dagger}$$

Maximally entangled states

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$$

A B

Orthonormal bases $\{|k\rangle_A\}_{k=1,2,\ldots,d}$ $\{|k\rangle_B\}_{k=1,2,\ldots,d}$

$$\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

Maximally entangled state

Relative states

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$$

Fix a maximally entangled state

$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A |k\rangle_B$$

A $\qquad$ B

Relative states

$$|\phi\rangle_A = \sum_k \alpha_k |k\rangle_A \longleftrightarrow |\phi^*\rangle_B = \sum_k \overline{\alpha_k} |k\rangle_B$$

$$= \sqrt{d}\, {}_B\langle\phi^*||\Phi\rangle_{AB} \qquad = \sqrt{d}\, {}_A\langle\phi||\Phi\rangle_{AB}$$

$$|\Phi\rangle_{AB} \begin{cases} A \longrightarrow |\phi\rangle_A \\ B \xrightarrow[\text{measurement}]{\text{Orthogonal}} \text{outcome } j = 1 \end{cases}$$

$$\{|v_j\rangle_B\}_{j=1,2,\ldots,d}$$

$$p_1 = \frac{1}{d}$$

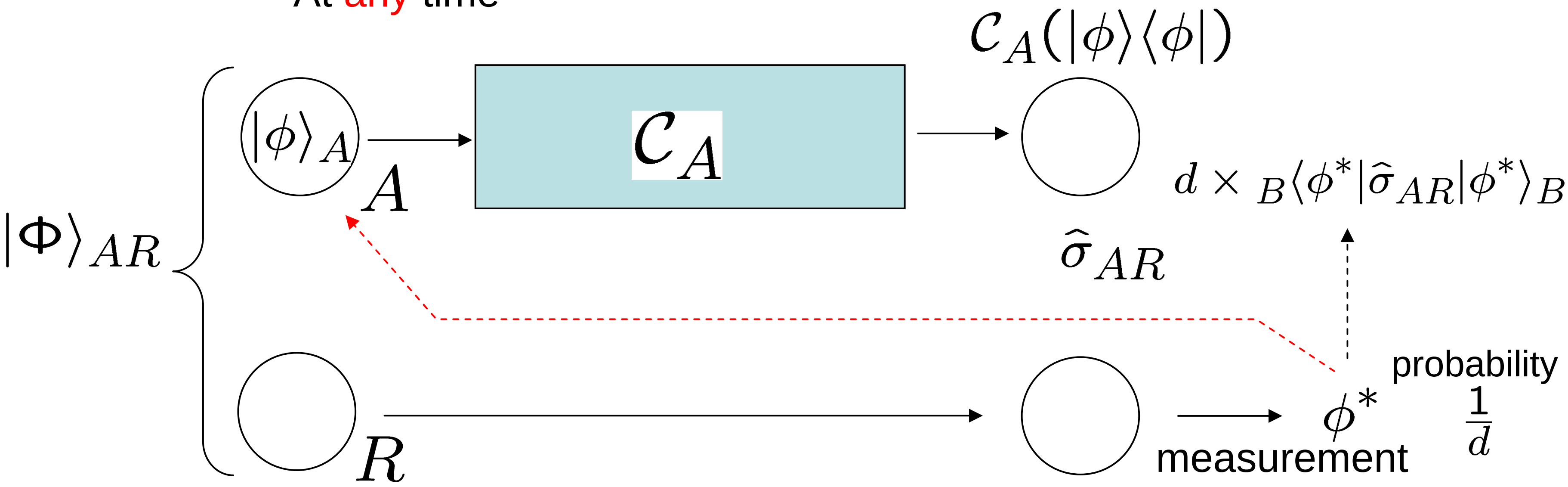
$$|v_1\rangle_B = |\phi^*\rangle_B$$

$$\boxed{\frac{1}{\sqrt{d}} |\phi\rangle_A = {}_B\langle\phi^*||\Phi\rangle_{AB}}$$

Quantum operation and bipartite state

We can remotely prepare system A in any state
with a nonzero success probability.

At any time



$$\hat{\sigma}_{AR} \equiv (\mathcal{C}_A \otimes \mathcal{I}_R)(|\Phi\rangle\langle\Phi|)$$

If this single state is known …

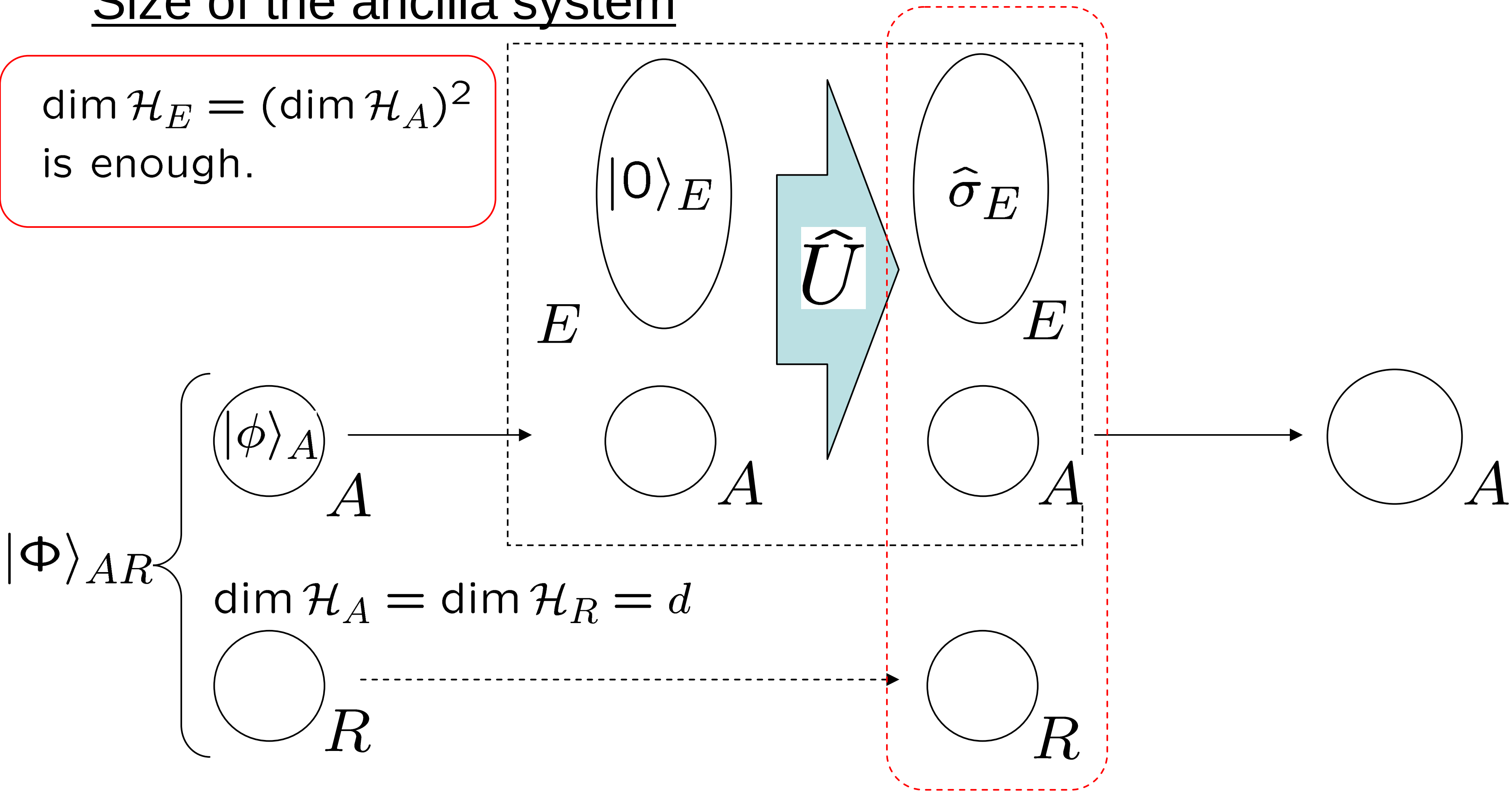
$$\frac{1}{d}\mathcal{C}_A(|\phi\rangle\langle\phi|) = {}_B\langle\phi^*|\hat{\sigma}_{AR}|\phi^*\rangle_B$$

Output for every input state is known!

Characterization of a process = Characterization of a state

Size of the ancilla system

$\dim \mathcal{H}_E = (\dim \mathcal{H}_A)^2$
is enough.

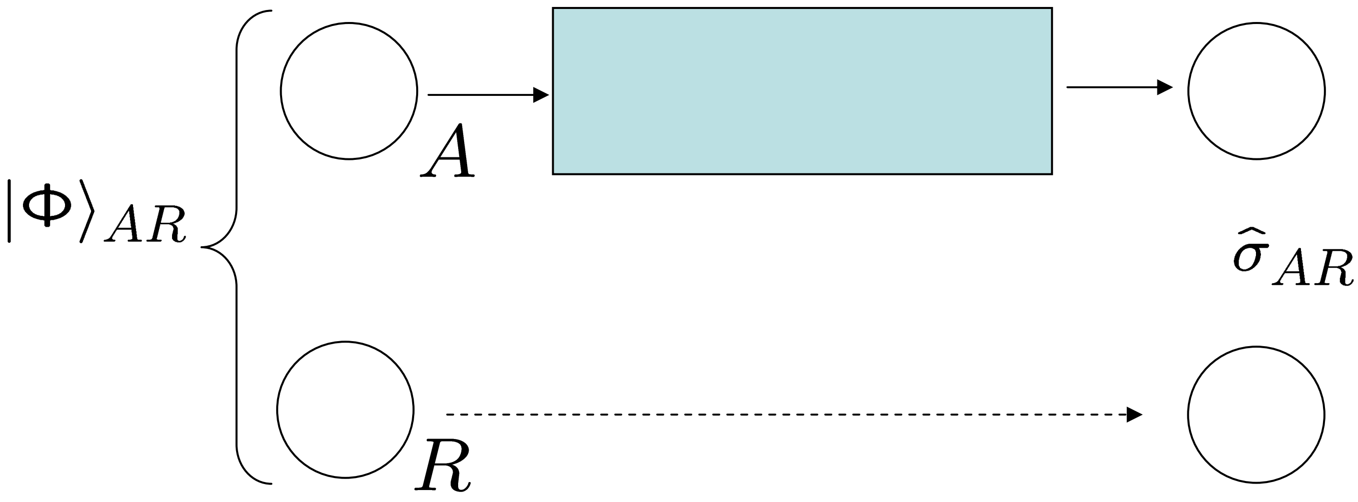


$$|\xi\rangle_{ARE} \equiv \hat{U}(|\Phi\rangle_{AR} \otimes |0\rangle_E)$$

$$\dim(\text{Ran } \hat{\rho}_E) = \dim(\text{Ran } \hat{\rho}_{AR}) \leq \dim \mathcal{H}_{AR} = d^2$$

$$\text{Ran } \hat{\sigma}_E \subset \text{Ran } \hat{\rho}_E$$

Kraus operators for the same CPTP map



$$\hat{\rho}_{\text{out}} = \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \quad \overset{\text{same}}{\underset{?}{\longleftrightarrow}} \quad \hat{\rho}_{\text{out}} = \sum_k \hat{N}^{(k)} \hat{\rho} \hat{N}^{(k)\dagger}$$

$$\Updownarrow$$

$$\sum_j \hat{M}^{(j)} |\Phi\rangle\langle\Phi| \hat{M}^{(j)\dagger} = \sum_k \hat{N}^{(k)} |\Phi\rangle\langle\Phi| \hat{N}^{(k)\dagger} = \hat{\sigma}_{AR}$$

Kraus operators for the same CPTP map

$$\hat{\rho}_{\text{out}} = \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \xleftrightarrow{\text{same}} \hat{\rho}_{\text{out}} = \sum_k \hat{N}^{(k)} \hat{\rho} \hat{N}^{(k)\dagger}$$

$$\Updownarrow$$

$$\sum_j \hat{M}^{(j)} |\Phi\rangle\langle\Phi| \hat{M}^{(j)\dagger} = \sum_k \hat{N}^{(k)} |\Phi\rangle\langle\Phi| \hat{N}^{(k)\dagger} = \hat{\sigma}_{AR}$$

$$\Updownarrow$$

$$\hat{M}^{(j)} |\Phi\rangle_{AR} = \sum_k u_{jk} \hat{N}^{(k)} |\Phi\rangle_{AR}$$

$$\Updownarrow$$

Apply ${}_R\langle\phi^*|$

$$\hat{M}^{(j)} |\phi\rangle_A = \sum_k u_{jk} \hat{N}^{(k)} |\phi\rangle_A$$

$$\hat{M}^{(j)} = \sum_k u_{jk} \hat{N}^{(k)}$$

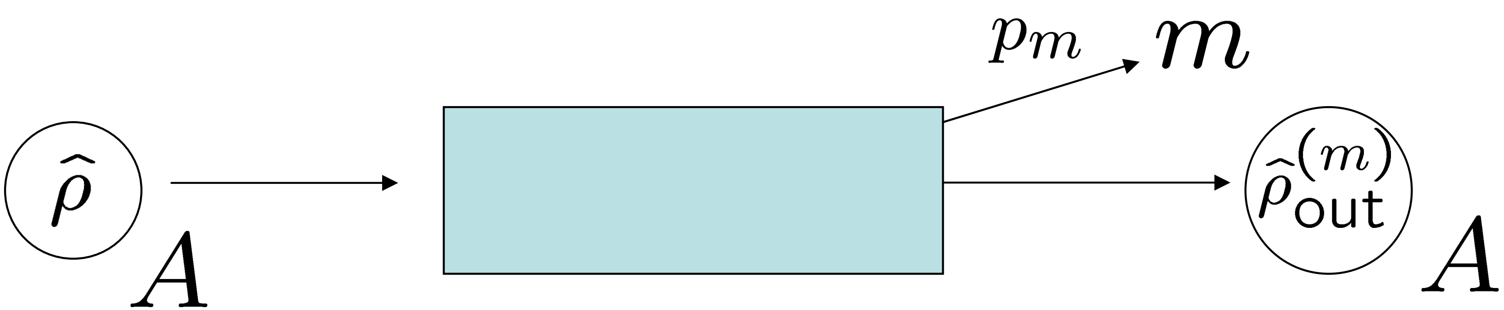
Unitary matrix

What can we do in principle?

We have seen what we can (at least) do by using an ancilla system.

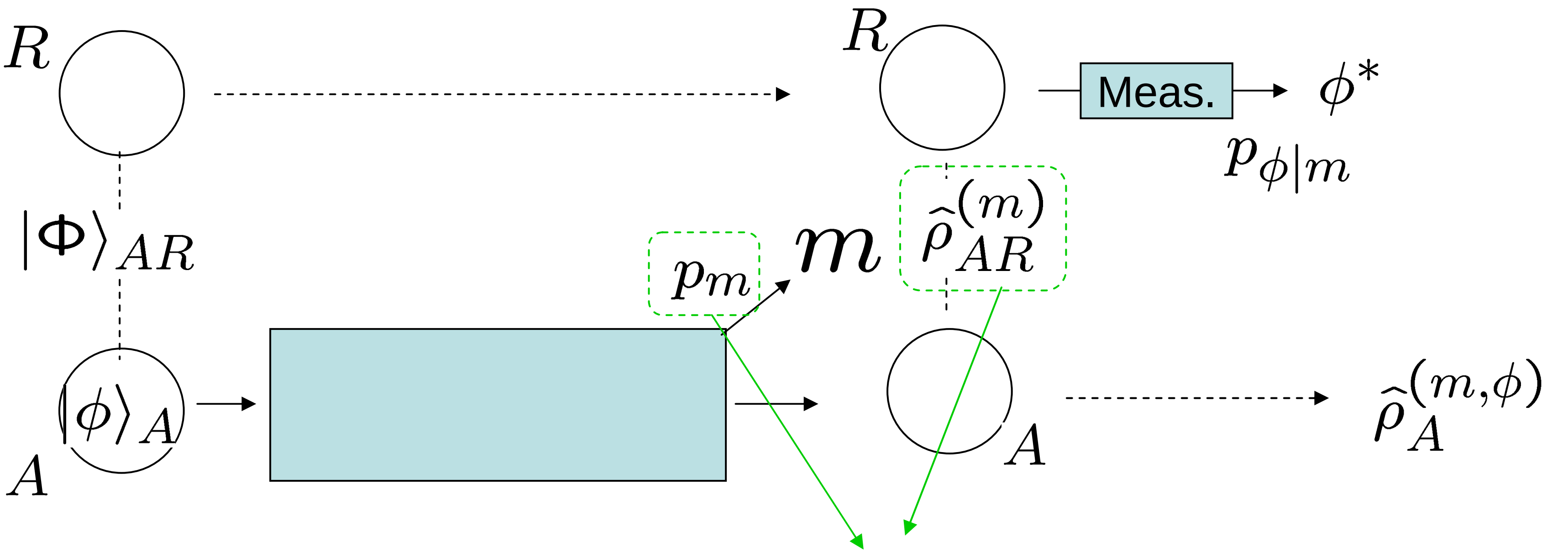
$$p_j \hat{\rho}^{(j)}_{\text{out}} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \text{ with } \textstyle\sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

We also want to know what we cannot do.



Black box with classical and quantum output

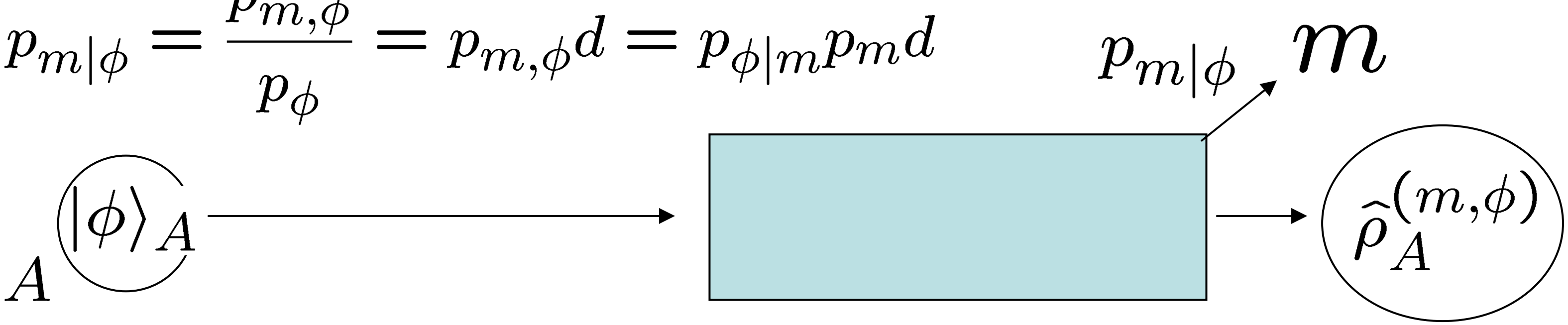
What can we do in principle?



These should tell us everything about the black box.

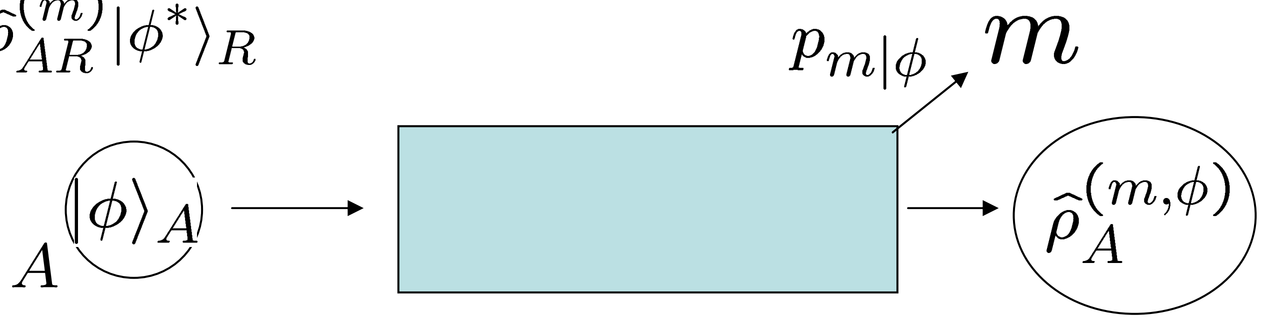
$$p_{\phi|m}\hat{\rho}_A^{(m,\phi)} = {}_R\langle\phi^*|\hat{\rho}_{AR}^{(m)}|\phi^*\rangle_R$$

$$p_{m|\phi} = \frac{p_{m,\phi}}{p_\phi} = p_{m,\phi}d = p_{\phi|m}p_m d$$



Some algebras...

$$\begin{cases} p_{\phi|m}\hat{\rho}_A^{(m,\phi)} = {}_R\langle\phi^*|\hat{\rho}_{AR}^{(m)}|\phi^*\rangle_R \\ p_{m|\phi} = p_{\phi|m}p_m d \end{cases}$$



$$p_{m|\phi}\hat{\rho}_A^{(m,\phi)} = \sqrt{d}\,{}_R\langle\phi^*|p_m\hat{\rho}_{AR}^{(m)}|\phi^*\rangle_R\sqrt{d}$$

$$p_m\rho_{AR}^{(m)} = \sum_k |\Psi_{k,m}\rangle_{AR}\,{}_{AR}\langle\Psi_{k,m}|$$

(unnormalized states)

$$\langle\Phi|\,|\phi\rangle_A \qquad |\Psi_{k,m}\rangle$$

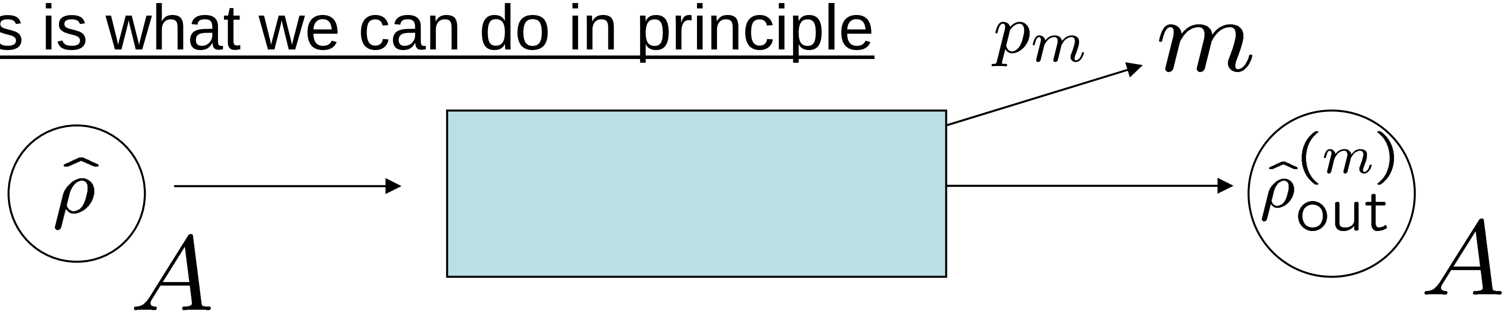
$${}_R\langle\phi^*| = \sqrt{d}\;\;{}_{AR}\langle\Phi||\phi\rangle_A$$

$$\longrightarrow \sqrt{d}\,{}_R\langle\phi^*||\Psi_{k,m}\rangle_{AR} = \hat{M}^{(k,m)}|\phi\rangle_A$$

$$\boxed{p_{m|\phi}\rho_A^{(m,\phi)} = \sum_k \hat{M}^{(k,m)}|\phi\rangle_{A\,A}\langle\phi|\hat{M}^{(k,m)\dagger}}$$

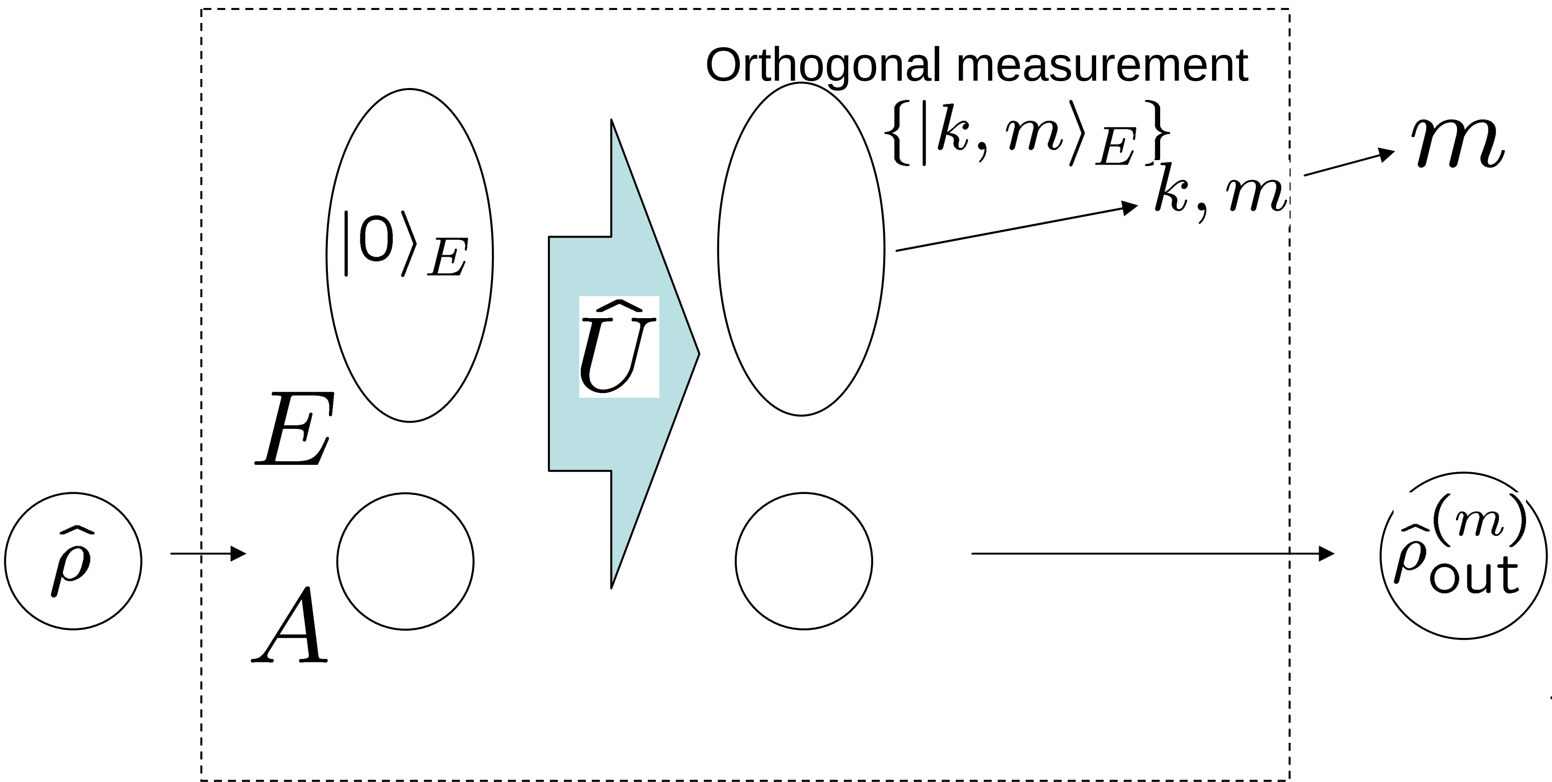
Applying $\sum_m \mathrm{Tr} \longrightarrow \sum_{m,k} {}_A\langle\phi|\hat{M}^{(k,m)\dagger}\hat{M}^{(k,m)}|\phi\rangle_A = 1 \longrightarrow \sum_{m,k}\hat{M}^{(k,m)\dagger}\hat{M}^{(k,m)} = \hat{1}_A$

This is what we can do in principle



Any physical process should be represented in the following form:

$$p_m \hat{\rho}_{\text{out}}^{(m)} = \sum_k \hat{M}^{(k,m)} \hat{\rho} \hat{M}^{(k,m)\dagger} \qquad \sum_{m,k} \hat{M}^{(k,m)\dagger} \hat{M}^{(k,m)} = \hat{1}_A$$



Universal NOT ? Spin reversal ?

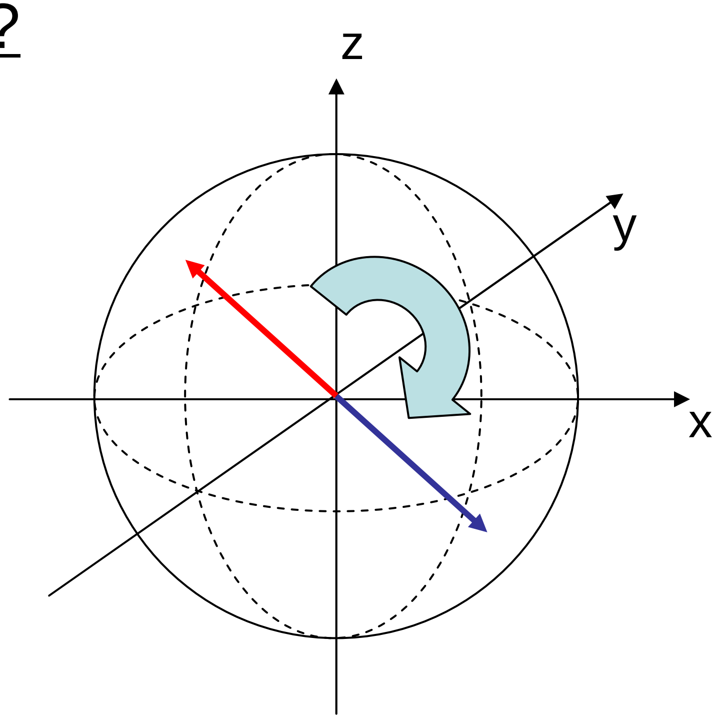
Bloch vector

$$\boldsymbol{P} \to -\boldsymbol{P}$$

linear map $\hat{\rho} \to \mathcal{C}(\hat{\rho})$

$$\mathcal{C}(\hat{1}) = \hat{1} \qquad \mathcal{C}(\hat{\sigma}_x) = -\hat{\sigma}_x$$

$$\mathcal{C}(\hat{\sigma}_y) = -\hat{\sigma}_y \qquad \mathcal{C}(\hat{\sigma}_z) = -\hat{\sigma}_z$$

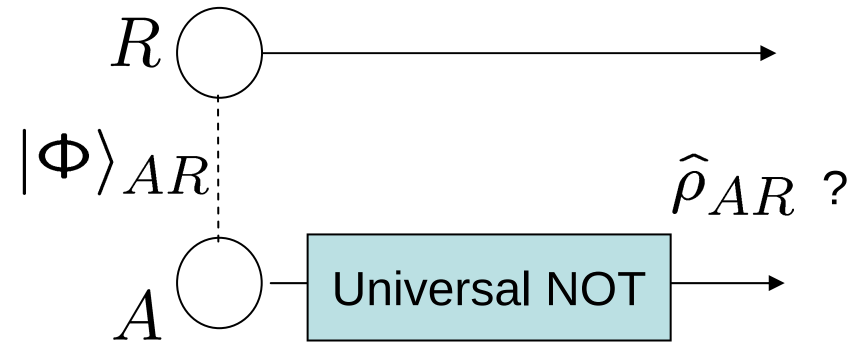


$$\mathcal{C}(|0\rangle\langle 0|) = |1\rangle\langle 1|$$
$$\mathcal{C}(|1\rangle\langle 1|) = |0\rangle\langle 0|$$
$$\mathcal{C}(|0\rangle\langle 1|) = -|0\rangle\langle 1|$$
$$\mathcal{C}(|1\rangle\langle 0|) = -|1\rangle\langle 0|$$

$$\hat{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$
$$\hat{\sigma}_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$
$$\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$
$$\hat{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

This map is positive, but...

Universal NOT ? Spin reversal ?



$$\mathcal{C}(|0\rangle\langle 0|) = |1\rangle\langle 1|$$

$$\mathcal{C}(|1\rangle\langle 1|) = |0\rangle\langle 0|$$

$$\mathcal{C}(|0\rangle\langle 1|) = -|0\rangle\langle 1|$$

$$\mathcal{C}(|1\rangle\langle 0|) = -|1\rangle\langle 0|$$

$$2|\Phi\rangle\langle\Phi| = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$
$$= |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|$$

$$2\widehat{\rho}_{AR} \equiv 2(\mathcal{C} \otimes \mathcal{I})|\Phi\rangle\langle\Phi| =$$
$$= |10\rangle\langle 10| - |00\rangle\langle 11| - |11\rangle\langle 00| + |01\rangle\langle 01|$$

$$2\widehat{\rho}_{AR}(|00\rangle + |11\rangle) = -|11\rangle - |00\rangle = -(|00\rangle + |11\rangle)$$

$\widehat{\rho}_{AR}$ has a negative eigenvalue! (The map is not completely positive.)

$\longrightarrow$ Universal NOT is impossible.

5. Distinguishability

Trace distance

- Trace norm and polar decomposition
- Minimum-error discrimination

Fidelity

- Local operations on a maximally entangled state
- Fidelity and distinguishability

Fidelity and trace distance

No-cloning theorem

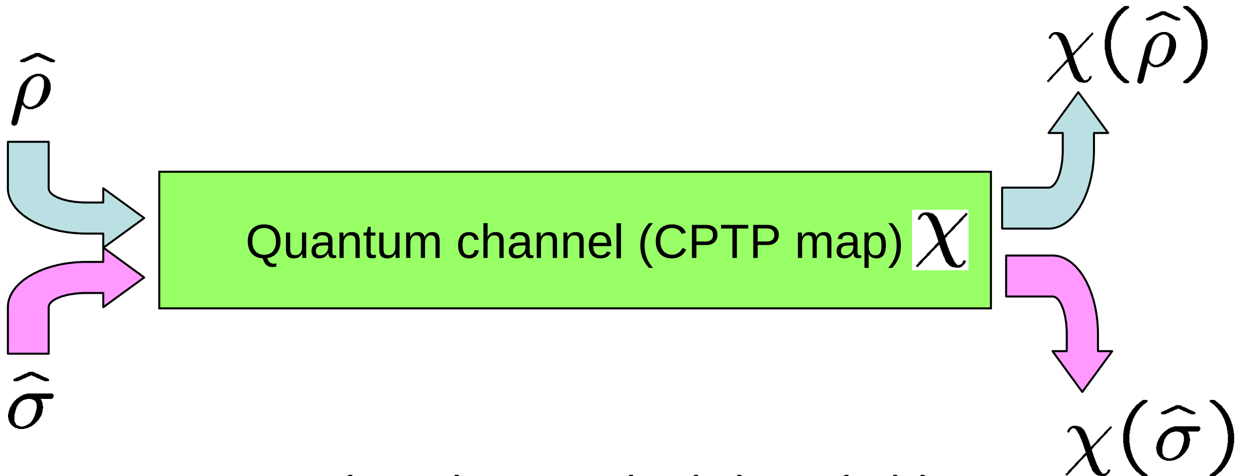
Distinguishability

Measure of distinguishability between two states $D(\hat{\rho}, \hat{\sigma})$

A quantity describing how we can distinguish between the two states in principle.

The distinguishability should never be improved by a quantum operation.

Monotonicity under quantum operations

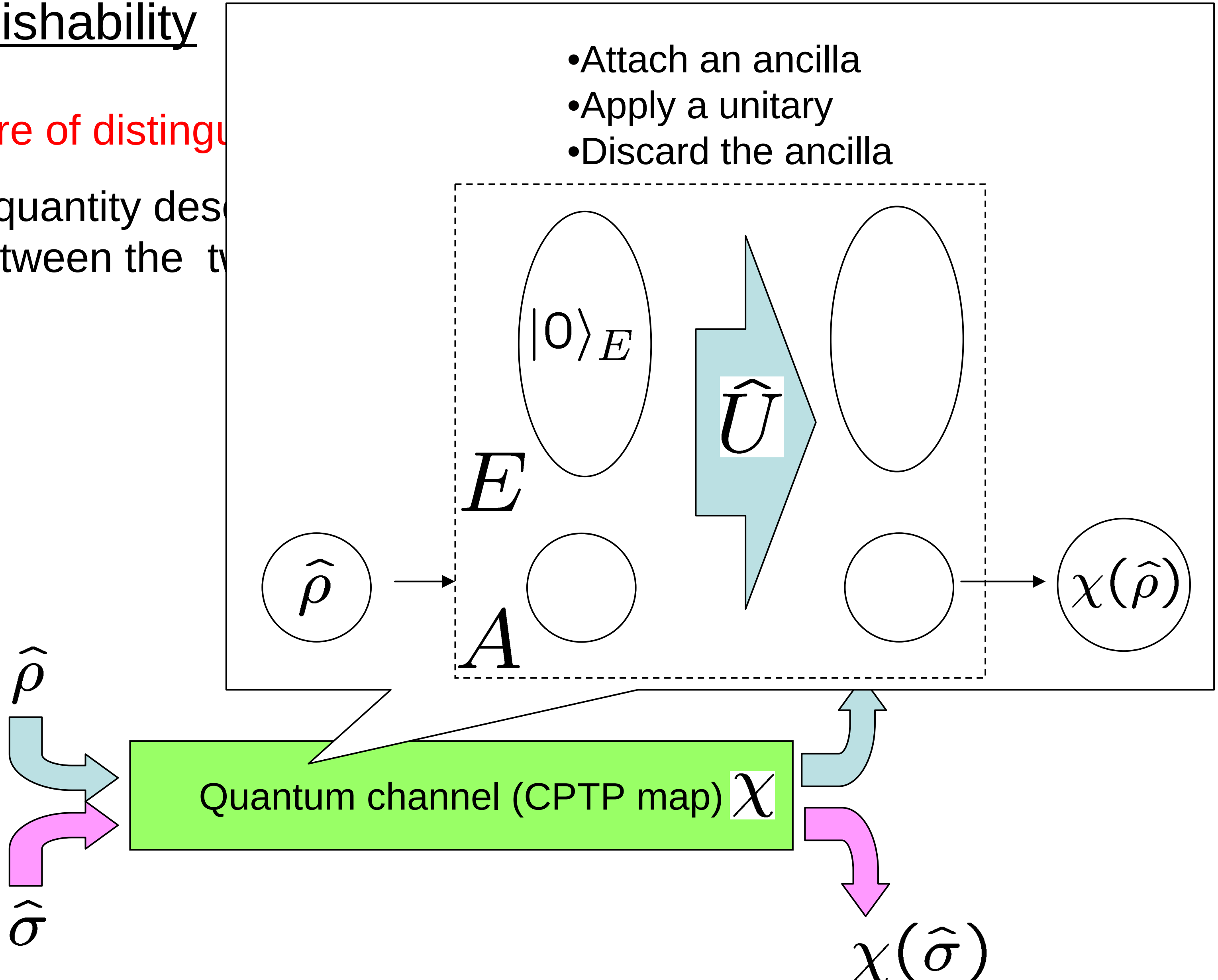


$$D(\hat{\rho}, \hat{\sigma}) \geq D(\chi(\hat{\rho}), \chi(\hat{\sigma}))$$

Distinguishability

Measure of distingu

A quantity desc
between the tw



$$D(\hat{\rho},\hat{\sigma}) \geq D(\chi(\hat{\rho}),\chi(\hat{\sigma}))$$

Trace norm

$$\|\hat{A}\| = \|\hat{A}\|_1 \equiv \mathrm{Tr}|\hat{A}| = \mathrm{Tr}\sqrt{\hat{A}^\dagger \hat{A}}$$

In particular, when $\hat{A}$ is normal (diagonalizable),

$\mathrm{Tr}(|\hat{A}|) = \sum_j |\lambda_j|$ $\quad \lambda_j$: Eigenvalues of $\hat{A}$

$$\|\hat{A}\| = \max_{\hat{U}} |\mathrm{Tr}(\hat{A}\hat{U})|$$

$|\hat{A}| = \sum_j \nu_j |j\rangle\langle j|$ $\qquad \|\hat{A}\| = \sum_j \nu_j$

$$\mathrm{Tr}(\hat{A}\hat{U}) = \mathrm{Tr}(\hat{V}|\hat{A}|\hat{U}) = \sum_j \nu_j \langle j|\hat{U}\hat{V}|j\rangle$$

$$|\langle j|\hat{U}\hat{V}|j\rangle| \le 1$$

$$\hat{U} = \hat{V}^\dagger \to |\langle j|\hat{U}\hat{V}|j\rangle| = 1$$

Polar decomposition

number $\qquad \alpha = e^{i\theta}|\alpha|$

linear operator $\qquad \hat{A} = \hat{V}|\hat{A}|$

unitary $\qquad$ positive

$$|\hat{A}| \equiv \sqrt{\hat{A}^\dagger \hat{A}}$$

$$\hat{V} \equiv \hat{A}|\hat{A}|^{-1}$$

(when $\hat{A}$ is invertible)

$$\hat{V}^\dagger \hat{V} = |\hat{A}|^{-1}\hat{A}^\dagger \hat{A}|\hat{A}|^{-1} = \hat{1}$$

Trace distance

$$\frac{1}{2}\|\hat{\rho}-\hat{\sigma}\|$$

Zero when $\hat{\rho}=\hat{\sigma}$ (the same state)

Unity when $\hat{\rho}\hat{\sigma}=0$ (perfectly distinguishable)

Monotonicity? $\boxed{\|\hat{\rho}-\hat{\sigma}\| \geq \|\chi(\hat{\rho})-\chi(\hat{\sigma})\|}$

- Attach an ancilla $\hat{\rho}\to\hat{\rho}\otimes\hat{\tau}$ $\hat{\sigma}\to\hat{\sigma}\otimes\hat{\tau}$

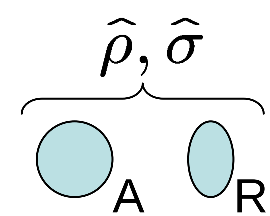
$$\mathrm{Tr}|\hat{A}\otimes\hat{B}|=\mathrm{Tr}(\sqrt{\hat{A}^\dagger\hat{A}}\otimes\sqrt{\hat{B}^\dagger\hat{B}})=\mathrm{Tr}|\hat{A}|\mathrm{Tr}|\hat{B}|$$

$$\|\hat{\rho}\otimes\hat{\tau}-\hat{\sigma}\otimes\hat{\tau}\|=\|(\hat{\rho}-\hat{\sigma})\otimes\hat{\tau}\|=\|\hat{\rho}-\hat{\sigma}\|\times\|\hat{\tau}\|=\|\hat{\rho}-\hat{\sigma}\|$$

- Apply a unitary $\hat{\rho}\to\hat{U}\hat{\rho}\hat{U}^\dagger$ $\hat{\sigma}\to\hat{U}\hat{\sigma}\hat{U}^\dagger$

$$\max_{\hat{V}}|\mathrm{Tr}(\hat{A}\hat{V})|=\max_{\hat{V}}|\mathrm{Tr}(\hat{U}\hat{A}\hat{U}^\dagger\hat{V})|$$

$$\|\hat{U}\hat{\rho}\hat{U}^\dagger-\hat{U}\hat{\sigma}\hat{U}^\dagger\|=\|\hat{U}(\hat{\rho}-\hat{\sigma})\hat{U}^\dagger\|=\|\hat{\rho}-\hat{\sigma}\|$$



- Discard the ancilla $\hat{\rho}\to\mathrm{Tr}_R(\hat{\rho})$ $\hat{\sigma}\to\mathrm{Tr}_R(\hat{\sigma})$

$$\max_{\hat{V}_A}|\mathrm{Tr}[(\mathrm{Tr}_R\hat{\rho}-\mathrm{Tr}_R\hat{\sigma})\hat{V}_A]|=\max_{\hat{V}_A}|\mathrm{Tr}[(\hat{\rho}-\hat{\sigma})(\hat{V}_A\otimes\hat{1}_R)]|$$

$$\leq\max_{\hat{U}_{AR}}|\mathrm{Tr}[(\hat{\rho}-\hat{\sigma})\hat{U}_{AR}]|$$

Trace distance

Monotonicity $$\boxed{\|\hat{\rho}-\hat{\sigma}\| \geq \|\chi(\hat{\rho})-\chi(\hat{\sigma})\|}$$

This rule also applies to a measurement with outcome j :

$\hat{\rho} \to \{p_j\}$ $\hat{\sigma} \to \{q_j\}$

$$\Leftrightarrow \hat{\rho}_{\text{mes}} \equiv \begin{pmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & p_3 & \\ 0 & & & \ddots \end{pmatrix} \qquad \Leftrightarrow \hat{\sigma}_{\text{mes}} \equiv \begin{pmatrix} q_1 & & & 0 \\ & q_2 & & \\ & & q_3 & \\ 0 & & & \ddots \end{pmatrix}$$

$$\frac{1}{2}\|\hat{\rho}-\hat{\sigma}\| \geq \frac{1}{2}\|\hat{\rho}_{\text{mes}}-\hat{\sigma}_{\text{mes}}\| = \frac{1}{2}\sum_j |p_j - q_j|$$

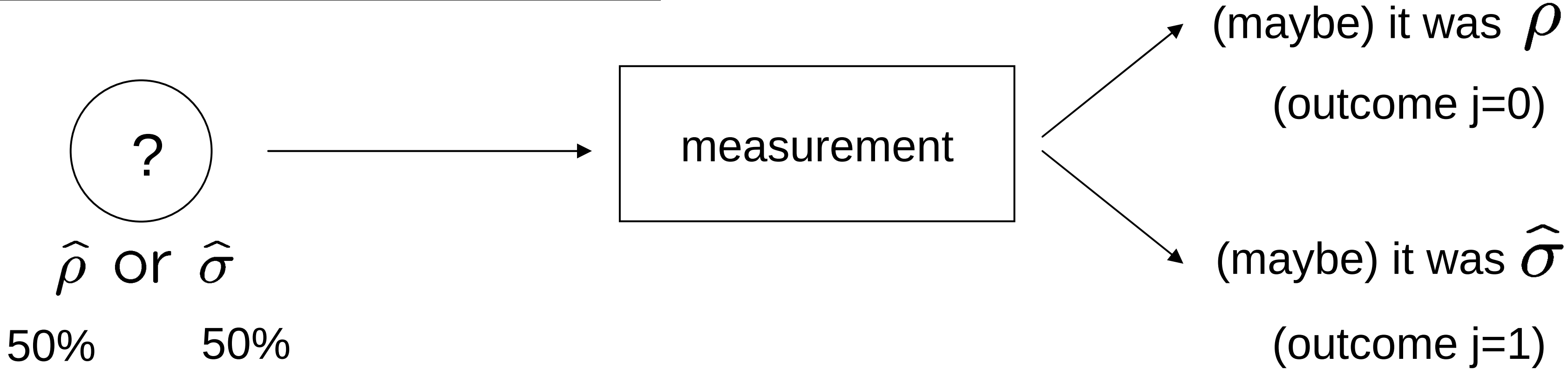
(total variation distance)

This must hold for any measurement

Note: The equality holds for the orthogonal measurement for the observable $\hat{\rho}-\hat{\sigma} = \sum \lambda_j |j\rangle\langle j|$

$$\frac{1}{2}\|\hat{\rho}-\hat{\sigma}\| = \frac{1}{2}\sum_j |p_j - q_j| = \sum_j |\lambda_j|$$

Minimum-error discrimination



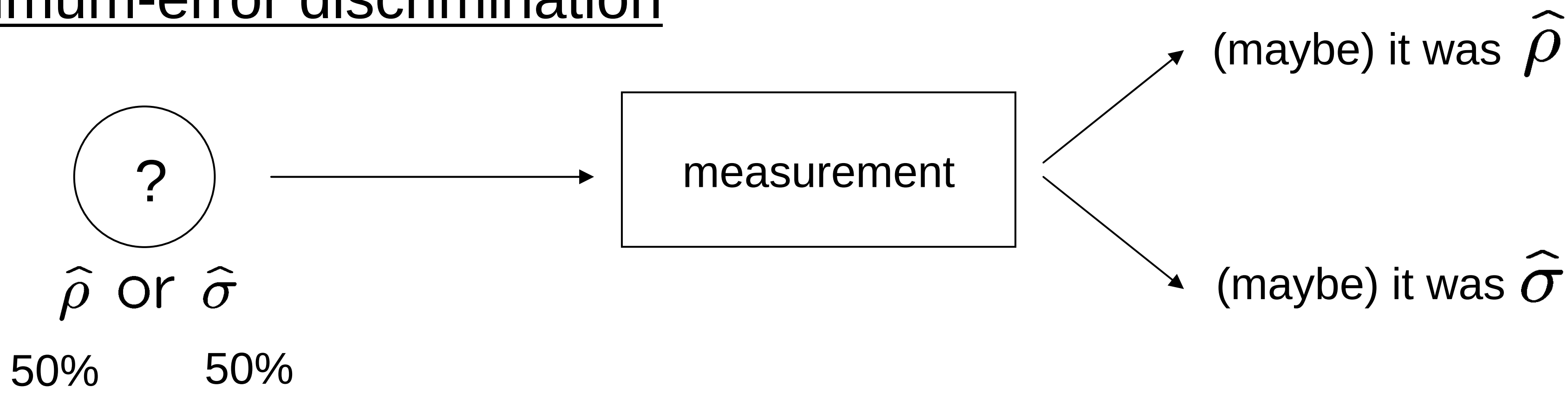
outcome

Input	It was $\hat{\rho}$ (j=0)	It was $\hat{\sigma}$ (j=1)
$\hat{\rho}$	$p_0 = 1 - \epsilon$	$p_1 = \epsilon$
$\hat{\sigma}$	$q_0 = \epsilon'$	$q_1 = 1 - \epsilon'$

$$p_{\mathrm{err}} = \frac{1}{2}(\epsilon + \epsilon')$$

$$\frac{1}{2}\|\hat{\rho} - \hat{\sigma}\| \geq \frac{1}{2}\sum_j |p_j - q_j| = |1 - \epsilon - \epsilon'| = 1 - 2p_{\mathrm{err}}$$

Minimum-error discrimination



Optimal measurement: orthogonal measurement $\{\hat{P}_0, \hat{P}_1\}$

$$\hat{\rho} - \hat{\sigma} = \sum_k \lambda_k |k\rangle\langle k|$$

$$\hat{P}_0 \equiv \sum_{k:\lambda_k \geq 0} |k\rangle\langle k| \qquad \hat{P}_1 \equiv \sum_{k:\lambda_k < 0} |k\rangle\langle k|$$

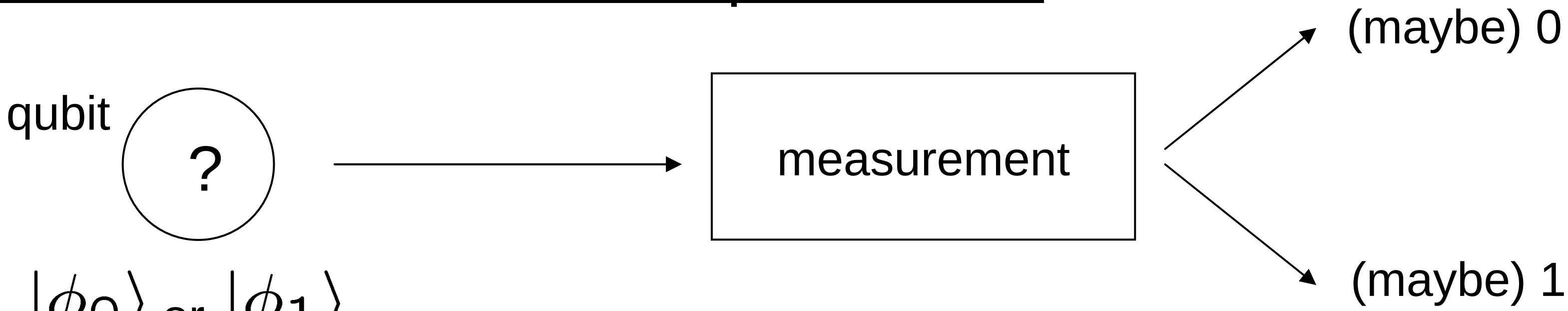
$$\frac{1}{2}(|p_0 - q_0| + |p_1 - q_1|) = \frac{1}{2}\left(\left|\sum_{k:\lambda_k \geq 0} \lambda_k\right| + \left|\sum_{k:\lambda_k < 0} \lambda_k\right|\right)$$

$$= \frac{1}{2}\sum_k |\lambda_k| = \frac{1}{2}\|\hat{\rho} - \hat{\sigma}\|$$

$$\boxed{\frac{1}{2}\|\hat{\rho} - \hat{\sigma}\| = 1 - 2p_{\mathrm{err}}^{(\mathrm{opt})}}$$

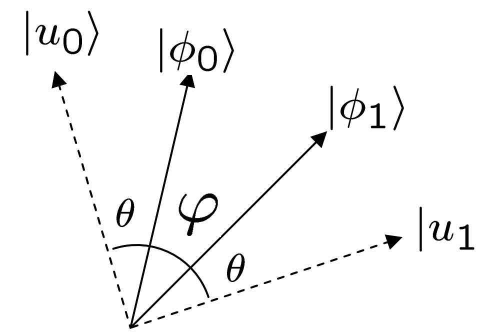
Operational meaning of the trace distance

Discrimination between two pure states



$|\phi_0\rangle$ or $|\phi_1\rangle$

50% 50%



$$\langle\phi_0|\phi_1\rangle = s = \cos\varphi = \sin 2\theta > 0$$

$$|\phi_0\rangle = \cos\theta|u_0\rangle + \sin\theta|u_1\rangle$$
$$|\phi_1\rangle = \sin\theta|u_0\rangle + \cos\theta|u_1\rangle$$

$$p_{\text{err}}^{(\text{opt})} = \sin^2\theta = \frac{1-\sqrt{1-s^2}}{2}$$

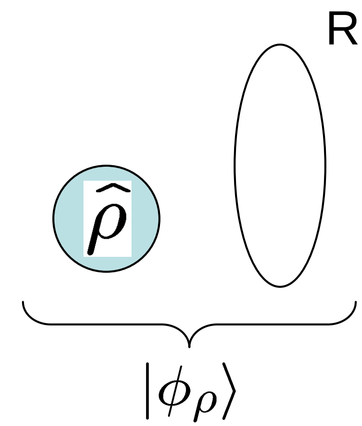
$$\hat{\rho}_0 := |\phi_0\rangle\langle\phi_0|$$
$$\hat{\rho}_1 := |\phi_1\rangle\langle\phi_1|$$

$$\hat{\rho}_0 - \hat{\rho}_1 = \cos 2\theta(|u_0\rangle\langle u_0| - |u_1\rangle\langle u_1|)$$

$$\frac{1}{2}\|\hat{\rho}_0 - \hat{\rho}_1\| = \cos 2\theta = \sqrt{1-s^2} = 1 - 2p_{\text{err}}^{(\text{opt})}$$

Fidelity

$$\boxed{F(\hat{\rho},\hat{\sigma}) \equiv \max |\langle\phi_\rho|\phi_\sigma\rangle|^2}$$



$$\mathrm{Tr}_R[|\phi_\rho\rangle\langle\phi_\rho|] = \hat{\rho}$$
$$\mathrm{Tr}_R[|\phi_\sigma\rangle\langle\phi_\sigma|] = \hat{\sigma}$$
(purifications)

$$\boxed{F(\hat{\rho},\hat{\sigma}) = \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|^2}$$

$|\psi_\rho\rangle \equiv \sum_k \sqrt{\hat{\rho}}|k\rangle \otimes |k\rangle_R$ is a purification of $\hat{\rho}$

$$\mathrm{Tr}_R|\psi_\rho\rangle\langle\psi_\rho| = \sum_{kl} \sqrt{\hat{\rho}}|k\rangle\langle l|\sqrt{\hat{\rho}} \times \mathrm{Tr}(|k\rangle_{RR}\langle l|) = \hat{\rho}$$

Any purification can be written as $|\phi_\rho\rangle = \sum_k \sqrt{\hat{\rho}}|k\rangle \otimes \hat{U}_R|k\rangle_R$

$$= \sum_k \sqrt{\hat{\rho}}\hat{U}'|k\rangle \otimes |k\rangle_R$$

$$F(\hat{\rho},\hat{\sigma}) = \max_{\hat{U},\hat{V}} \left| \sum_{kl} \langle k|\hat{U}^\dagger \sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\hat{V}|l\rangle \times {}_R\langle k|l\rangle_R \right|^2$$

$$= \max_{\hat{U},\hat{V}} \left| \mathrm{Tr}(\hat{U}^\dagger \sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\hat{V}) \right|^2 = \max_{\hat{V}} \left| \mathrm{Tr}(\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\hat{V}) \right|^2$$

Local operations on a maximally entangled state

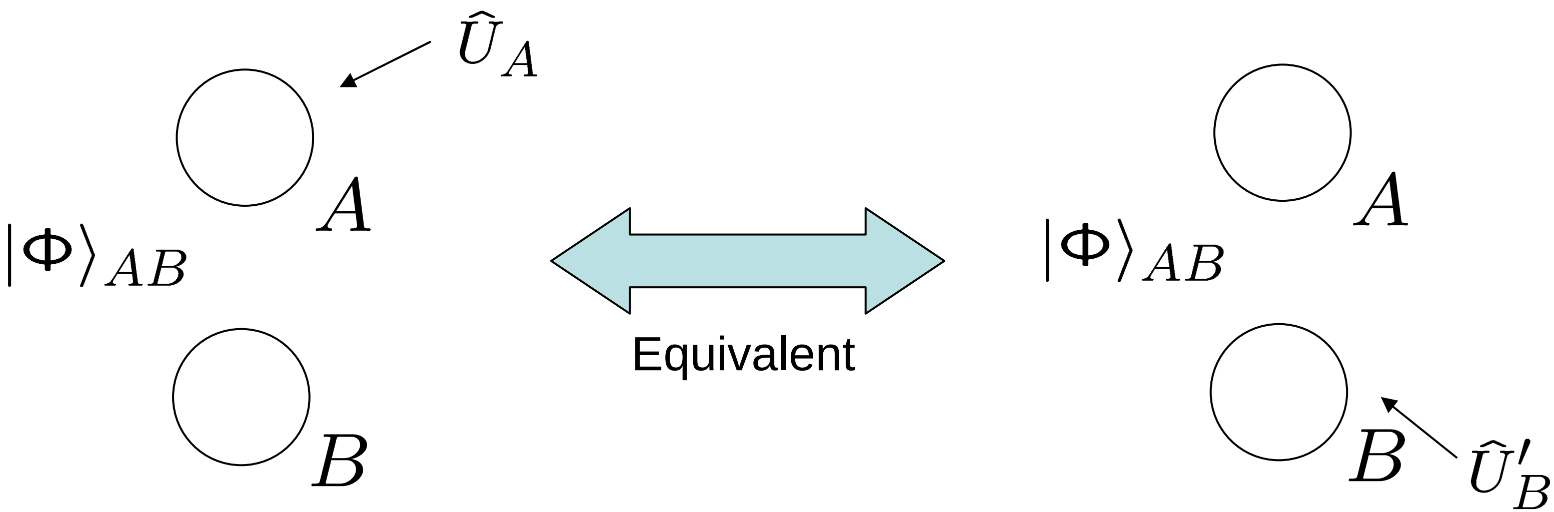
$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

$$(\hat{T}_A \otimes \hat{1}_B)|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{T}'_B)|\Phi\rangle_{AB}$$

${}_A\langle l| \otimes {}_B\langle k|$

$${}_A\langle l|\hat{T}_A|k\rangle_A = {}_B\langle k|\hat{T}'_B|l\rangle_B$$

transpose



Fidelity

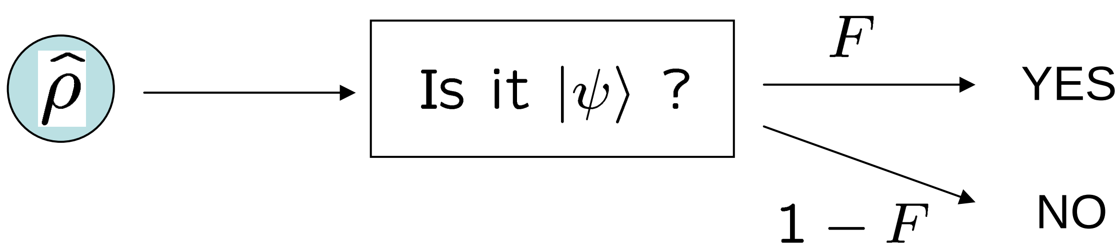
$$F(\hat{\rho},\hat{\sigma}) \equiv \max|\langle\phi_\rho|\phi_\sigma\rangle|^2 = \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\hat{\sigma}}\hat{\rho}\sqrt{\hat{\sigma}}}\right)^2$$

$F(\hat{\rho},\hat{\sigma}) = 1$ when $\hat{\rho} = \hat{\sigma}$ (the same state)

$F(\hat{\rho},\hat{\sigma}) = 0$ when $\hat{\rho}\hat{\sigma} = 0$ (perfectly distinguishable)

$$F(\hat{\rho},|\psi\rangle\langle\psi|) = \langle\psi|\hat{\rho}|\psi\rangle$$

$$\mathrm{Tr}\sqrt{\sqrt{|\psi\rangle\langle\psi|}\hat{\rho}\sqrt{|\psi\rangle\langle\psi|}} = \mathrm{Tr}\sqrt{\langle\psi|\hat{\rho}|\psi\rangle|\psi\rangle\langle\psi|} = \sqrt{\langle\psi|\hat{\rho}|\psi\rangle}$$



Operational meaning of the fidelity

Fidelity

$$F(\hat{\rho},\hat{\sigma}) \equiv \max|\langle\phi_\rho|\phi_\sigma\rangle|^2 = \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\hat{\sigma}}\hat{\rho}\sqrt{\hat{\sigma}}}\right)^2$$

$F(\hat{\rho},\hat{\sigma}) = 1$ when $\hat{\rho} = \hat{\sigma}$ (the same state)

$F(\hat{\rho},\hat{\sigma}) = 0$ when $\hat{\rho}\hat{\sigma} = 0$ (perfectly distinguishable)

$F(\hat{\rho},|\psi\rangle\langle\psi|) = \langle\psi|\hat{\rho}|\psi\rangle$ Operational meaning of the fidelity

But not applicable to general $F(\hat{\rho},\hat{\sigma})$

$F(|\phi\rangle\langle\phi|,|\psi\rangle\langle\psi|) = |\langle\phi|\psi\rangle|^2$ Direct generalization of the magnitude of the inner product

$F(\hat{\rho}_1\otimes\hat{\rho}_2,\hat{\sigma}_1\otimes\hat{\sigma}_2) = F(\hat{\rho}_1,\hat{\sigma}_1)F(\hat{\rho}_2,\hat{\sigma}_2)$ Multiplicativity

$1 - F(\hat{\rho},\hat{\sigma})$ is a measure of distinguishability. (not a distance)

Classical case $\hat{\rho}\to\{p_j\}$ $\hat{\sigma}\to\{q_j\}$

$$F = \left(\sum_j \sqrt{p_j}\sqrt{q_j}\right)^2$$

Hard to find a operational meaning…

There exists a measurement that preserves the fidelity: Measure σ/ρ

Projection to the range of $\hat{\rho}$

Measure the observable $\hat{\rho}^{-1/2}|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}|\hat{\rho}^{-1/2}$

Fidelity and distinguishability

$$F(\hat\rho,\hat\sigma) \equiv \max |\langle \phi_\rho | \phi_\sigma \rangle|^2 = \|\sqrt{\hat\rho}\sqrt{\hat\sigma}\|^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\hat\sigma}\hat\rho\sqrt{\hat\sigma}}\right)^2$$

$$F(\hat\rho,\hat\sigma) = 1 \text{ when } \hat\rho = \hat\sigma \qquad F(\hat\rho,\hat\sigma) = 0 \text{ when } \hat\rho\hat\sigma = 0$$

$1 - F(\hat\rho,\hat\sigma)$ is a measure of distinguishability. (not a distance)

Monotonicity $\boxed{F(\hat\rho,\hat\sigma) \le F(\chi(\hat\rho),\chi(\hat\sigma))}$

- Attach an ancilla

$$F(\hat\rho\otimes\hat\tau,\hat\sigma\otimes\hat\tau) = F(\hat\rho,\hat\sigma)F(\hat\tau,\hat\tau) = F(\hat\rho,\hat\sigma)$$

- Apply a unitary

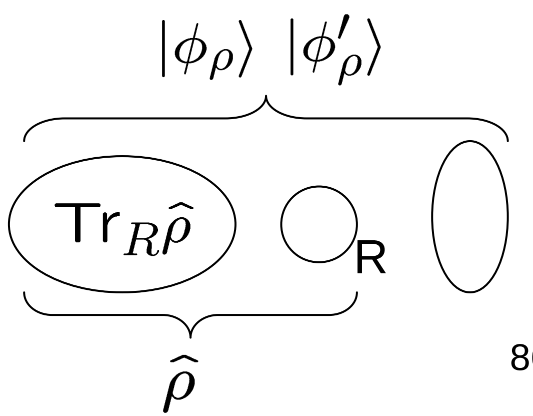
$$F(\hat U\hat\rho\hat U^\dagger,\hat U\hat\sigma\hat U^\dagger) = \|\hat U\sqrt{\hat\rho}\sqrt{\hat\sigma}\hat U^\dagger\|^2 = \|\sqrt{\hat\rho}\sqrt{\hat\sigma}\|^2 = F(\hat\rho,\hat\sigma)$$

- Discard the ancilla

$$F(\hat\rho,\hat\sigma) = \max |\langle \phi_\rho | \phi_\sigma \rangle|^2$$

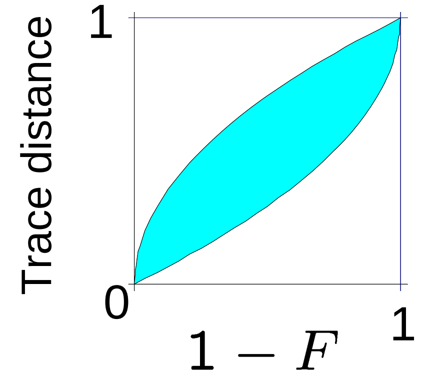
$$F(\mathrm{Tr}_R\hat\rho,\mathrm{Tr}_R\hat\sigma) = \max |\langle \phi'_\rho | \phi'_\sigma \rangle|^2$$

$$\max |\langle \phi_\rho | \phi_\sigma \rangle|^2 \le \max |\langle \phi'_\rho | \phi'_\sigma \rangle|^2$$



Fidelity and trace distance

$$1-\sqrt{F(\hat\rho,\hat\sigma)} \le \frac{1}{2}\|\hat\rho-\hat\sigma\| \le \sqrt{1-F(\hat\rho,\hat\sigma)}$$



Measurement preserving the fidelity

$$\hat\rho \to \{p_j\} \quad \hat\sigma \to \{q_j\}$$

$$\frac{1}{2}\|\hat\rho-\hat\sigma\| \ge \frac{1}{2}\sum_j |p_j - q_j|$$

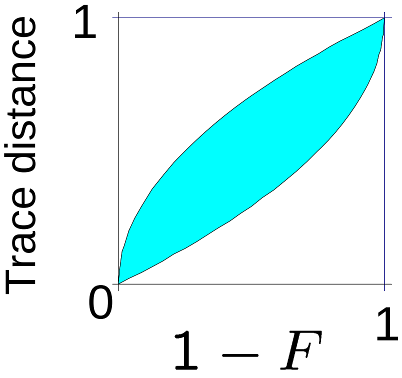
$$=\frac{1}{2}\sum_j |\sqrt{p_j}-\sqrt{q_j}|(\sqrt{p_j}+\sqrt{q_j})$$

$$\ge \frac{1}{2}\sum_j(\sqrt{p_j}-\sqrt{q_j})^2 = \frac{1}{2}\left(\sum_j p_j + \sum_j q_j - 2\sum_j \sqrt{p_j}\sqrt{q_j}\right)$$

$$=1-\sqrt{F}$$

Fidelity and trace distance

$$1-\sqrt{F(\hat{\rho},\hat{\sigma})}\le\frac{1}{2}\|\hat{\rho}-\hat{\sigma}\|\le\sqrt{1-F(\hat{\rho},\hat{\sigma})}$$



Purifications satisfying $F(\hat{\rho},\hat{\sigma})=|\langle\phi_\rho|\phi_\sigma\rangle|^2$

The fidelity is preserved in the physical process $|\phi_\rho\rangle\to\hat{\rho}$, $|\phi_\sigma\rangle\to\hat{\sigma}$

$$\frac{1}{2}\||\phi_\rho\rangle\langle\phi_\rho|-|\phi_\sigma\rangle\langle\phi_\sigma|\|\ge\frac{1}{2}\|\hat{\rho}-\hat{\sigma}\|$$

$$\|$$

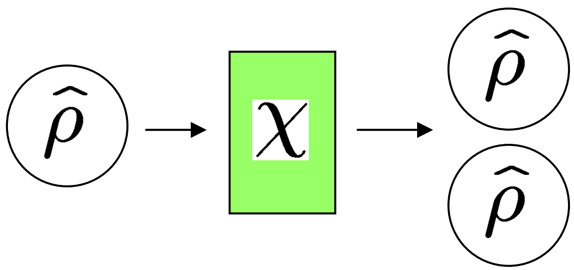
$$\sqrt{1-|\langle\phi_\rho|\phi_\sigma\rangle|^2}=\sqrt{1-F}$$

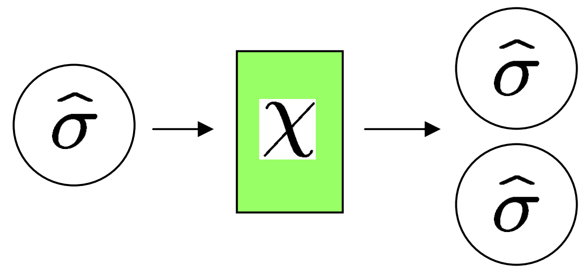
No-cloning theorem

$$F(\hat{\rho}_1 \otimes \hat{\rho}_2, \hat{\sigma}_1 \otimes \hat{\sigma}_2) = F(\hat{\rho}_1, \hat{\sigma}_1) F(\hat{\rho}_2, \hat{\sigma}_2)$$ Multiplicativity

$$F(\hat{\rho}, \hat{\sigma}) \leq F(\chi(\hat{\rho}), \chi(\hat{\sigma}))$$ Monotonicity

Is it possible to realize $\begin{aligned} \chi(\hat{\rho}) &= \hat{\rho} \otimes \hat{\rho} \\ \chi(\hat{\sigma}) &= \hat{\sigma} \otimes \hat{\sigma} \end{aligned}$?





$$F(\hat{\rho}, \hat{\sigma}) \leq F(\chi(\hat{\rho}), \chi(\hat{\sigma})) = F(\hat{\rho} \otimes \hat{\rho}, \hat{\sigma} \otimes \hat{\sigma}) = F(\hat{\rho}, \hat{\sigma})^2$$

Possible only when $F(\hat{\rho}, \hat{\sigma}) = 0$ or 1

It is impossible to create independent copies of two inputs that are neither distinguishable nor identical.

No-cloning theorem for classical case?

It is impossible to create independent copies of two inputs that are neither distinguishable nor identical.

If we allow mixed states, partial distinguishability is not rare even in classical states.

$$\hat{\rho} = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \qquad \hat{\sigma} = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$$

It is possible to create correlated copies. (Broadcasting)

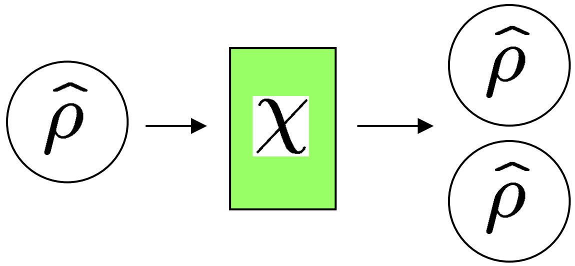
$$\chi(\hat{\rho}) = \frac{2}{3}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

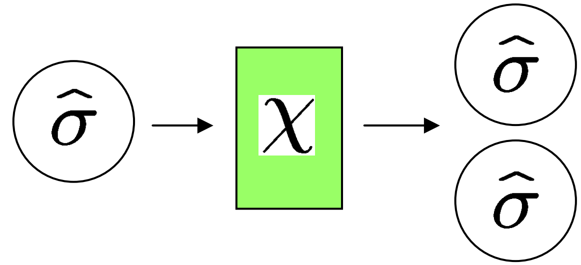
$$\chi(\hat{\sigma}) = \frac{1}{3}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

The marginal states are the same as the input.

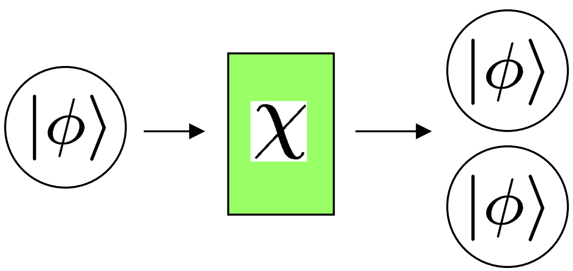
No-cloning theorem for pure states

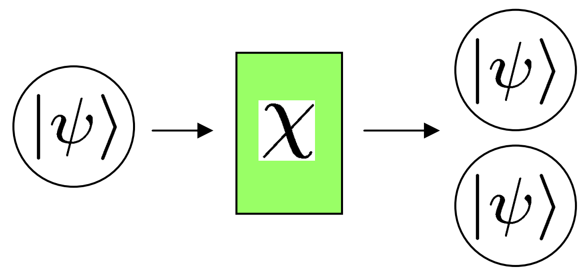
It is impossible to create independent copies of two inputs that are neither distinguishable nor identical.





If the marginal state is pure, the subsystem has no correlation to other systems.





It is impossible to create copies of two nonorthogonal and nonidentical pure states.

Of course, it is impossible to create copies of unknown pure states.

What is peculiar about quantum mechanics?

Partially distinguishable $\longrightarrow$ No independent copies

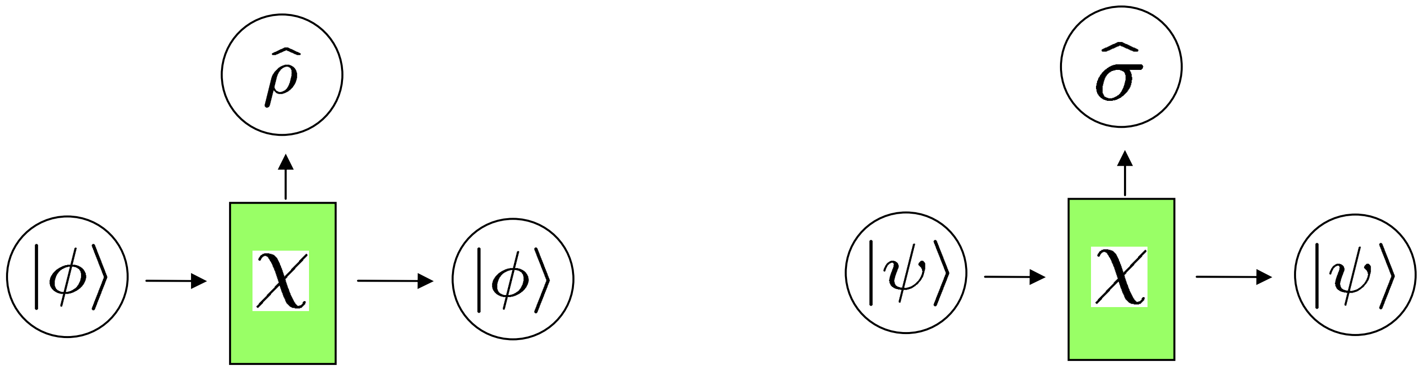
Pure $\longrightarrow$ No correlation

These implications are not unique to quantum mechanics.

In quantum mechanics, there are cases where states are partially distinguish and pure.

Information – disturbance tradeoff

Suppose that $|\langle\phi|\psi\rangle| > 0$



$$|\langle\phi|\psi\rangle|^2 \le F(|\phi\rangle\langle\phi| \otimes \hat{\rho}, |\psi\rangle\langle\psi| \otimes \hat{\sigma})$$
$$= |\langle\phi|\psi\rangle|^2 F(\hat{\rho}, \hat{\sigma})$$

$$\longrightarrow F(\hat{\rho}, \hat{\sigma}) = 1 \qquad \hat{\rho} = \hat{\sigma}$$

If a process causes absolutely no disturbance on two nonorthogonal states, it leaves no trace about which of the states has been fed to the input.

Basic principle for a quantum cryptography scheme, called B92 protocol.

6. Communication resources

Classical channel

Quantum channel

Entanglement

- How does the state evolve under LOCC?
- Properties of maximally entangled states
- Bell basis

Quantum dense coding

Quantum teleportation

- Entanglement swapping

Resource conversion protocols and bounds

Classical channel

0 - - - - - - - - - - → 0
1   1

Ideal classical channel: faithful transfer of any signal chosen from d symbols

Parallel use of channels

0	- - - →	0		0a	0a
1		1		0b	0b
				0c	- - - → 0c
a		a		1a	1a
b	- - - →	b		1b	1b
c		c		1c	1c

d-symbol ideal classical channel
d'-symbol ideal classical channel
(dd')-symbol ideal classical channel

Measure of usefulness

d-symbol ideal classical channel → (log d) bits

Additive for ideal channels

Quantum channel

$\alpha|0\rangle + \beta|1\rangle$ $\alpha|0\rangle + \beta|1\rangle$

Ideal quantum channel: faithful transfer of any state (including unknown states) of an d-level system (Hilbert space of dimension d)

Faithful transfer of any state $\longrightarrow$ Faithful transfer of any correlation

$\hat{\rho}_A^{(\text{in})}$

$\rho_A^{(\text{out})}$

$$\left.\begin{array}{l} \hat{\rho}_A^{(\text{out})} = \sum_j \hat{M}_j \hat{\rho}_A^{(\text{in})} \hat{M}_j^\dagger \\ \hat{\rho}_A^{(\text{out})} = \hat{\rho}_A^{(\text{in})} \quad \text{for any input} \end{array}\right\} \longrightarrow \hat{M}_j \propto \hat{1}_A$$

$\hat{\rho}_{AR}^{(\text{in})}$

$\hat{\rho}_{AR}^{(\text{out})}$

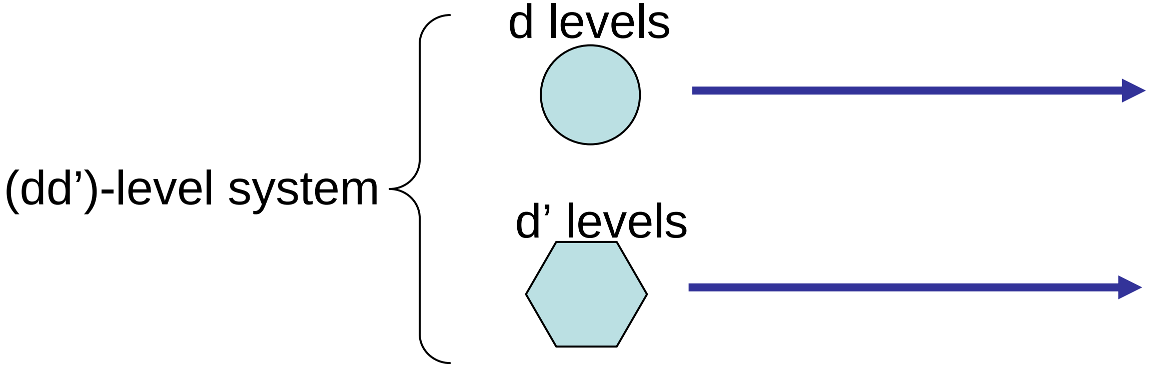
$$\hat{\rho}_{AR}^{(\text{out})} = \sum_j (\hat{M}_j \otimes \hat{1}_R)\hat{\rho}_{AR}^{(\text{in})}(\hat{M}_j \otimes \hat{1}_R)^\dagger = \hat{\rho}_{AR}^{(\text{in})}$$

Quantum channel

$\alpha|0\rangle + \beta|1\rangle$ $\qquad$ $\alpha|0\rangle + \beta|1\rangle$

Ideal quantum channel: faithful transfer of any state of an d-level system (Hilbert space of dimension d)

Parallel use of channels



Measure of usefulness

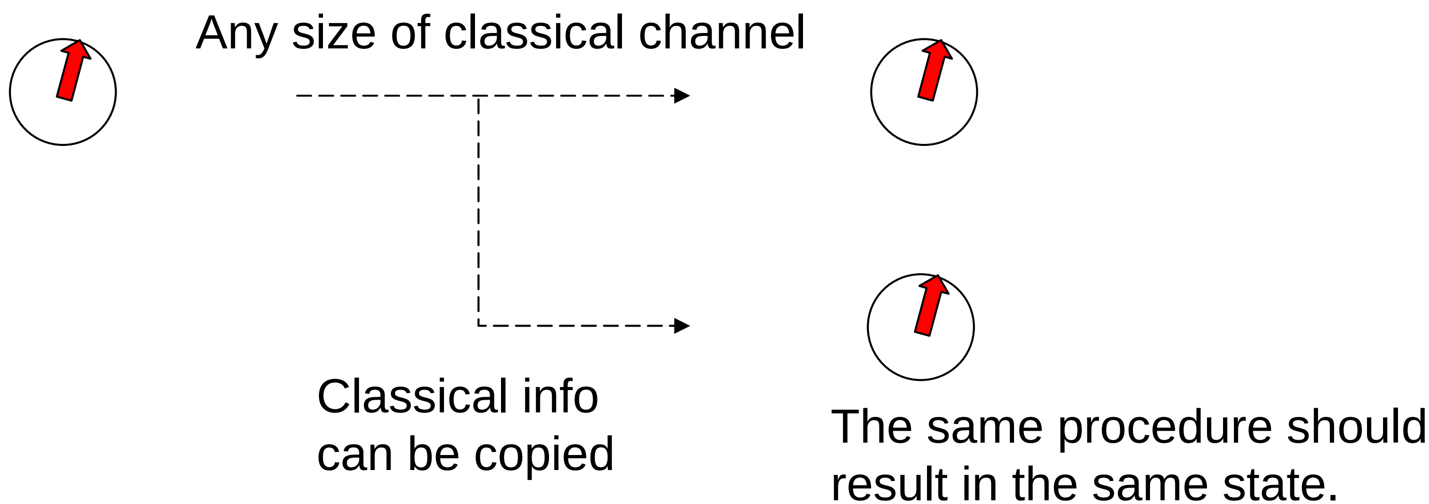
d-level ideal quantum channel $\longrightarrow$ (log d) qubits

Additive for ideal channels

Can classical channels substitute a quantum channel?

NO (with no other resources)

Suppose that it was possible …



This amounts to the cloning of unknown quantum states,
which is forbidden.

Can a quantum channel substitute a classical channel?

Of course yes.

But not so bizarre (with no other resources).

n-qubit ideal quantum channel can only substitute a n-bit classical channel.

(Holevo bound)

Suppose that transfer of an d-level system can convey any signal from s symbols faithfully.

Measurement

$$j = 1, 2, \ldots, s \qquad \hat{\rho}_j \longrightarrow \hat{\rho}_j \longrightarrow j'$$

$$\dim \mathcal{H} = d$$

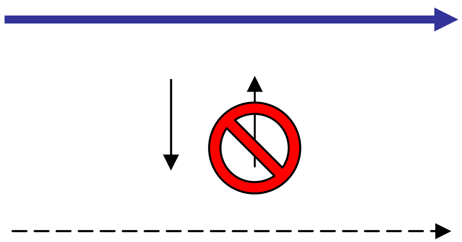
Always $j' = j$

Recall that any measurement must be described by a POVM. $\sum_{j'} \hat{F}_{j'} = \hat{1}$

$$\mathrm{Tr}(\hat{F}_j \hat{\rho}_j) = 1$$

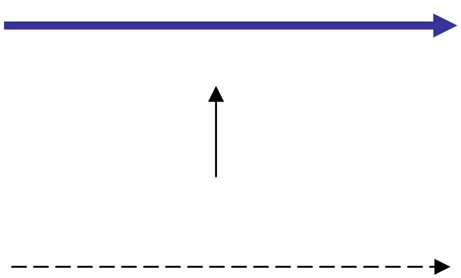
$$s = \sum_j \mathrm{Tr}(\hat{F}_j \hat{\rho}_j) \le \sum_j \mathrm{Tr}(\hat{F}_j \hat{1}) = \sum_j \mathrm{Tr}(\hat{F}_j) \le \sum_{j'} \mathrm{Tr}(\hat{F}_{j'}) = \mathrm{Tr}(\hat{1}) = d$$

Difference between quantum and classical channels



We have seen that a quantum channel is more powerful than a classical channel.

Can we pin down what is missing in a classical channel?



+
Other resource
Entanglement

I've already bought a classical channel, but now I want to use a quantum channel. Do I have to buy the quantum channel?

Oh, you can buy this optional package for a cheaper price, and upgrade the classical channel to a quantum channel!

Operational definition of entanglement

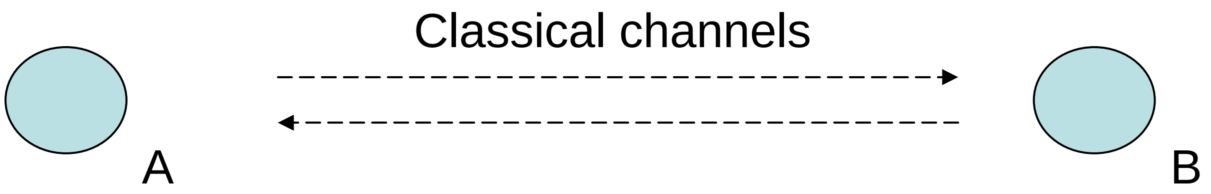
"Correlations that cannot be created over classical channels"

LOCC: Local operations and classical communication

Alice has a subsystem A, and Bob has a subsystem B.

Operations (including measurements) on a local subsystem are free.

Communication between Alice and Bob only uses classical channels.



Separable states: The states that can be created under LOCC from scratch.

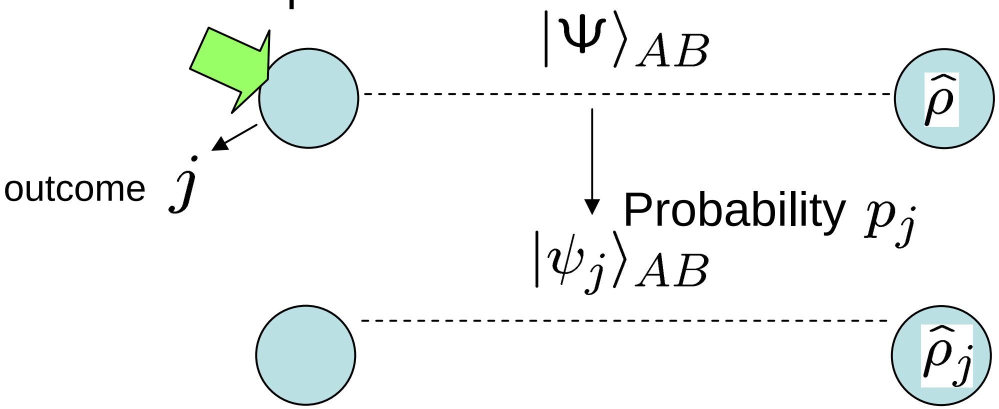
Entangled states: The states that cannot be created under LOCC from scratch.

How does the state evolve under LOCC?

Any LOCC procedure can be made a sequential one:

- Alice appies local operations
- Alice communicates to Bob
- Bob applies local operations
- Bob communicates to Alice
- Alice .....

When Alice operates



$$\boxed{\sum_j p_j \hat{\rho}_j = \hat{\rho}}$$

Ran $\hat{\rho} \supset$ Ran $\hat{\rho}_j$

Schmidt number never increases under LOCC (even probabilistically)

Schmidt number >1 $\longrightarrow$ Impossible under LOCC

If a concave functional S only depends on the eigenvalues,

$$S(\hat{\rho}) \geq \sum_j p_j S(\hat{\rho}_j)$$

Any such functional of the marginal density operator (e.g., von Neumann entropy) is monotone decreasing under LOCC on average.

Maximally entangled states (MES)

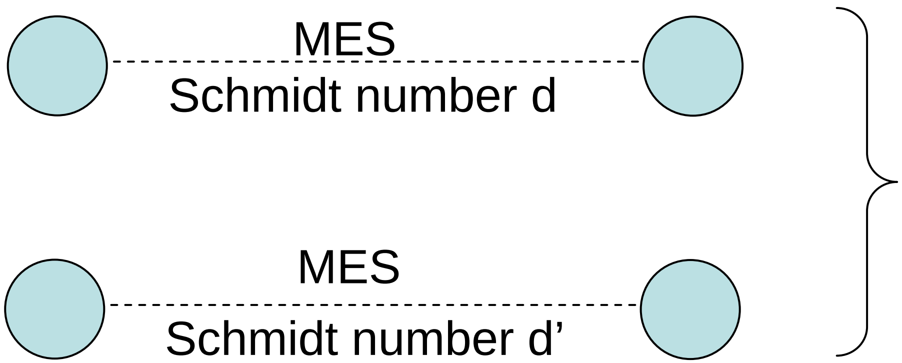
"ideal" entangled states

$$\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

$$p_1 = p_2 = \cdots = p_d = \frac{1}{d}$$

Schmidt number = d

Putting two MESs together



MES with

Schmidt number dd'

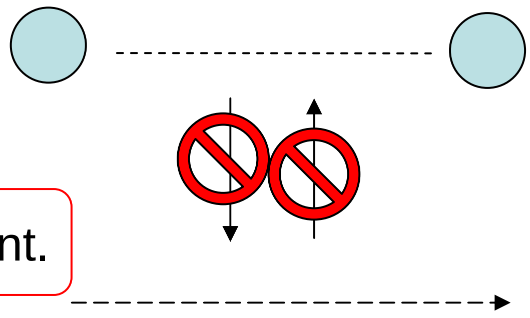
$$\left(\sum_{j=1}^{d} \frac{1}{\sqrt{d}} |j\rangle_A \otimes |j\rangle_B \right) \otimes \left(\sum_{k=1}^{d'} \frac{1}{\sqrt{d'}} |k\rangle_{A'} \otimes |k\rangle_{B'} \right) = \sum_{j,k} \frac{1}{\sqrt{dd'}} |jk\rangle_{AA'} \otimes |jk\rangle_{BB'}$$

Measure of entanglement

MES with Schmidt number d $\longrightarrow$ (log d) ebits

Additive for MESs

Ebits and bits are mutually exclusive



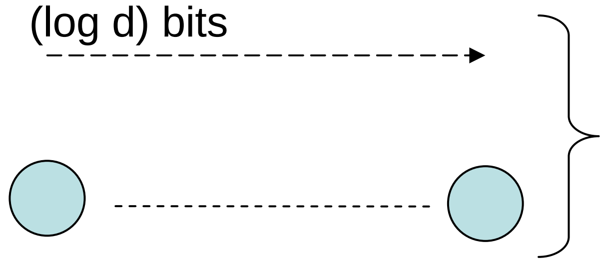
Schmidt number never increases under LOCC.

Classical channels cannot increase (ideal) entanglement.

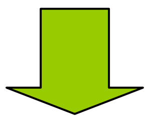
d-symbol ideal classical channel

The outcome can be correctly predicted with probability at least 1/d.

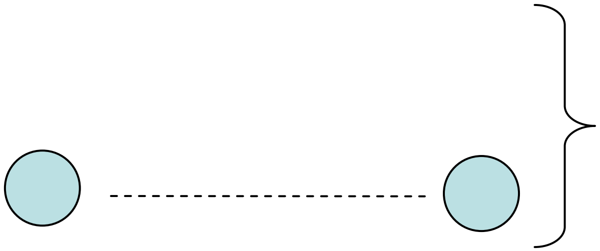
Transfer of d’(>d) symbols



Success probability 1



Transfer of d’(>d) symbols



Success probability 1/d

Entanglement cannot assist (ideal) classical channels

Resource conversion protocols

Conversion to ebits

Entanglement sharing

1 qubit → 1 ebit

Conversion to bits

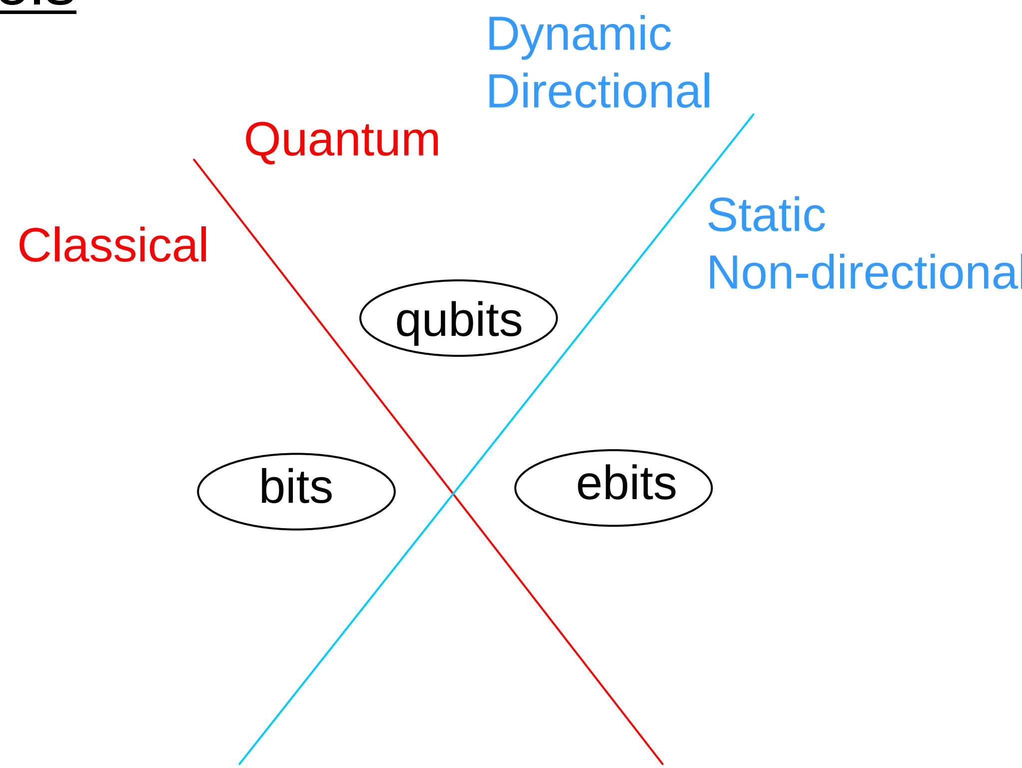
Quantum dense coding

1 qubit + 1 ebit → 2 bits

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit → 1 qubit



Restrictions

bits alone → no ebits

ebits alone → no bits

1 qubit alone → no more than 1 bit

Properties of maximally entangled states

$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

Pair of local states (relative states)

$$\boxed{\frac{1}{\sqrt{d}} |\phi\rangle_A = {}_B\langle\phi^*||\Phi\rangle_{AB}}$$

$$|\phi\rangle_A = \textstyle\sum_k \alpha_k |k\rangle_A$$

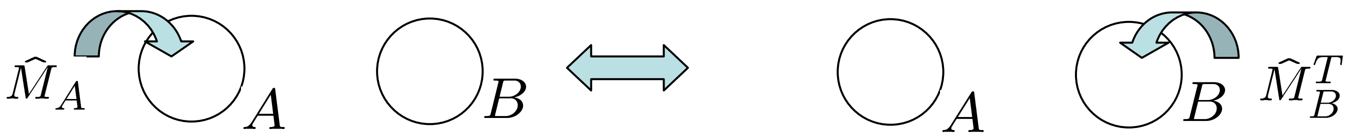
$$|\phi^*\rangle_B = \textstyle\sum_k \overline{\alpha_k} |k\rangle_B$$

measurement

$p = 1/d$

Pair of local operations

$$\boxed{(\hat{M}_A \otimes \hat{1}_B)|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{M}_B^T)|\Phi\rangle_{AB}}$$



Locally maximally mixed

$$\boxed{\hat{\rho}_A = \mathrm{Tr}_B |\Phi\rangle\langle\Phi| = \frac{1}{d}\hat{1}_A}$$

Convertibility via local unitary

$$\boxed{|\Phi'\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Phi\rangle_{AB}}$$

Orthonormal basis (Bell basis)

$$\boxed{\langle\Phi_j|\Phi_k\rangle = \delta_{jk} \; (j,k = 1,\ldots d^2)}$$

There exists an orthonormal basis composed of MESs.

Bell basis for a pair of qubits

$(d = 2)$

$\hat{Z} \equiv \hat{\sigma}_z, \ \hat{X} \equiv \hat{\sigma}_x$

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B) = \hat{Z}_B|\Phi_+\rangle$$

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B) = \hat{X}_A|\Phi_+\rangle$$

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B) = (\hat{X}_A \otimes \hat{Z}_B)|\Phi_+\rangle$$

Bell basis

$$\beta \equiv \exp[2\pi i/d] \quad (\beta^d = \beta^0 = 1, \beta^{-1} = \overline{\beta})$$

Basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\} \quad (|d\rangle = |0\rangle)$

$$\hat{X} \equiv \sum_{j=0}^{d-1} |j+1\rangle\langle j| \qquad \hat{Z} \equiv \sum_{j=0}^{d-1} \beta^j |j\rangle\langle j|$$

(Unitary)

$$\hat{X}^T = \hat{X}^{-1} \qquad \hat{Z}^T = \hat{Z}$$

$$\hat{Z}^d = \hat{X}^d = \hat{1}$$

Eigenvalues: $1, \beta, \beta^2, \ldots, \beta^{d-1}$

$$\hat{Z}\hat{X} = \beta\hat{X}\hat{Z} \qquad \hat{Z}^m\hat{X}^l = \beta^{lm}\hat{X}^l\hat{Z}^m$$

$$|\Phi_{0,0}\rangle \equiv \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

$$(\hat{X}_A \otimes \hat{X}_B)|\Phi_{0,0}\rangle = |\Phi_{0,0}\rangle$$
$$(\hat{Z}_A \otimes \hat{Z}_B^{-1})|\Phi_{0,0}\rangle = |\Phi_{0,0}\rangle$$

Bell basis: $\{|\Phi_{l,m}\rangle\}$ $(l = 0, 1, \ldots, d-1;\ m = 0, 1, \ldots, d-1)$

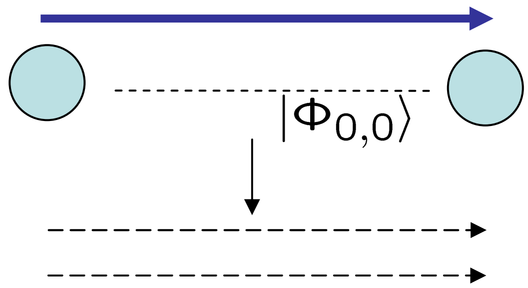
$$|\Phi_{l,m}\rangle \equiv (\hat{X}_A^l \otimes \hat{Z}_B^m)|\Phi_{0,0}\rangle$$

$$\left.\begin{aligned} (\hat{X}_A \otimes \hat{X}_B)|\Phi_{l,m}\rangle &= \beta^{-m}|\Phi_{l,m}\rangle \\ (\hat{Z}_A \otimes \hat{Z}_B^{-1})|\Phi_{l,m}\rangle &= \beta^{l}|\Phi_{l,m}\rangle \end{aligned}\right\} \longrightarrow$$

All states are orthogonal.

Quantum dense coding

1 qubit + 1 ebit $\longrightarrow$ 2 bits
n qubits + n ebits $\longrightarrow$ 2n bits



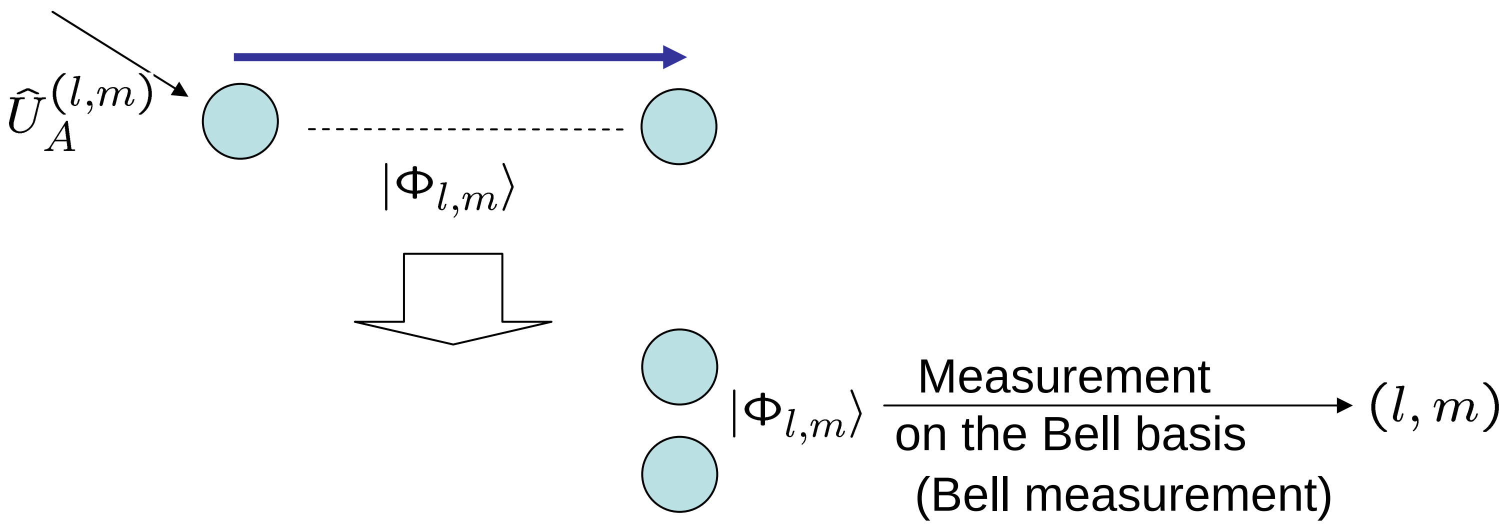
(Dimension d) + (Schmidt number d)
$\rightarrow$ (d^2 symbols)

MES

Convertibility via local unitary

Orthonormal basis (Bell basis)

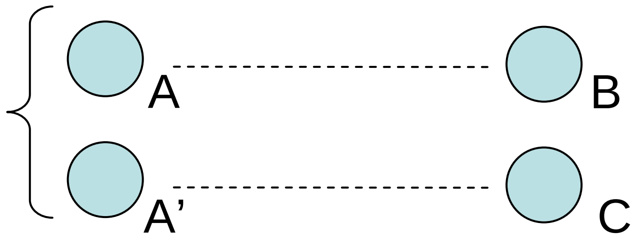
d^2 symbols (l, m)



Creating entanglement by nonlocal measurement

measurement

$|\Psi\rangle_{AA'} \longleftarrow$



$\dashrightarrow |\Psi^*\rangle_{BC}$

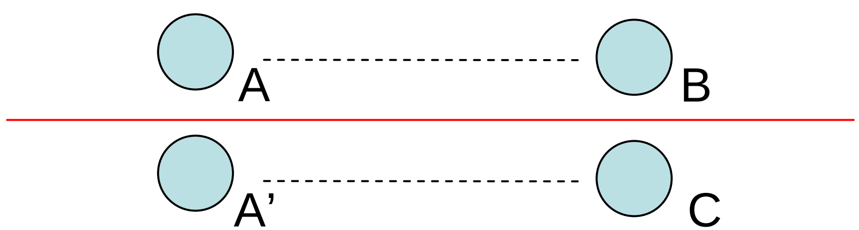
Relative state of $|\Psi\rangle_{AA'}$

Same entanglement

(More precisely, obtaining an outcome corresponding to a POVM element $\mu|\Psi\rangle\langle\Psi|$)

$$\left(\sum_{j=1}^{d}\frac{1}{\sqrt{d}}|j\rangle_A\otimes|j\rangle_B\right)\otimes\left(\sum_{k=1}^{d}\frac{1}{\sqrt{d}}|k\rangle_{A'}\otimes|k\rangle_{B'}\right)=\sum_{j,k}\frac{1}{\sqrt{d^2}}|jk\rangle_{AA'}\otimes|jk\rangle_{BB'}$$

When $|\Psi\rangle_{AA'}$ is an entangled state, (e.g., Bell measurement)



Initially no entanglement $\longrightarrow$ entangled $|\Psi^*\rangle_{BC}$

The measurement cannot be implemented over LOCC.

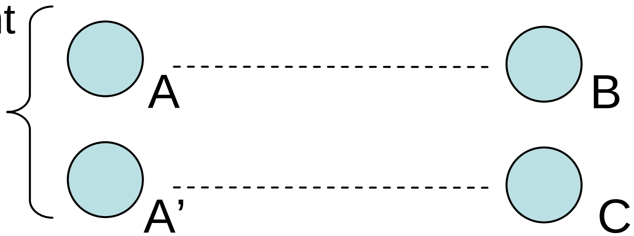
Entanglement swapping

$$|\Phi_{0,0}\rangle \equiv \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle \otimes |k\rangle$$

$$|\Theta\rangle_{AA'BC} = |\Phi_{0,0}\rangle_{AB} |\Phi_{0,0}\rangle_{A'C} = \sum_{j,k} \frac{1}{\sqrt{d^2}} |jk\rangle_{AA'} \otimes |jk\rangle_{BC}$$

Bell
measurement

$|\Phi_{0,0}\rangle_{AA'} \leftarrow$

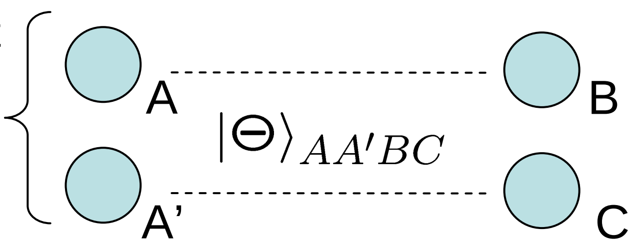


$\dashrightarrow |\Phi_{0,0}\rangle_{BC}$

$$_{AA'}\langle\Phi_{0,0}||\Theta\rangle_{AA'BC} = \frac{1}{\sqrt{d^2}} |\Phi_{0,0}\rangle_{BC}$$

Bell
measurement

$|\Phi_{l,m}\rangle_{AA'} \leftarrow$

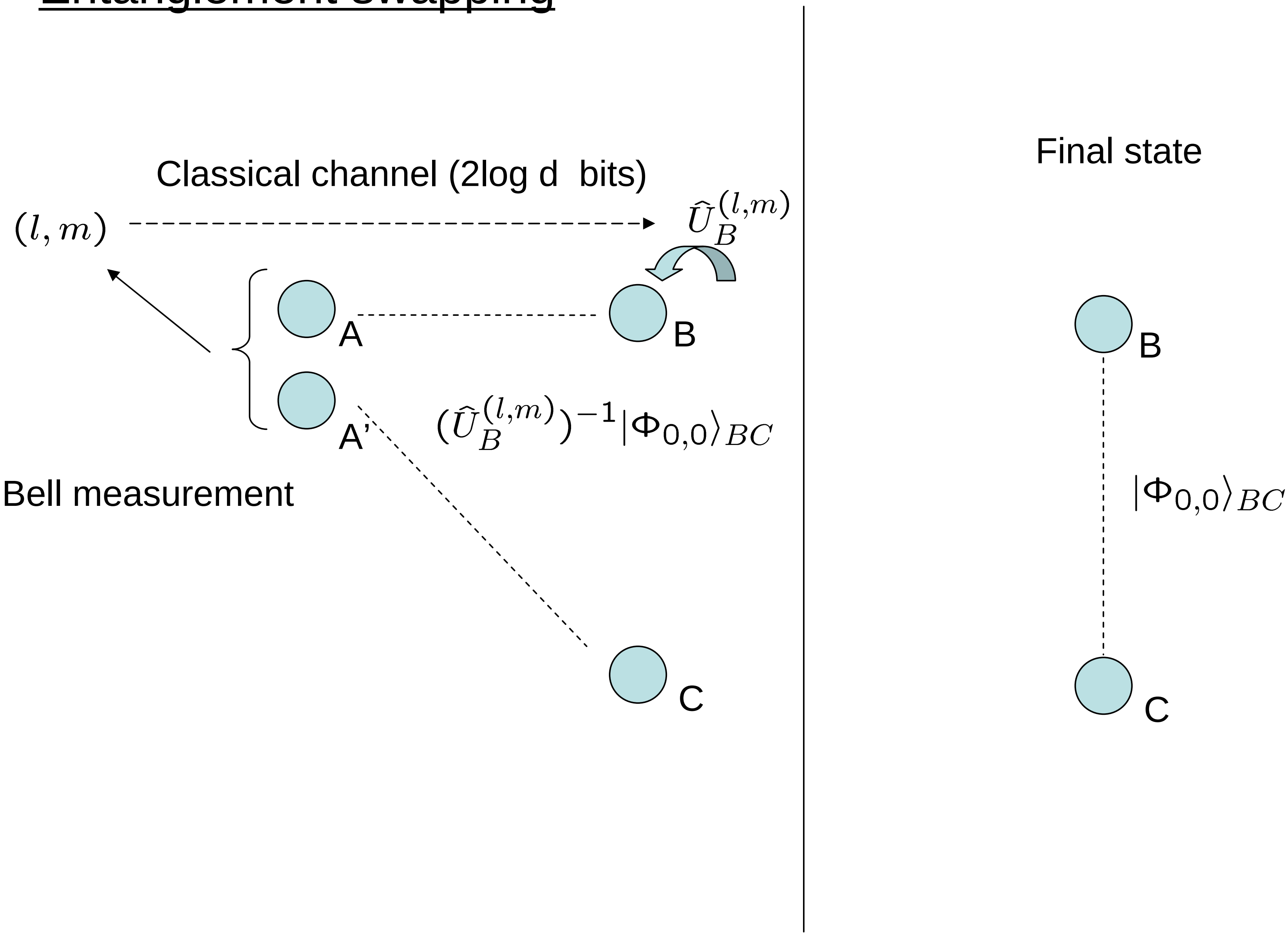


$\dashrightarrow$? $(\hat{U}_B^{(l,m)})^{-1} |\Phi_{0,0}\rangle_{BC}$

$$|\Phi_{l,m}\rangle_{AA'} = \hat{V}_A |\Phi_{0,0}\rangle_{AA'}$$

$$\begin{aligned} _{AA'}\langle\Phi_{l,m}||\Theta\rangle_{AA'BC} &= {}_{AA'}\langle\Phi_{0,0}|\hat{V}_A^{\dagger}|\Theta\rangle_{AA'BC} \\ &= {}_{AA'}\langle\Phi_{0,0}|\hat{V}_B^{*}|\Theta\rangle_{AA'BC} \\ &= \hat{V}_B^{*} \left[{}_{AA'}\langle\Phi_{0,0}||\Theta\rangle_{AA'BC} \right] \end{aligned}$$

Entanglement swapping



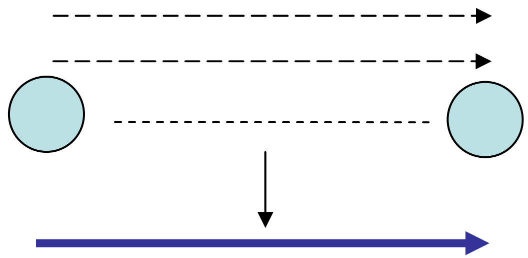
Quantum teleportation

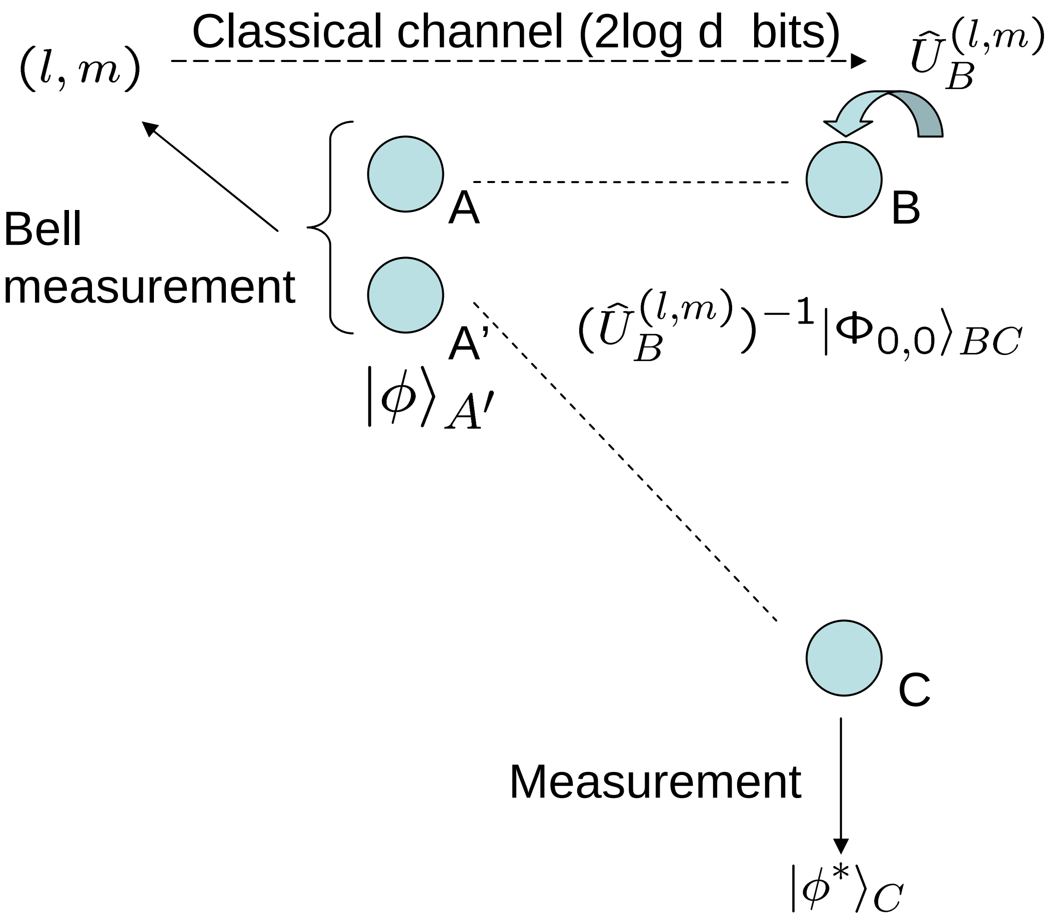
1 ebit + 2 bit $\longrightarrow$ 1 qubit

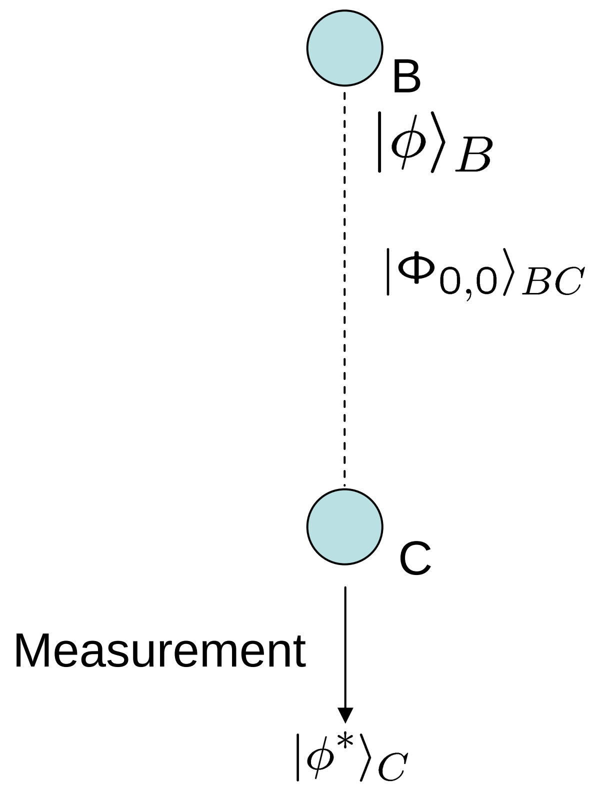
n ebits + 2n bits $\longrightarrow$ n qubits

$(d^2$ symbols$)$ + (Schmidt number d)

$\rightarrow$ (Dimension d)

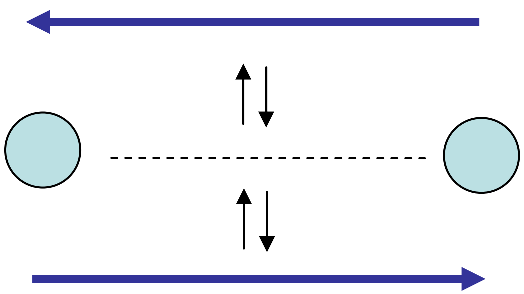






Quantum teleportation

If the cost of classical communication is neglected …



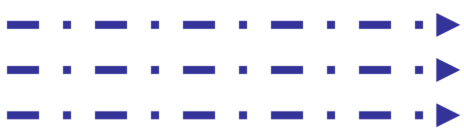
One can reserve the quantum channel by storing a quantum state.

One can use a quantum channel in the opposite direction.

A convenient way of quantum error correction (failure $\longrightarrow$ retry).

Noisy quantum channel

$|\phi\rangle$



$\hat{\rho} \rightarrow |\phi\rangle$

Recovering

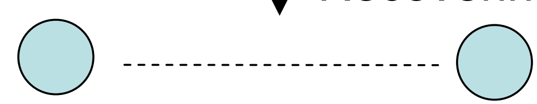
Failure $\longrightarrow$ no recovery.

Noisy quantum channel

$\downarrow$

Noisy entanglement

$\downarrow$ Recovering



Resource conversion protocols and bounds

We can do the following…

Conversion to ebits

Entanglement sharing

1 qubit ⟶ 1 ebit

$$(\Delta q, \Delta e, \Delta c) = (-1, 1, 0)$$

Conversion to bits

Quantum dense coding

1 qubit + 1 ebit ⟶ 2 bits

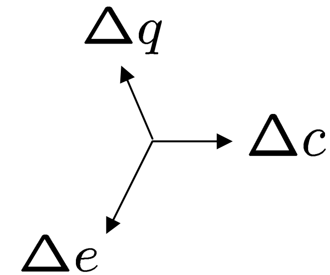
$$(\Delta q, \Delta e, \Delta c) = (-1, -1, 2)$$

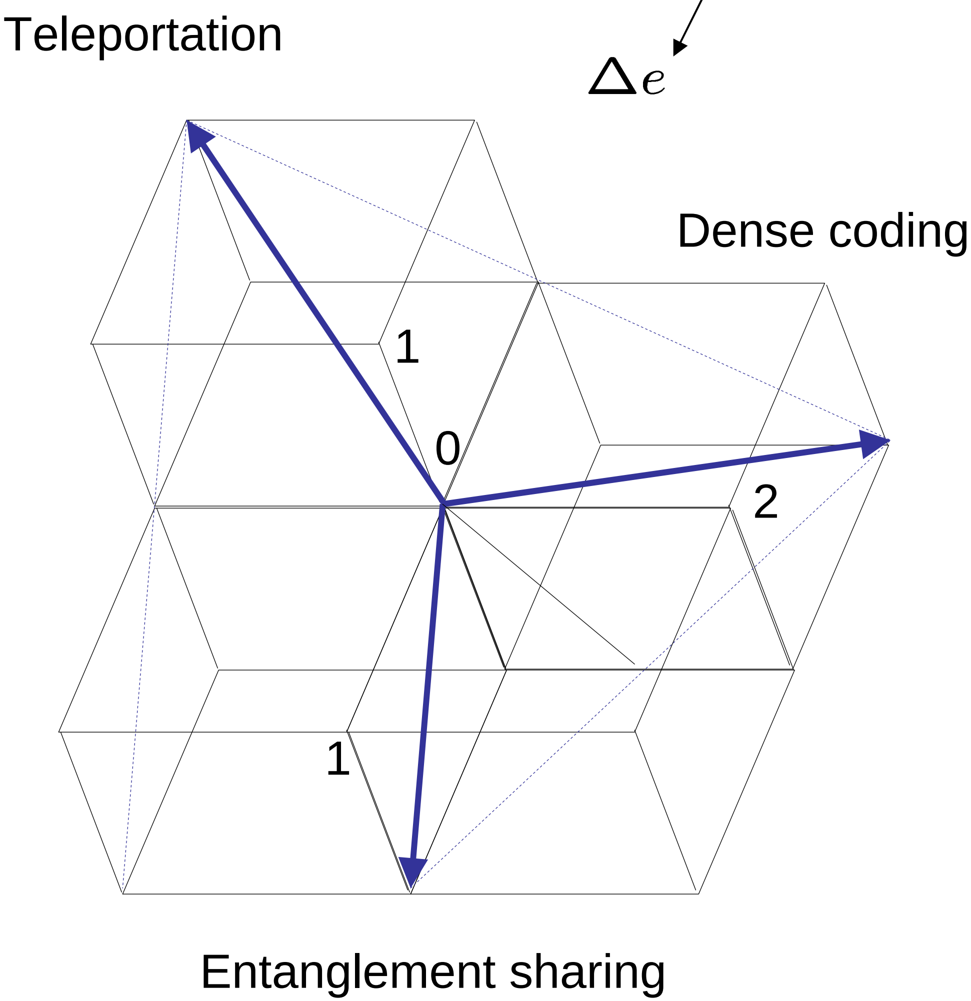
Conversion to qubits

Quantum teleportation

2 bits + 1 ebit ⟶ 1 qubit

$$(\Delta q, \Delta e, \Delta c) = (1, -1, -2)$$





Resource conversion protocols and bounds

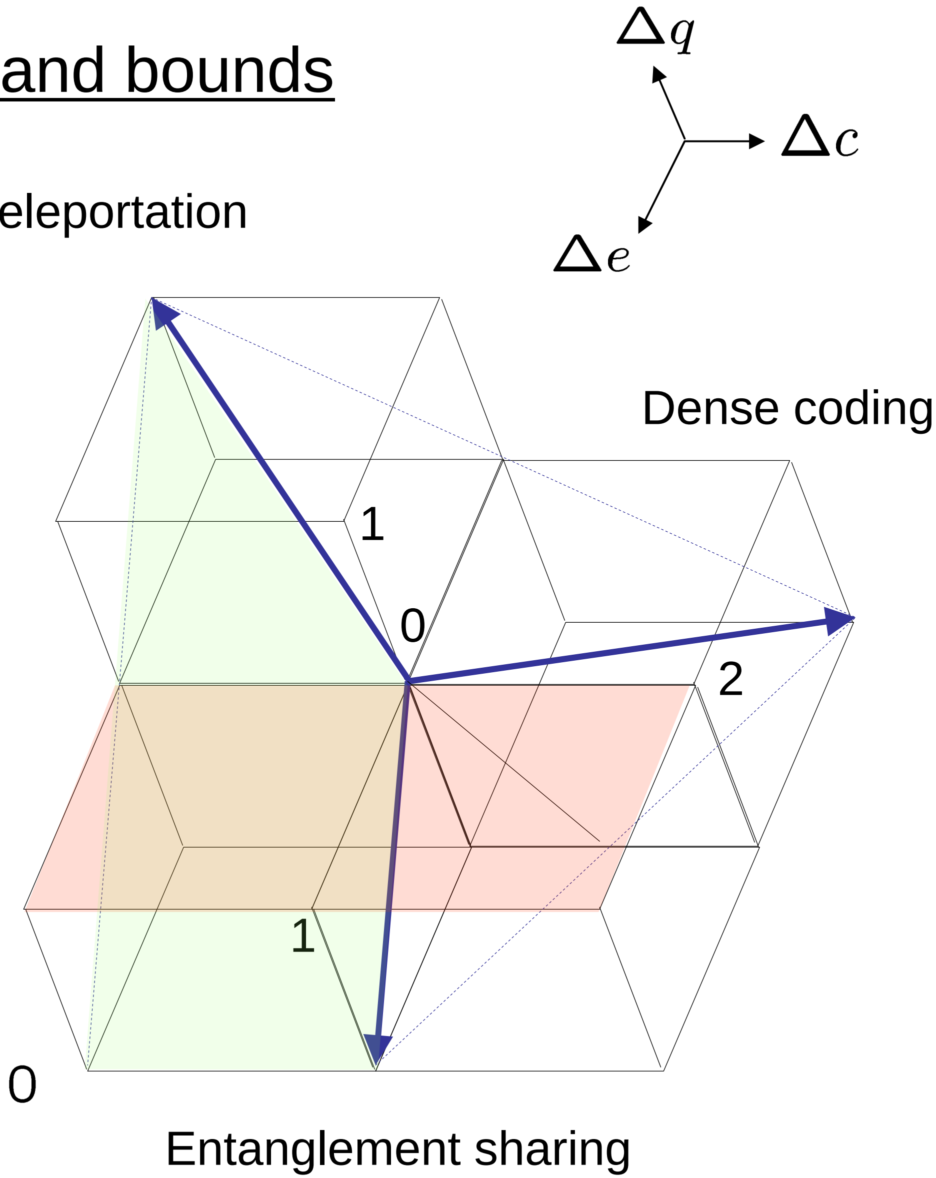
We can do the following…

Restrictions

bits alone $\longrightarrow$ no ebits

ebits alone $\longrightarrow$ no bits

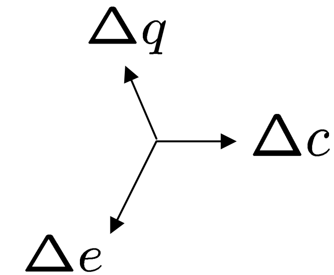
1 qubit alone $\longrightarrow$ no more than 1 bit



$$\Delta e + \Delta q \leq 0$$

Resource conversion protocols and bounds

We can do the following…



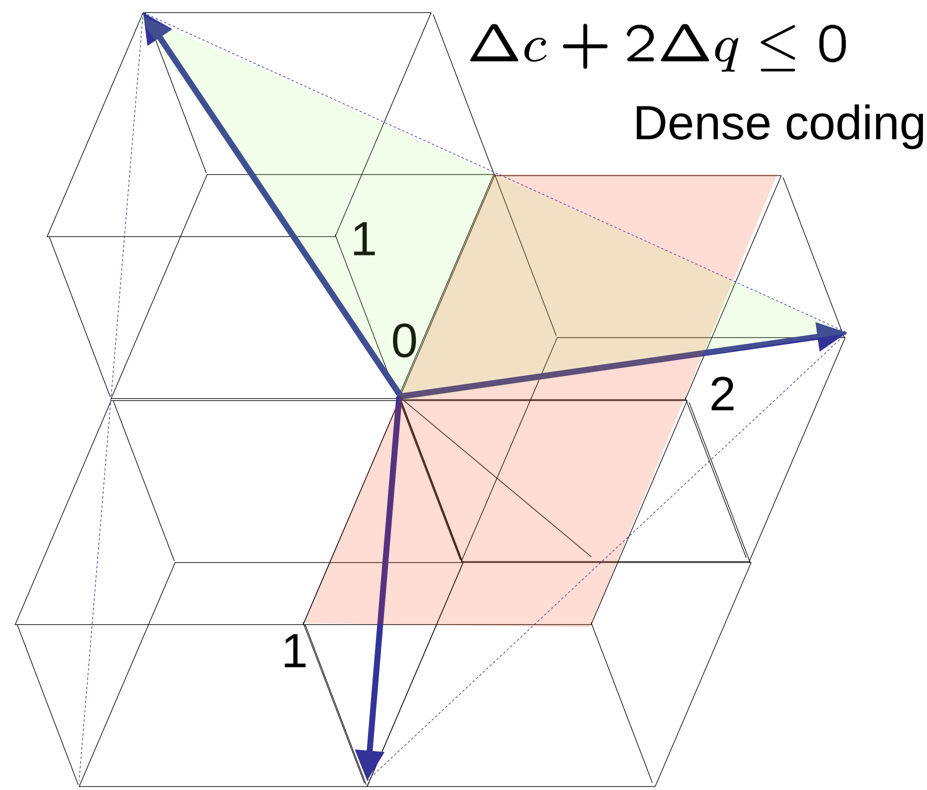
Restrictions

bits alone $\longrightarrow$ no ebits

ebits alone $\longrightarrow$ no bits

1 qubit alone $\longrightarrow$ no more than 1 bit

$$\Delta c + 2\Delta q \leq 0$$



Resource conversion protocols and bounds

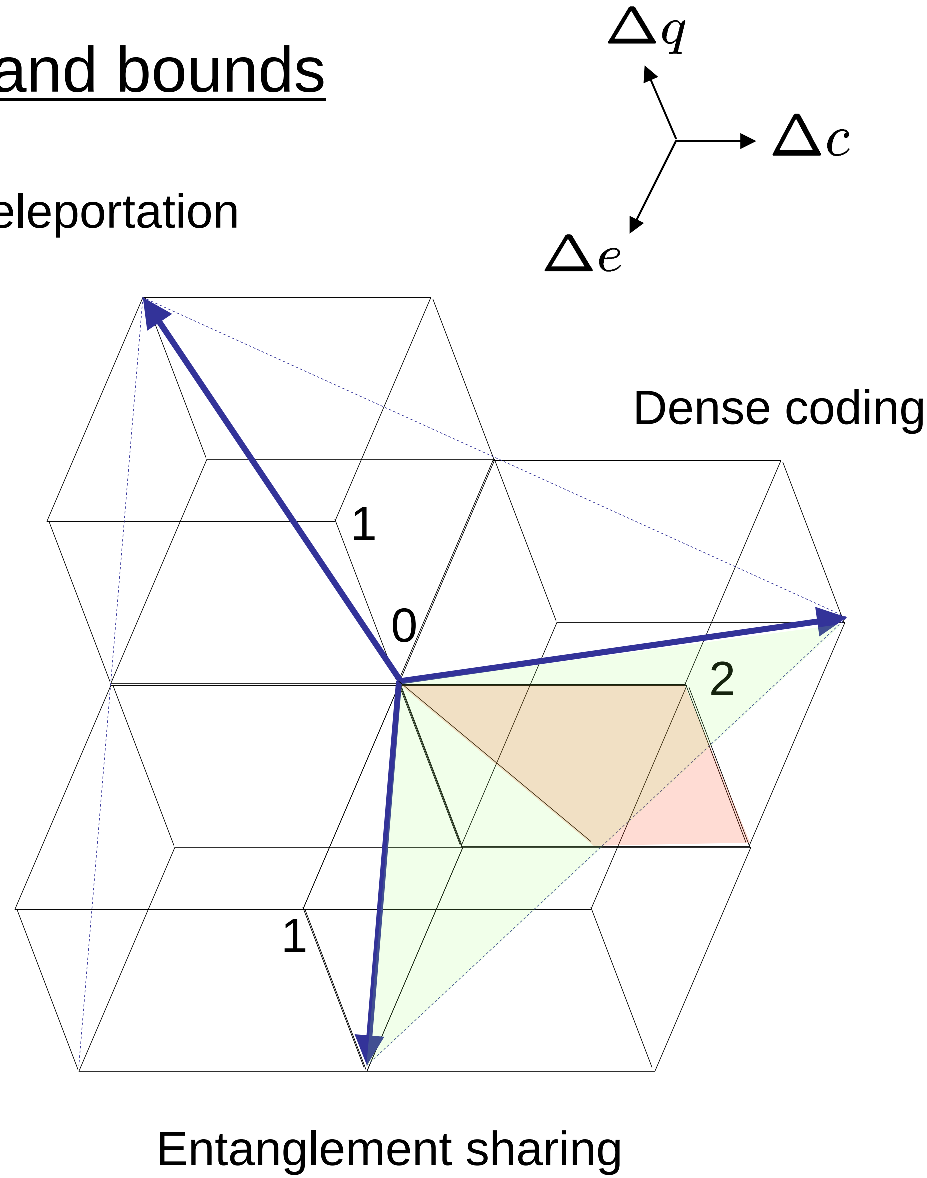
We can do the following…

Restrictions

bits alone $\longrightarrow$ no ebits

ebits alone $\longrightarrow$ no bits

1 qubit alone $\longrightarrow$ no more than 1 bit



$$\Delta c + \Delta q + \Delta e \leq 0$$

Resource conversion protocols and bounds

We can do the following…

Conversion to ebits

Entanglement sharing

1 qubit $\longrightarrow$ 1 ebit

$$(\Delta q, \Delta e, \Delta c) = (-1, 1, 0)$$

Conversion to bits

Quantum dense coding

1 qubit + 1 ebit $\longrightarrow$ 2 bits

$$(\Delta q, \Delta e, \Delta c) = (-1, -1, 2)$$

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit $\longrightarrow$ 1 qubit

$$(\Delta q, \Delta e, \Delta c) = (1, -1, -2)$$

We cannot violate the following …

Entanglement never assists classical channels

+ QD,QT

$$\Delta c + 2\Delta q \leq 0$$

Classical channels cannot increase entanglement

+ QT,ES

$$\Delta e + \Delta q \leq 0$$

Holevo + ES,QD

$$\Delta q + \Delta e + \Delta c \leq 0$$

Resource conversion protocols and bounds