Chapter 8

Quantum statistical properties of lasers

A laser is a non-equilibrium open system. It goes through a second order phase transition at oscillation threshold, where the phase or spectral linewidth is stabilized to well below the limit imposed by a cold cavity bandwidth and the amplitude is also stabilized to well below the limit imposed by a thermal photon distribution. In general, a macroscopic order established via a second order phase transition is determined by the balance between two counteracting forces: a system's stabilizing force and a reservoir's fluctuating force. In the case of a laser, the system's stabilizing force is the stimulated emission of photons. The amplitude is stabilized around its steady state value via relaxation oscillation between the photon field and the population inversion. The phase is stabilized by the generation of photons with identical phase.

However, both photon field and atomic inversion systems dissipate continuously to external reservoirs. The field decays due to an output coupling loss through a cavity mirror and an absorption loss induced by ground state atoms. The atomic dipole moment decays due to spontaneous emission and collision with other atoms. The population inversion is also subject to dissipation induced by spontaneous emission and external pumping. These dissipation processes inevitably introduce fluctuating forces into the system. Figure 8.1 summarizes such an open system model for a laser.

We have already formulated the theoretical framework for an open dissipative system in chapter 7. An important departure in the quantum theory of a laser from those presented in the previous chapter is the onset of the system's ordering force, gain saturation which is a key element separating a linear amplifier and a nonlinear oscillator. We will derive the density operator master equation for a laser in Sec. 8.1. The steady state solutions for averaged photon number and population inversion will be obtained in Sec. 8.2. The photon statistics and the spectral linewidth will be calculated also in Sec. 8.2. The quantum mechanical Fokker-Planck equation will be derived in Sec. 8.3, by which the same conclusions can be recovered. In Sec. 8.4 we will develop the quantum mechanical Langevin equation and calculate the noise spectrum.

In a standard model of a laser, it is assumed that a photon decay rate $\frac{\omega}{Q}$ is a slowest decay process. In such a case, the atomic variables can be adiabatically eliminated and the system is described by only field variables. This is called a slaving principle and means

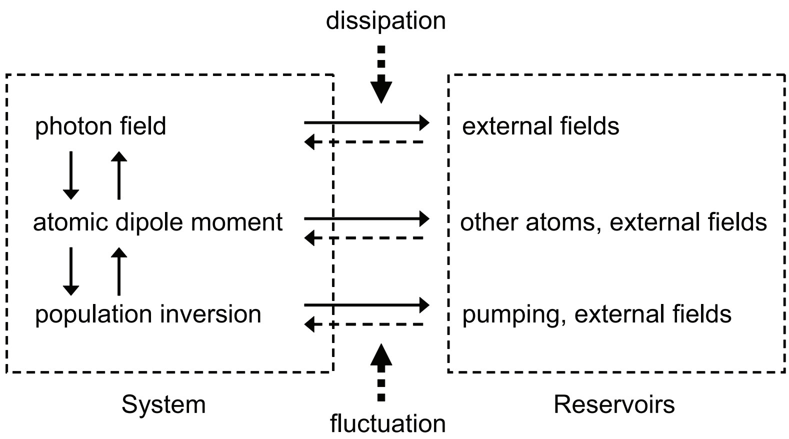

Figure 8.1: An open system model for a laser.

that a slowest variable governs the dynamics of a whole system. It is also assumed in a standard model of a laser that a pump process accompanies full shot noise. In some important cases, however, the above assumptions are not satisfied. Next chapter will treat such a case of a sub-Poissonian laser. Next chapter will also discuss an injection-locked laser and phase-locked-loop laser.

8.1 Density operator master equation [1]

Our model for a laser is schematically shown in Fig. 8.2. A cavity field interacts with an atomic beam prepared in an excited state through electric dipole coupling during a finite time interval τ. It is assumed that the cavity field changes appreciably only in a coarse-grained time constant Δt which is much longer than the atomic transit time lifetime τ. The atom-field interaction occurs as a Poisson point process and the interaction time τ obeys the exponential distribution

$$P(\tau) = \Gamma \exp(-\Gamma\tau) \quad , \tag{8.1}$$

where Γ^{-1} stands for a average interaction time of the atom with the cavity field.

8.1.1 Complete master equation

The atom-field interaction Hamiltonian is given by an electric dipole type,

$$\mathcal{H}_I = i\hbar g \left(\hat{\sigma}_+\hat{a} - \hat{a}^+\hat{\sigma}_-\right) \quad , \tag{8.2}$$

where $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$ are the atomic raising and lowering operators. The Liouville-von Neumann equation for the combined atomic-field density operator $\hat{\rho}$ is given by

$$\begin{aligned} \frac{d}{dt}\hat{\rho} &= \frac{1}{i\hbar}\left[\hat{\mathcal{H}}_I, \hat{\rho}\right] \qquad (8.3) \\ &= g\left[\hat{\sigma}_+\hat{a}\hat{\rho} - \hat{a}^+\hat{\sigma}_-\hat{\rho} - \hat{\rho}\hat{\sigma}_+\hat{a} + \hat{\rho}\hat{a}^+\hat{\sigma}_-\right] \quad , \end{aligned}$$

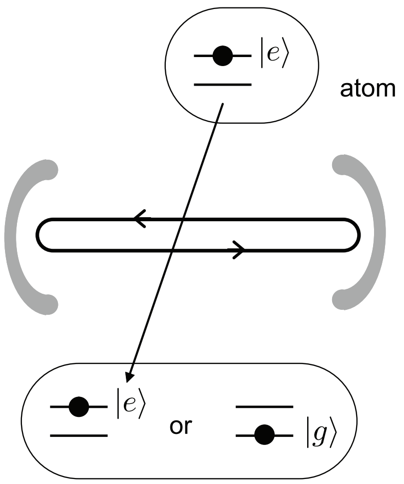

Figure 8.2: An theoretical model of a laser, in which an excited state atom interacts with a cavity field for a time duration τ.

in the interaction picture. The change of the density operator by the external pumping is already formulated in chapter 7 as

$$\frac{d}{dt}\hat{\rho} = -\frac{R_e}{2}\left[\hat{\sigma}_{00}\hat{\rho} + \hat{\rho}\hat{\sigma}_{00} - 2\hat{\sigma}_{e0}\hat{\rho}\hat{\sigma}_{0e}\right] \quad . \tag{8.4}$$

Here we assume the four-level atom shown in Fig. 8.3, in which the ground state $|0\rangle$ is pumped to the upper lasing transition level $|e\rangle$ due to rapid relaxation from the excited energy level $|1\rangle$ to $|e\rangle$ before it is injected into the cavity. We use the relation $\hat{\sigma}_{00} = \hat{\sigma}_{0e}\hat{\sigma}_{e0}$.

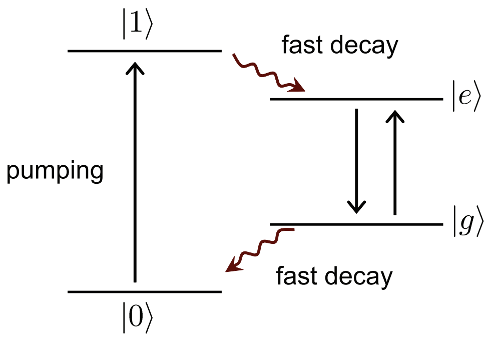

Figure 8.3: A four-level atom, in which the pump field excites the atom from $|0\rangle$ to $|e\rangle$ before the atom is injected into the cavity.

The excited state atom is subject to the spontaneous emission into external field reservoirs with continuous spectra. The master equation for this dissipation process is given by

$$\frac{d}{dt}\hat{\rho} = -\frac{\Gamma_{eg}}{2}\left[\hat{\sigma}_{+}\hat{\sigma}_{-}\hat{\rho} + \hat{\rho}\hat{\sigma}_{+}\hat{\sigma}_{-} - 2\hat{\sigma}_{-}\hat{\rho}\hat{\sigma}_{+}\right] \quad , \tag{8.5}$$

where Γ_{eg} is a Fermi's golden-rule decay rate for spontaneous emission. $\hat{\sigma}_{+}\hat{\sigma}_{-} \equiv \hat{\sigma}_{ee} = |e\rangle\langle e|$ and $\hat{\sigma}_{-}\hat{\sigma}_{+} = \hat{\sigma}_{gg} = |g\rangle\langle g|$ are the atomic population operators in the upper and lower states for the lasing transition. An energy-conserving scattering of atoms by external degrees of freedom dephase the atomic dipole moment beyond the fundamental decoherence limit imposed by the spontaneous decay rate Γ_{eg}. If such an extrinsic decoherence mechanism exists in the system, we can express it by the master equation

$$\frac{d}{dt}\hat{\rho} = -\frac{\gamma}{2}\left[\hat{\sigma}_{ee}\hat{\rho}\hat{\sigma}_{gg} + \hat{\sigma}_{gg}\hat{\rho}\hat{\sigma}_{ee}\right] \quad . \tag{8.6}$$

Combining (8.3), (8.4), (8.5) and (8.6), together with the master equation for a linear photon loss due to output coupling through a cavity mirror, we finally obtain the complete master equation for a laser:

$$\begin{aligned}\frac{d}{dt}\hat{\rho} \;=\; & g\left[\hat{\sigma}_{+}\hat{a}\hat{\rho} - \hat{a}^{+}\hat{\sigma}_{-}\hat{\rho} - \hat{\rho}\hat{\sigma}_{+}\hat{a} + \hat{\rho}\hat{a}^{+}\hat{\sigma}_{-}\right] \\ & -\frac{1}{2}\left(\frac{\omega}{Q}\right)\left[\hat{a}^{+}\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^{+}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^{+}\right] \\ & -\frac{1}{2}R_e\left[\hat{\sigma}_{00}\hat{\rho} + \hat{\rho}\sigma_{00} - 2\hat{\sigma}_{e0}\hat{\rho}\hat{\sigma}_{0e}\right] \\ & -\frac{1}{2}\Gamma_{eg}\left[\hat{\sigma}_{+}\hat{\sigma}_{-}\hat{\rho} + \hat{\rho}\hat{\sigma}_{+}\hat{\sigma}_{-} - 2\hat{\sigma}_{-}\hat{\rho}\hat{\sigma}_{+}\right] \\ & -\frac{1}{2}\gamma\left[\hat{\sigma}_{ee}\hat{\rho}\hat{\sigma}_{gg} + \hat{\sigma}_{gg}\hat{\rho}\hat{\sigma}_{ee}\right] \quad .\end{aligned} \tag{8.7}$$

8.1.2 Master equation for reduced field density operator

The master equation for the reduced density operator for a cavity field is obtained by taking a trace over atomic coordinates:

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_f \;&=\; Tr_{\text{atom}}\left[\frac{d}{dt}\hat{\rho}\right] \\ &=\; \sum_{i=0,g,e}\left\langle i\left|\frac{d}{dt}\hat{\rho}\right|i\right\rangle \quad .\end{aligned} \tag{8.8}$$

Substituting (8.7) into (8.8), we obtain

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_f \;=\; & g\left[\hat{a}\hat{\rho}_{ge} - \hat{a}^{+}\hat{\rho}_{eg} - \hat{\rho}_{ge}\hat{a} + \hat{\rho}_{eg}\hat{a}^{+}\right] \\ & -\frac{1}{2}\left(\frac{\omega}{Q}\right)\left[\hat{a}^{+}\hat{a}\hat{\rho}_f + \hat{\rho}_f\hat{a}^{+}\hat{a} - 2\hat{a}\hat{\rho}_f\hat{a}^{+}\right] \quad .\end{aligned} \tag{8.9}$$

Here $\hat{\rho}_{ge}$ satisfies the following equation

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_{ge} \;&\equiv\; \langle g|\frac{d}{dt}\hat{\rho}|e\rangle \\ &=\; g\left(\hat{\rho}_{gg}\hat{a}^{+} - \hat{a}^{+}\hat{\rho}_{ee}\right) - \frac{\gamma_T}{2}\hat{\rho}_{ge} \quad ,\end{aligned} \tag{8.10}$$

and similar equation holds for $\hat{\rho}_{eg} \equiv \langle e|\hat{\rho}|g\rangle$. The total decay rate of the atomic dipole is given by

$$\gamma_T = \gamma + \Gamma_{eg} \quad . \tag{8.11}$$

If $\gamma_T \gg \frac{\omega}{Q}$, we can adiabatically eliminate $\hat{\rho}_{ge}$ and $\hat{\rho}_{eg}$ by assuming $\frac{d}{dt}\hat{\rho}_{ge} = 0$:

$$\hat{\rho}_{ge} = \frac{2g}{\gamma_T}\left(\hat{\rho}_{gg}\hat{a}^+ - \hat{a}^+\hat{\rho}_{ee}\right) \quad . \tag{8.12}$$

Using (8.12) in (8.9), we obtain

$$\begin{aligned} \frac{d}{dt}\hat{\rho}_f &= \frac{2g^2}{\gamma_T}\left[\hat{a}\hat{\rho}_{gg}\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho}_{gg} + \hat{a}^+\hat{\rho}_{ee}\hat{a} - \hat{a}\hat{a}^+\hat{\rho}_{ee}\right] \\ &\quad + \frac{1}{2}\left(\frac{\omega}{Q}\right)\left[\hat{a}\hat{\rho}_f\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho}_f\right] + h.c. \quad . \end{aligned} \tag{8.13}$$

Here $\hat{\rho}_{ee}$ and $\hat{\rho}_{gg}$ satisfy the following equations:

$$\frac{d}{dt}\hat{\rho}_{ee} = \frac{2g^2}{\gamma_T}\left[2\hat{a}\hat{\rho}_{gg}\hat{a}^+ - \hat{a}\hat{a}^+\hat{\rho}_{ee} - \hat{\rho}_{ee}\hat{a}\hat{a}^+\right] - \Gamma_{eg}\hat{\rho}_{ee} + R_e\hat{\rho}_{00} \quad , \tag{8.14}$$

$$\frac{d}{dt}\hat{\rho}_{gg} = \frac{2g^2}{\gamma_T}\left[2\hat{a}^+\hat{\rho}_{ee}\hat{a} - \hat{a}^+\hat{a}\hat{\rho}_{gg} - \hat{\rho}_{gg}\hat{a}^+\hat{a}\right] + \Gamma_{eg}\hat{\rho}_{ee} \quad , \tag{8.15}$$

If the lower energy level population is negligible due to the fast energy relaxation from $|g\rangle$ to the ground state $|0\rangle$ and weak gain saturation,

$$\hat{\rho}_{gg} \simeq 0 \quad , \tag{8.16}$$

$$\hat{\rho}_{00} \simeq \hat{\rho}_f \quad . \tag{8.17}$$

Then we can solve $\hat{\rho}_{ee}$ by the perturbation technique to the second order:

$$\hat{\rho}_{ee} \simeq \frac{R_e}{\Gamma_{eg}}\hat{\rho}_f - \frac{2g^2R_e}{\gamma_T\Gamma_{eg}^2}\left[\hat{a}\hat{a}^+\hat{\rho}_f + \hat{\rho}_f\hat{a}\hat{a}^+\right] \quad . \tag{8.18}$$

Substituting (8.16), (8.17) and (8.18) into (8.13), we finally obtain

$$\begin{aligned} \frac{d}{dt}\hat{\rho}_f &= \frac{1}{2}\left(\frac{\omega}{Q}\right)\left[\hat{a}\hat{\rho}_f\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho}_f\right] \\ &\quad + \frac{2g^2R_e}{\gamma_T\Gamma_{eg}^2}\left[\hat{a}^+\hat{\rho}_f\hat{a} - \hat{a}\hat{a}^+\hat{\rho}_f\right] \\ &\quad + \frac{4g^4R_e}{\gamma_T^2\Gamma_{eg}^2}\left[\hat{\rho}_f\left(\hat{a}\hat{a}^+\right)^2 + \hat{a}\hat{a}^+\hat{\rho}_f\hat{a}\hat{a}^+ - 2\hat{a}^+\hat{\rho}_f\hat{a}\hat{a}^+\hat{a}\right] \\ &\quad + h.c. \quad . \end{aligned} \tag{8.19}$$

This is the master equation for the reduced field density operator. The first, second and third terms of R. H. S. of (8.19) represent the cavity loss, linear gain and gain saturation processes, respectively.

8.1.3 Photon number representation of the master equation

The change of the reduced field density operator in photon number representation over a coarse-grained time Δt is written as

$$\dot{\rho}_{nm} \equiv \frac{\rho_{nm}(t+\Delta t)-\rho_{nm}(t)}{\Delta t} = R_e \int_0^\infty d\tau \Gamma \exp(-\Gamma\tau) \left\{ \sum_\alpha \rho_{\alpha n,\alpha m}(t+\tau) - \rho_{nm}(t) \right\} \quad . \tag{8.20}$$

Here R_e is the rate of excited-state atom injection per second, $\rho_{\alpha n,\alpha m}(t+\tau)$ is the density matrix element with the atomic coordinate $\alpha = e$ or g at a time $t+\tau$, and $\rho_{nn}(t)$ is the density matrix element at a time t.

We assume the initial state of the atom-field system at a time t is in a pure state:

$$|\psi_{af}(t)\rangle = |e\rangle \sum_n C_n(t)|n\rangle \quad . \tag{8.21}$$

We introduced the above assumption for simplicity. The final master equation is independent of the choice of an initial state and we obtain the identical result if we start with a mixed state. The unitary time evolution is expressed by

$$|\psi_{af}(t+\tau)\rangle = \sum_n \left[C_{en}(t+\tau)|e\rangle|n\rangle + C_{g,n+1}(t+\tau)|g\rangle|n+1\rangle\right] \quad , \tag{8.22}$$

$$C_{en}(t+\tau) = C_n(t)\cos(g\sqrt{n+1}t) \quad , \tag{8.23}$$

$$C_{g,n+1}(t+\tau) = -iC_n(t)\sin(g\sqrt{n+1}t) \quad . \tag{8.24}$$

The matrix element $\rho_{en,em}(t+\tau)$ corresponds to the initial state $\rho_{nm}(t)$ and no photon emission, and thus it is given by

$$\begin{aligned} \rho_{en,em}(t+\tau) &= C_{en}(t+\tau)C^*_{em}(t+\tau) \\ &= \rho_{nm}(t)\cos(g\sqrt{n+1}\tau)\cos(g\sqrt{m+1}\tau) \quad . \end{aligned} \tag{8.25}$$

On the other hand, the matrix element $\rho_{gn,gm}(t+\tau)$ corresponds to the initial state $\rho_{n-1,m-1}(t)$ and one photon emission, and thus it is given by

$$\begin{aligned} \rho_{gn,gm}(t+\tau) &= C_{gn}(t+\tau)C^*_{gm}(t+\tau) \\ &= \rho_{n-1,m-1}(t)\sin(g\sqrt{n}\tau)\sin(g\sqrt{m}\tau) \quad . \end{aligned} \tag{8.26}$$

Substituting (8.25) and (8.26) into (8.20), we obtain

$$\begin{aligned} \dot{\rho}_{nm} &= -R_e\rho_{nm}\left[1-\Gamma\int_0^\infty d\tau \exp(-\Gamma\tau)\cos\left(g\sqrt{n+1}\tau\right)\cos(g\sqrt{m+1}\tau)\right] \\ &\quad +R_e\rho_{n-1,m-1}\Gamma\int_0^\infty d\tau\exp(-\Gamma\tau)\sin(g\sqrt{n}\tau)\sin(g\sqrt{m}\tau) \\ &= -\frac{N'_{nm}A}{1+N_{nm}\,(B/A)}\rho_{nm} + \left[\frac{\sqrt{nm}A}{1+N_{n-1,m-1}\,(B/A)}\right]\rho_{n-1,m-1} \end{aligned} \tag{8.27}$$

Here

$$A = 2R_e\left(\frac{g}{\gamma_T}\right)^2 \quad , \tag{8.28}$$

$$B = \left(\frac{g}{\gamma_T}\right)^2 A \quad , \tag{8.29}$$

$$N_{nm} = \frac{1}{2}(n+m+2) + \frac{1}{16}(n-m)^2\frac{B}{A} \quad , \tag{8.30}$$

$$N'_{nm} = \frac{1}{2}(n+m+2) + \frac{1}{8}(n-m)^2\frac{B}{A} \quad , \tag{8.31}$$

An actual laser has a photon loss due to output coupling through a cavity mirror and internal absorption. These dissipation processes have been already formulated as a master equation in chapter 7. If we include them in (8.27), we have a master equation for a laser in photon number representation,

$$\begin{aligned} \dot{\rho}_{nm} &= -\left(\frac{N'_{nm}A}{1+N_{nm}\frac{B}{A}}\right)\rho_{nm} + \left(\frac{\sqrt{nm}A}{1+N_{n+1,m-1}\frac{B}{A}}\right)\rho_{n-1,m-1} \\ &\quad -\frac{1}{2}\frac{\omega}{Q}(n+m)\rho_{nm} + \frac{\omega}{Q}\left[(n+1)(m+1)\right]^{1/2}\rho_{n+1,m+1} \quad , \end{aligned} \tag{8.32}$$

where no thermal photon contribution, $n_{th} = 0$, is assumed.

We can easily derive the master equation for the diagonal element ρ_{nn} by substituting $N'_{nm} = N_{nm} = n+1$ in (8.32):

$$\begin{aligned} \dot{\rho}_{nn} &= -\left[\frac{(n+1)A}{1+(n+1)\frac{B}{A}}\right]\rho_{nn} + \left(\frac{nA}{1+n\frac{B}{A}}\right)\rho_{n-1,n-1} \\ &\quad -\left(\frac{\omega}{Q}\right)n\rho_{nn} + \frac{\omega}{Q}(n+1)\rho_{n+1,n+1} \quad . \end{aligned} \tag{8.33}$$

If we assume a mild gain saturation, $n\frac{B}{A} \ll 1, in(8.33)$, we can obtain the approximate master equation

$$\begin{aligned} \dot{\rho}_{nn} &= -\left[A - B(n+1)\right](n+1)\rho_{nn} + (A - Bn)n\rho_{n-1,n-1} \\ &\quad -\left(\frac{\omega}{Q}\right)n\rho_{nn} + \left(\frac{\omega}{Q}\right)(n+1)\rho_{n+1,n+1} \quad . \end{aligned} \tag{8.34}$$

The interpretation of (8.34) is schematically shown in Fig. 8.4. An up-ward flow proportional to $A(n+1)$ expresses the stimulated and spontaneous emission of photons, and two down-ward flows proportional to Bn^2 and $\frac{\omega}{Q}n$ stand for the nonlinear atomic absorption (gain saturation) and linear photon decay from the cavity.

8.2 Steady state solutions

8.2.1 Rate equations

We can derive the rate equation for the average photon number $\bar{n} = \sum n\rho_{nn}$ from (8.34):

$$\frac{d}{dt}\bar{n} = -\left(\frac{\omega}{Q}\right)\bar{n} + E_{cv}\left(\bar{n}+1\right) \quad , \tag{8.35}$$

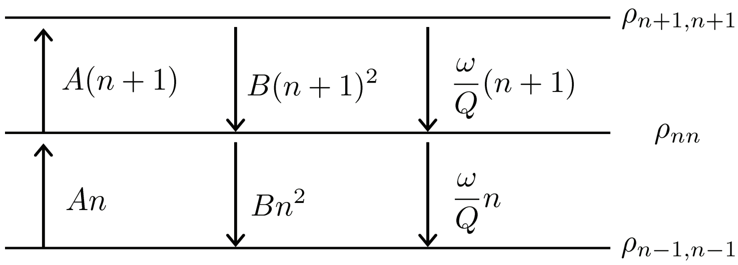

Figure 8.4: The flow of the diagonal elements ρ_{nn}.

where the nonlinear gain coefficient $E_{cv} = A - B(n+1)$ is expressed by

$$E_{cv} = \beta\frac{\bar{N}}{\tau_s} \quad . \tag{8.36}$$

Here β is the fractional spontaneous emission coupling efficiency, N is the average number of atoms in the state $|e\rangle$ and $\tau_s = 1/\Gamma_{eg}$ is the spontaneous emission lifetime for the $|e\rangle$ to $|g\rangle$ transition. We can also derive the rate equation for $\bar{N}$ from (8.14) and (8.15):

$$\frac{d}{dt}\bar{N} = P - \frac{\bar{N}}{\tau_s} - E_{cv}\,(\bar{n}+1) \quad , \tag{8.37}$$

where $P = R_e$ is the pump rate.

The steady state solutions for n and N are obtained by setting $\frac{d}{dt}n = \frac{d}{dt}N = 0$ in (8.35) and (8.37),

$$-\left(\frac{\omega}{Q}\right)\bar{n} + \beta\frac{\bar{N}}{\tau_s}(\bar{n}+1) = 0 \quad , \tag{8.38}$$

$$P - \frac{\bar{N}}{\tau_s} - \beta\frac{\bar{N}}{\tau_s}(\bar{n}+1) = 0 \quad . \tag{8.39}$$

8.2.2 Oscillation threshold

A standard definition of an oscillation threshold is given by the balanced cavity loss and stimulated emission gain,

$$\frac{\omega}{Q} = \beta\frac{\bar{N}_{th}}{\beta} \simeq p_{th}\beta \quad . \tag{8.40}$$

Here the second equality is obtained by the approximation that the stimulated and spontaneous emission rates $\beta\frac{N_{th}}{\tau_s}(n+1)$ into a single lasing mode is negligibly small compared to the spontaneous emission rate N_{th}/τ_s into all continuum modes. From (8.40), the threshold pump rate is given by

$$p_{th} = \left(\frac{\omega}{Q}\right)/\beta \quad . \tag{8.41}$$

An alternative definition of an oscillation threshold is given by the condition that the average photon number reaches one, where the stimulated emission rate exceeds the spontaneous emission rate, i.e.

$$\bar{n} = 1 \rightarrow \frac{\omega}{Q} = 2\beta\frac{N_{th}}{\tau_s} \quad . \tag{8.42}$$

Using (8.42) in (8.39), we obtain

$$p_{th} = \frac{1+\beta}{2\beta}\left(\frac{\omega}{Q}\right) \quad . \tag{8.43}$$

When $\beta \ll 1$, the difference between (8.41) and (8.43) is only a factor of two. On the other hand, if β is close to one, the two threshold conditions become identical, $p_{th} = \left(\frac{\omega}{Q}\right)$.

By solving (8.38) and (8.39), we obtain the average photon number and atom number as a function of normalized pump rate $r = p/p_{th}$:

$$\bar{n} = \frac{F(r,\beta)}{1-F(r,\beta)} \quad , \tag{8.44}$$

$$\bar{N} = \left(\frac{\omega}{Q}\right)\frac{\tau_s}{\beta}\cdot F(r,\beta) \quad , \tag{8.45}$$

where

$$F(r,\beta) = \frac{(r+1)-\sqrt{(r+1)^2-4(1-\beta)r}}{2(1-\beta)} \quad . \tag{8.46}$$

The average photon number n and the average atom number N vs. the (effective) pump rate $I = ep$ are plotted in Figs. 8.5 and 8.6 as a function of β.

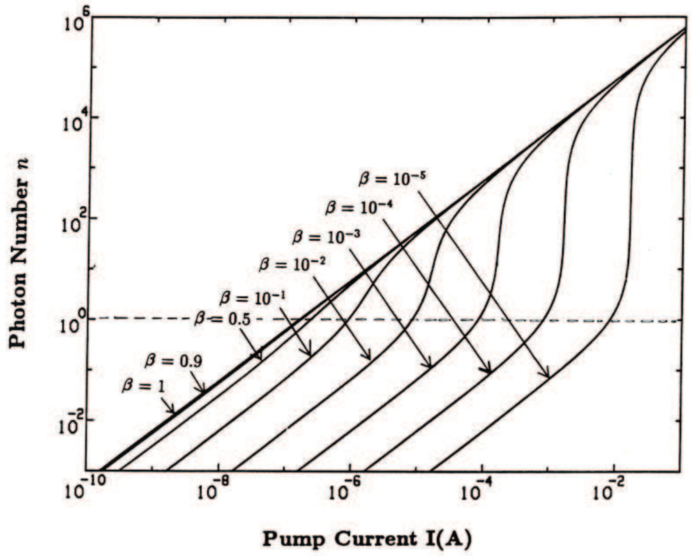

Figure 8.5: n vs. $I = ep$ for various β.

8.2.3 Functional spontaneous emission coupling efficiency

Equation (8.37) tells us that the injected atoms in the state $|e\rangle$ is depleted by the two competing processes, spontaneous emission $\frac{N}{\tau_s}$ and stimulated emission $\beta\frac{N}{\tau_s}n$. A saturation

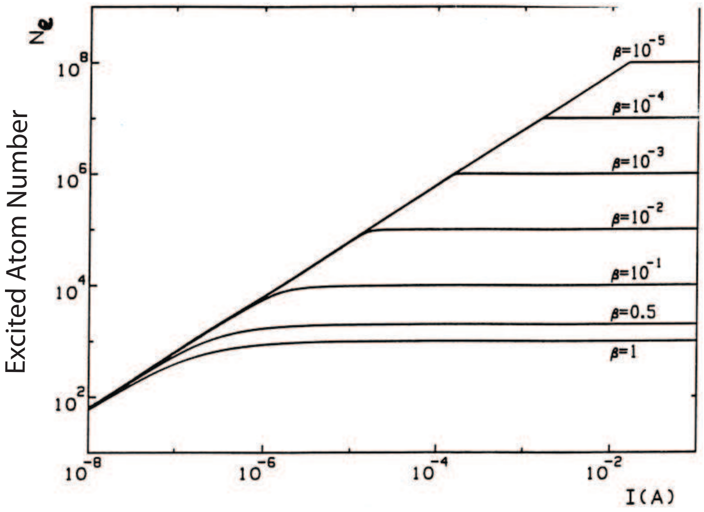

Figure 8.6: N vs. $I = ep$ for various β.

photon number n_s is defined by the equal contribution from $\frac{N}{\tau_s}$ and $\beta\frac{N}{\tau_s}n_s$, so that we obtain

$$n_s = \frac{1}{\beta} = \frac{A}{B} \quad . \tag{8.47}$$

The fractional spontaneous emission coupling efficiency β (or inverse saturation photon number n_s^{-1}) is calculated by counting the effective number M of optical field modes in the gain bandwidth and inside the active volume as shown in Fig. 8.7. If the active volume where the injected atoms exist is V_a and the gain bandwidth is $\Delta\omega_L$, then the effective number of optical field modes is given by

$$\begin{aligned} M &= \rho(\omega)\Delta\omega_L V_a \\ &= \frac{\omega^2}{\pi^2 c^3}\Delta\omega_L V_a \quad , \end{aligned} \tag{8.48}$$

where $\rho(\omega)$ is the 3D field density of states. Using (8.48), we obtain

$$\beta = \frac{1}{n_s} = \frac{1}{M} = \frac{\pi^2 c^3}{\omega^2 \Delta\omega_L V_a} \quad . \tag{8.49}$$

8.2.4 Photon statistics

In the steady state condition, the net flow of the adjacent diagonal elements should be identically equal to zero to sustain the time invariant ρ_{nn}. This condition is called a detailed balance and is given by

$$-\left[\frac{A(n+1)}{1+(n+1)\frac{B}{A}}\right]\rho_{nn} + \frac{\omega}{Q}(n+1)\rho_{n+1,n+1} = 0 \quad . \tag{8.50}$$

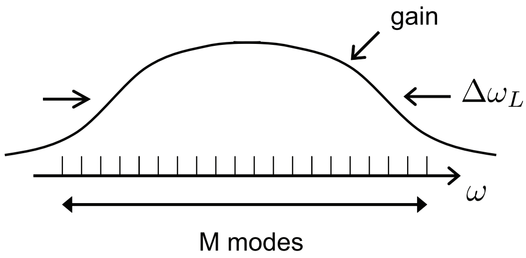

Figure 8.7: Optical field modes trapped in the active volume V_a in the gain bandwidth $\Delta\omega_L$.

From (8.50), we can write a recursion relation for $\rho_{n+1,n+1}$as

$$\rho_{n+1,n+1} = \frac{A^2}{\left(\frac{\omega}{Q}\right)[A+(n+1)B]}\rho_{nn} \quad . \tag{8.51}$$

By repetitive use of the above recursion relation, we can express ρ_{nn} in terms of the zero photon probability ρ_{00}:

$$\rho_{nn} = \left[\frac{A^2}{\left(\frac{\omega}{Q}\right)B}\right]^n \frac{\rho_{00}}{(n+\frac{A}{B})!} \quad . \tag{8.52}$$

At a pump rate below oscillation threshold, the average photon number is much smaller than one, i.e.

$$\bar{n} = \frac{\beta}{\left(\frac{\omega}{Q}\right)}P = \frac{A}{\left(\frac{\omega}{Q}\right)} \ll 1 \quad , \tag{8.53}$$

In this case, (8.51) is reduced to

$$\rho_{n+1,n+1} \simeq \frac{A}{\left(\frac{\omega}{Q}\right)}\rho_{nn} \quad , \tag{8.54}$$

and we obtain

$$\begin{aligned} \rho_{nn} &\simeq \left[\frac{A}{\left(\frac{\omega}{Q}\right)}\right]^n \rho_{00} \\ &= \bar{n}^n \rho_{00} \\ &= \frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n \quad . \end{aligned} \tag{8.55}$$

This is the photon statistics of a single-mode thermal state. An effective temperature T_{eff} of a laser is determined by the photon number $\bar{n}$:

$$\bar{n} = \frac{1}{\exp\left(\hbar\omega/k_B T_{eff}\right) - 1} \quad . \tag{8.56}$$

At an oscillation threshold, $\bar{n} = 1$, the effective temperature is about $T_{eff} \sim 10000$K! The photon statistics at oscillation threshold has a unique form of

$$\rho_{nn} = \left(\frac{1}{2}\right)^{n+1} \quad . \tag{8.57}$$

At well above threshold, the following approximations can be made:

$$\bar{n} \simeq \frac{p}{\left(\frac{\omega}{Q}\right)} \gg 1 \quad , \tag{8.58}$$

$$\bar{n} \gg \frac{A}{B} \quad , \tag{8.59}$$

$$\rho_{nn} \simeq \rho_{n-1,n-1} \quad . \tag{8.60}$$

Using these relations, we obtain

$$\begin{aligned} \bar{n} &\simeq \frac{A}{\left(\frac{\omega}{Q}\right)} \cdot \frac{A - \left(\frac{\omega}{Q}\right)}{B} \\ &\simeq \frac{A^2}{\left(\frac{\omega}{Q}\right) B} \quad . \end{aligned} \tag{8.61}$$

Substituting (8.58)-(8.61) into (8.52), we obtain the Poisson statistics

$$\begin{aligned} \rho_{nn} &\simeq \frac{\bar{n}^n}{n!}\rho_{00} \\ &= e^{-\bar{n}}\frac{\bar{n}^n}{n!} \quad . \end{aligned} \tag{8.62}$$

Fig. 8.8 shows the photon statistics of a laser at $p < p_{th}, p = p_{th}$ and $p > p_{th}$.

8.2.5 Spectral linewidth

The electric field operator in a Fabry-Perot cavity is expressed by

$$\hat{\varepsilon} = i\sqrt{\frac{\hbar\omega}{2\varepsilon V}}\left(\hat{a} - \hat{a}^+\right)\sin(kz) \quad , \tag{8.63}$$

where $k = \frac{2\pi}{\lambda}$ is the wavenumber of a laser mode. Quantum average of (8.63) is

$$\begin{aligned} \langle\hat{\varepsilon}(t)\rangle &= Tr\left(\hat{\rho}_f\hat{\varepsilon}\right) \\ &= \sum_n \langle n|\hat{\rho}_f\hat{\varepsilon}|n\rangle \\ &= i\sqrt{\frac{\hbar\omega}{2\varepsilon V}}\sin(kz)\sum_n\left(-\sqrt{n+1}\rho_{n,n+1} + \sqrt{n}\rho_{n,n-1}\right) \quad . \end{aligned} \tag{8.64}$$

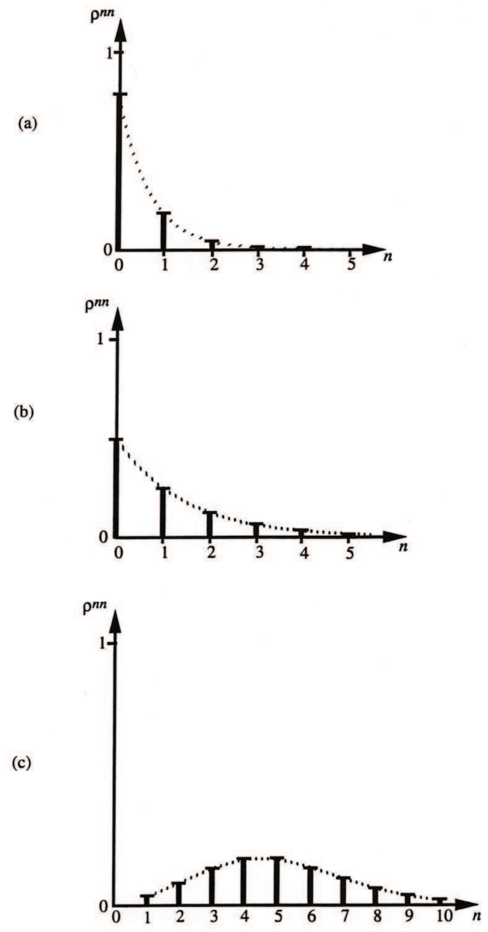

Figure 8.8: Photon statistics of a laser at $p < p_{th}, p = p_{th}$ and $p > p_{th}$.

If we substitute $m = n + 1$ and assume $n\frac{B}{A} \ll 1$ in (8.32), we obtain

$$\begin{aligned}\dot{\rho}_{n,n+1} \quad = \quad & \left\{\left[A - B\left(n+\frac{3}{2}\right)\right]\left(n+\frac{3}{2}\right) - \frac{1}{8}B - \frac{\omega}{Q}\left(n+\frac{1}{2}\right)\right\}\rho_{n,n+1} \\ & + \left[A - B\left(n+\frac{1}{2}\right)\right]\sqrt{n(n+1)}\rho_{n-1,n} \\ & + \frac{\omega}{Q}\sqrt{(n+1)(n+2)}\rho_{n+1,n+2}\end{aligned} \qquad (8.65)$$

When a laser is pumped at well above threshold, (8.65) is reduced to

$$\dot{\rho}_{n,n+1} = -\frac{1}{4}\frac{\left(\frac{\omega}{Q}\right)}{\bar{n}}\rho_{n,n+1} \quad . \qquad (8.66)$$

If we introduce a phase diffusion constant $D = \frac{1}{2}\frac{\left(\frac{\omega}{Q}\right)}{\bar{n}}$, (8.64) is reduced to

$$\langle\hat{\varepsilon}(t)\rangle = \langle\hat{\varepsilon}(0)\rangle\cos(\omega t)e^{-\frac{1}{2}Dt} \quad . \qquad (8.67)$$

The Fourier transform of (8.67) results in the Lorentzian field power spectrum centered at angular frequency ω and spectral linewidth (full width at half-maximum)

$$\Delta\omega_{1/2} = D = \frac{1}{2}\frac{\left(\frac{\omega}{Q}\right)}{\bar{n}} \quad . \tag{8.68}$$

Figure 8.9 shows the spectral linewidth D of a laser as a function of pump current $I = ep$. At below threshold, the spectral linewidth is equal to the cold cavity bandwidth $D = \frac{\omega}{Q}$. At threshold, the photon number $\bar{n}$ jumps from 1 to $1/\beta$, so that the spectral line width drops suddenly by a factor of β. At above threshold, the photon number increases in proportion to $p/p_{th} - 1$, so that the spectral linewidth decreases as $(p/p_{th} - 1)^{-1}$.

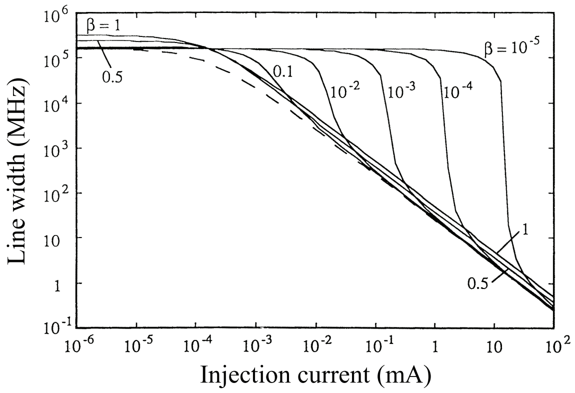

Figure 8.9: Spectral linewidth D of a laser vs. normalized pump rate $I = ep$.

8.3 Quantum mechanical Fokker-Planck equation [2]

The master equation for the reduced field density operator given by (8.19) is rewritten as

$$\begin{aligned} \frac{d}{dt}\hat{\rho}_f &= \frac{1}{2}\left(\frac{\omega}{Q}\right)\left[\hat{a}\hat{\rho}_f\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho}_f\right] \\ &\quad +\frac{1}{2}A\left[\hat{a}^+\hat{\rho}_f\hat{a} - \hat{a}\hat{a}^+\hat{\rho}_f\right] \\ &\quad +\frac{1}{2}B\left[\hat{\rho}_f\left(\hat{a}\hat{a}^+\right)^2 + \hat{a}\hat{a}^+\hat{\rho}_f\hat{a}\hat{a}^+ - 2\hat{a}^+\hat{\rho}_f\hat{a}\hat{a}^+\hat{a}\right] \\ &\quad +h.c. \quad , \end{aligned} \tag{8.69}$$

where $A = \frac{4g^2R_e}{\gamma_T\Gamma_{eg}}$ is a linear gain coefficient and $B = \frac{8g^4R_e}{\gamma_T^2\Gamma_{eg}^2}$ is a gain saturation coefficient. We now introduce the diagonal $P(\alpha)$ representation of coherent states for $\hat{\rho}_f$:

$$\hat{\rho}_f = \int P(\alpha)|\alpha\rangle\langle a|\frac{d^2\alpha}{\pi} \quad . \tag{8.70}$$

Following the procedures described in chapter 7, we have the quantum mechanical Fokker-Planck equation for a laser,

$$\begin{aligned}\frac{d}{dt}P(\alpha) &= -\frac{1}{2}\left\{\frac{\partial}{\partial\alpha}\left[\left(A-\frac{\omega}{Q}-B\alpha^2\right)\alpha P(\alpha)\right]+c.c.\right\} \\ &\quad +A\frac{\partial^2}{\partial\alpha\partial\alpha^*}P(\alpha) \quad .\end{aligned} \tag{8.71}$$

We can easily convert (8.71) into the polar coordinate by using $\alpha = re^{i\theta}$:

$$\begin{aligned}\frac{d}{dt}P(r,\theta) &= -\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\left[r^2\left(A-\frac{\omega}{Q}-Br^2\right)P(r,\theta)\right] \\ &\quad +\frac{1}{4}A\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)P(r,\theta) \quad .\end{aligned} \tag{8.72}$$

The first term and the second term of R.H.S. of (8.72) are the drift term and diffusion term as identified in chapter 7.

The steady state solution of (8.72) is obtained by setting $\frac{d}{dt}P(r,\theta)=0$. If we assume an average amplitude is r_0, the drift term should disappear at $r=r_0$ so that we obtain

$$A-\frac{\omega}{Q}-Br_0^2=0 \quad . \tag{8.73}$$

We now introduce a small amplitude noise by

$$\Delta r = r - r_0 \quad . \tag{8.74}$$

Then we have

$$r^2\left(A-\frac{\omega}{Q}-Br^2\right) = -2r\left(Br_0^2\right)(r-r_0) \quad . \tag{8.75}$$

Substituting (8.75) into (8.72) together with $\frac{d}{dt}P(r,\theta)=0$ and using the ansatz $P(r,\theta) = P(r)\Phi(\theta)$, we have

$$\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\left[r2Br_0^2(r-r_0)P(r)\right]+\frac{A}{4}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}P(r)\right]=0 \quad , \tag{8.76}$$

or

$$\frac{\partial}{\partial r}P(r) = -\frac{4Br_0^2}{A}(r-r_0)P(r) \quad . \tag{8.77}$$

The solution of (8.77) is given by the Gaussian distribution

$$P(r) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(r-r_0)^2}{2\sigma^2}\right] \quad , \tag{8.78}$$

where the variance in the amplitude is given by

$$\sigma^2 = \frac{A}{4Br_0^2} \quad . \tag{8.79}$$

At well above threshold, the average amplitude r_0 satisfies

$$r_0^2 = \frac{A - \frac{\omega}{Q}}{B} \simeq \frac{A}{B} \quad . \tag{8.80}$$

Therefore, the variance is reduced to

$$\sigma^2 = \frac{1}{4} \quad . \tag{8.81}$$

This is the amplitude noise of a coherent state, corresponding to the Poisson photon statistics. Thus we recovered the same result that was obtained by the master equation in photon number representation. Equation (8.79) can also be expressed as

$$\sigma^2 = \frac{A}{4\left(A - \frac{\omega}{Q}\right)} = \frac{p/p_{th}}{4\,(p/p_{th} - 1)} \quad , \tag{8.82}$$

where the relation $A/\left(\frac{\omega}{Q}\right) = p/p_{th}$ is used. Since the photon number noise Δn^2 is given by

$$\Delta n^2 = 4\bar{n}\sigma^2 = n_s \left(\frac{p}{p_{th}}\right) \quad , \tag{8.83}$$

at well above threshold, where $\bar{n} = n_s\,(p/p_{th} - 1)$ is used.

The steady state solution for the angular distribution function $\Phi(\theta)$ is from (8.72),

$$\frac{d}{d\theta}\Phi(\theta) = 0 \quad . \tag{8.84}$$

The above result suggests that the laser phase is uniformly distributed in $[0, 2\pi]$ under the steady state condition. This conclusion originates from the fact that a laser has a restoring force only for the amplitude r and does not have a restoring force for the phase θ. A gain saturation term Br^2 in (8.72) only depends on the amplitude r, so that a laser is a phase insensitive oscillator. Nevertheless, the field emitted by a laser has a distinct feature from a thermal field not only in the amplitude (or photon number) distribution but also in the phase noise. The time dependent phase distribution function satisfies

$$\frac{d}{dt}\Phi(\theta) = \frac{A}{4}\frac{1}{r_0^2}\frac{\partial^2}{\partial\theta^2}\Phi(\theta) \quad . \tag{8.85}$$

This is the diffusion equation for which a diffusion coefficient D is given by

$$D = \frac{\left(\frac{\omega}{Q}\right)}{2\bar{n}} \quad . \tag{8.86}$$

This is an identical result to (8.68) that was obtained by the master equation in photon number representation.

8.4 Heisenberg-Langevin equation [3]

8.4.1 Derivation of the equation

We will now switch to the Heisenberg picture to describe a laser, in which the field and atom operators evolve in time. The operator Langevin equations for the upper and lower level occupation operators $\hat{\sigma}_e = |e\rangle\langle e|$ and $\hat{\sigma}_g = |g\rangle\langle g|$ are given by

$$\frac{d}{dt}\hat{\sigma}_e = \Lambda - \gamma_e \hat{\sigma}_e + ig\left(\hat{a}^+\hat{\sigma} - \hat{\sigma}^+\hat{a}\right) + \hat{F}_e(t) \quad , \tag{8.87}$$

$$\frac{d}{dt}\hat{\sigma}_g = -\gamma_g \hat{\sigma}_g - ig\left(\hat{a}^+\hat{\sigma} - \hat{\sigma}^+\hat{a}\right) + \hat{F}_g(t) \quad , \tag{8.88}$$

where Λ is the external pump rate into the upper state, γ_e and γ_g are the spontaneous decay rates, and $\hat{F}_e(t)$ and $\hat{F}_g(t)$ are the noise operators. The atom-field coupling term is governed by the electric dipole operators $\hat{\sigma} = |g\rangle\langle e|$ and $\hat{\sigma}^+ = |e\rangle\langle g|$ which obey the operator Langevin equation,

$$\frac{d}{dt}\hat{\sigma} = -\left[\gamma + i(\omega - \nu)\right]\hat{\sigma} + ig\left(\hat{\sigma}_e - \hat{\sigma}_g\right)\hat{a} + \hat{F}_\sigma(t) \quad , \tag{8.89}$$

where γ is the dipole moment decay rate, ω and ν are the oscillation frequencies of the atomic dipole and the cavity field, respectively, and $\hat{F}_\sigma(t)$ is the noise operator. Finally, the cavity field operator $\hat{a}$ satisfies the operator Langevin equation,

$$\frac{d}{dt}\hat{a} = -\left[\frac{1}{2}\left(\frac{\omega}{Q}\right) + i(\Omega - \nu)\right]\hat{a} - igN\hat{\sigma} + \hat{F}(t) \quad , \tag{8.90}$$

where Ω is the empty cavity resonant frequency, N is the total number of atoms and $\hat{F}(t)$ is the noise operator.

We now introduce the population difference operator by

$$\hat{\sigma}_z \equiv \hat{\sigma}_e - \hat{\sigma}_g \quad , \tag{8.91}$$

and the associated noise operator by

$$\hat{F}_z(t) \equiv \hat{F}_e(t) - \hat{F}_g(t) \quad . \tag{8.92}$$

We further assume $\gamma_e = \gamma_g = \gamma_z$, so that the operator Langevin equation for $\hat{\sigma}_z$ is given by

$$\frac{d}{dt}\hat{\sigma}_z = \Lambda - \gamma_z \hat{\sigma}_z + 2ig\left(\hat{a}^+\hat{\sigma} - \hat{\sigma}^+\hat{a}\right) + \hat{F}_z(t) \quad . \tag{8.93}$$

Equations (8.89), (8.90) and (8.93) from a complete set of operator Langevin equations for a laser. The associated noise operators satisfy the following two time correlation functions:

$$\langle \hat{F}_z(t)\hat{F}_z(s)\rangle = \delta(t-s)\left[\Lambda + \gamma_z\left(\langle\hat{\sigma}_e\rangle + \langle\hat{\sigma}_e\rangle\right)\right] \quad , \tag{8.94}$$

$$\langle \hat{F}_\sigma^+(t)\hat{F}_\sigma(s)\rangle = \delta(t-s)\left[2\gamma\langle\hat{\sigma}_e\rangle + \Lambda - \gamma_z\langle\hat{\sigma}_e\rangle\right] \quad , \tag{8.95}$$

$$\langle \hat{F}_\sigma(t)\hat{F}_\sigma^+(s)\rangle = \delta(t-s)\left[2\gamma\langle\hat{\sigma}_g\rangle - \gamma_z\langle\hat{\sigma}_g\rangle\right] \quad , \tag{8.96}$$

$$\langle \hat{F}^{+}(t)\hat{F}(s)\rangle = \delta(t-s)\frac{\omega}{Q}n_{th} \quad , \tag{8.97}$$

$$\langle \hat{F}(t)\hat{F}^{+}(s)\rangle = \delta(t-s)\frac{\omega}{Q}\left(1+n_{th}\right) \quad . \tag{8.98}$$

If the atomic dipole decay rate γ is much greater than the population difference decay rate γ_z and the photon decay rate $\frac{\omega}{Q}$, we can adiabatically eliminate the dipole operator $\hat{\sigma}$ by

$$\hat{\sigma} \simeq \frac{1}{\gamma}\left[ig\hat{\sigma}_z\hat{a} + \hat{F}_\sigma(t)\right] \quad , \tag{8.99}$$

where $\omega = \nu$ is assumed. Substituting (8.99) into (8.90) and (8.93), we obtain

$$\frac{d}{dt}\hat{a} = -\frac{1}{2}\left(\frac{\omega}{Q}\right)\hat{a} + \frac{g^2}{\gamma}N\hat{\sigma}_z\hat{a} + \hat{F}_a(t) \quad , \tag{8.100}$$

$$\frac{d}{dt}\hat{\sigma}_z = \Lambda - \gamma_z\hat{\sigma}_z - \frac{4g^2}{\gamma}\hat{\sigma}_z\left(\hat{a}^{+}\hat{a}+1\right) + \hat{F}_z + i\frac{2g}{\gamma}\left(\hat{a}^{+}\hat{F}_\sigma - \hat{F}_\sigma^{+}\hat{a}\right) \quad , \tag{8.101}$$

where $\hat{F}_a(t) = \hat{F}(t) - i\frac{g}{\gamma}N\hat{F}_\sigma(t)$.

If N identical atoms interact with the cavity field, (8.101) is transformed to the Langevin equation for a population inversion operator $\hat{N}_c \equiv N\hat{\sigma}_z$:

$$\frac{d}{dt}\hat{N}_c = p - \gamma_z\hat{N}_c - \frac{4g^2}{\gamma}\hat{N}_c\left(\hat{n}+1\right) + \hat{F}_N(t) \quad , \tag{8.102}$$

where $\hat{F}_N(t)$ is the noise operators associated with the total pump rate $p = N\Lambda$, the total spontaneous decay rate $-\gamma_z\hat{N}_c$, and the spontaneous and stimulated emission rate into a lasing mode $\hat{E}_{cv} = \frac{4g^2}{\gamma}\hat{N}_c$. The photon number operator $\hat{n} = \hat{a}^{+}\hat{a}$ satisfies the Langevin equation,

$$\frac{d}{dt}\hat{n} = -\left(\frac{\omega}{Q}\right)\hat{n} + \frac{4g^2}{\gamma}\hat{N}_c\left(\hat{n}+1\right) + \hat{F}_n(t) \quad , \tag{8.103}$$

where $\hat{F}_n(t)$ is the noise operator associated with the photon decay rate $-\left(\frac{\omega}{Q}\right)\hat{n}$ and the spontaneous and stimulated emission rate $\frac{4g^2}{\gamma}\hat{N}_c\left(\hat{n}+1\right)$ into a lasing mode. Equations (8.102) and (8.103) are often refereed to as quantum mechanical rate equations. The two time correlation functions in (8.102) and (8.103) are given by

$$\langle \hat{F}_n(t)\hat{F}_n(s)\rangle = \delta(t-s)\left[\left(\frac{\omega}{Q}\right)\bar{n} + E_{cv}\left(\bar{n}+1\right)\right] \quad , \tag{8.104}$$

$$\langle \hat{F}_N(t)\hat{F}_N(s)\rangle = \delta(t-s)\left[p + \gamma_z\bar{N}_c + E_{cv}\left(\bar{n}+1\right)\right] \quad , \tag{8.105}$$

$$\langle \hat{F}_n(t)\hat{F}_N(s)\rangle = \langle \hat{F}_N(t)\hat{F}_n(s)\rangle = -\delta(t-s)E_{cv}\left(\bar{n}+1\right) \quad , \tag{8.106}$$

where $E_{cv} = \frac{4g^2}{\gamma}\bar{N}_c$ is the average spontaneous emission rate into a lasing mode, $\bar{n}$ and $\bar{N}_c$ are the average photon number and population inversion.

8.4.2 Linearization

The Heisenberg-Langevin equations are nonlinear coupled differential equations and cannot be solved analytically. At well above threshold, both field and population inversion operators can be split into the (c-number) average values and small fluctuation operators:

$$\hat{a} = \left(\bar{A} + \Delta\hat{A}\right) e^{-i\Delta\hat{\phi}} \quad , \tag{8.107}$$

$$\hat{N}_c = \bar{N}_c + \Delta\hat{N}_c \quad , \tag{8.108}$$

$$\hat{n} = \bar{A}^2 = 2\bar{A}\Delta\hat{A} = \bar{n} + \Delta\hat{n} \quad . \tag{8.109}$$

The rate equations for $\bar{n}$ and $\bar{N}_c$ are

$$\frac{d}{dt}\bar{n} = -\left(\frac{\omega}{Q}\right)\bar{n} + E_{cv}\left(\bar{n}+1\right) \quad , \tag{8.110}$$

$$\frac{d}{dt}\bar{N}_c = p - \gamma_z \bar{N}_c - E_{cv}\left(\bar{n}+1\right) \quad , \tag{8.111}$$

where we used $\langle\hat{F}_n\rangle = \langle\hat{F}_N\rangle = 0$. The steady state solutions for (8.110) and (8.111) are obtained by assuming $\frac{d}{dt}\bar{n} = \frac{d}{dt}\bar{N}_c = 0$ to the first order:

$$\frac{\omega}{Q} \simeq E_{cv} \quad , \tag{8.112}$$

$$p \simeq E_{cv}\bar{n} \quad . \tag{8.113}$$

Here we neglected the spontaneous emission rate E_{cv} and $\gamma_z\bar{N}_c$, because they are small compared to the stimulated emission rate $E_{cv}\bar{n}$ at well above threshold.

Using (8.107) and (8.108) in (8.100) and (8.102), we have the linearized equations for fluctuation operators:

$$\frac{d}{dt}\Delta\hat{A} = \frac{1}{2\bar{A}\tau_{st}}\Delta\hat{N}_c + \frac{1}{2}\left[\hat{F}_a e^{i\Delta\hat{\phi}} + e^{-i\Delta\hat{\phi}}\hat{F}_a^+\right] \quad , \tag{8.114}$$

$$\frac{d}{dt}\Delta\hat{\phi} = \frac{i}{2\bar{A}}\left[\hat{F}_a e^{i\Delta\hat{\phi}} - e^{-i\Delta\hat{\phi}}\hat{F}_a^+\right] \quad , \tag{8.115}$$

$$\frac{d}{dt}\Delta\hat{N}_c = -\left(\frac{1}{\tau_{sp}} + \frac{1}{\tau_{st}}\right)\Delta\hat{N}_c - 2\left(\frac{\omega}{Q}\right)\bar{A}\Delta\hat{A} + \hat{F}_N \quad . \tag{8.116}$$

Here the spontaneous emission lifetime τ_{sp} and stimulated emission lifetime τ_{st} are defined by

$$\tau_{sp} = 1/\gamma_z \quad , \tag{8.117}$$

$$\tau_{st} = \frac{\gamma}{4g^2\bar{A}^2} \quad . \tag{8.118}$$

We now introduce the Fourier analysis for a wavepacket with a during of T, for instance, the amplitude fluctuation operator is Fourier transformed by,

$$\Delta\hat{A}(T,\Omega) = \frac{1}{\sqrt{T}}\int_{-\frac{T}{2}}^{\frac{T}{2}} \Delta\hat{A}(t)e^{-i\Omega t}dt \quad . \tag{8.119}$$

Since the amplitude noise is a statistically stationary process, we can calculate the power spectral density using the ordinary Wiener-Khintchine theorem [4]:

$$S_{\Delta A}(\Omega) = \lim_{T\to\infty} \langle \Delta\hat{A}^{+}(T,\Omega)\, \Delta\hat{A}(T,\Omega) \rangle \quad . \tag{8.120}$$

From (8.114), (8.119) and (8.120), we obtain

$$S_{\Delta A}(\Omega) = \frac{1}{\left(\Omega^2 + A_2 A_3\right)^2 + \Omega^2 A_1^2} \left[A_3^2 S_{F_N}(\Omega) + \left(\Omega^2 + A_1^2\right) S_{F_a}(\Omega) - 2A_1 A_3 S_{F_a F_N}(\Omega) \right] \quad . \tag{8.121}$$

where

$$A_1 = \frac{1}{\tau_{sp}} + \frac{1}{\tau_{st}} \quad , \tag{8.122}$$

$$A_2 = 2\left(\frac{\omega}{Q}\right)\bar{A} \quad , \tag{8.123}$$

$$A_3 = \frac{1}{2\bar{A}\tau_{st}} \quad . \tag{8.124}$$

The power spectral densities of the noise operators can be obtained from their two time correlation functions,

$$S_{F_N}(\Omega) = 2\left[p + \frac{\bar{N}_c}{\tau_{sp}} + E_{cv}\left(\bar{n}+1\right)\right] \quad , \tag{8.125}$$

$$2\bar{A}^2 S_{F_a}(\Omega) = S_{F_n}(\Omega) = 2\left[\frac{\omega}{Q}\bar{n} + E_{cv}\left(\bar{n}+1\right)\right] \quad , \tag{8.126}$$

$$S_{F_N} \cdot 2\bar{A}^2 S_{F_a}(\Omega) = -2E_{cv}\left(\bar{n}+1\right) \quad . \tag{8.127}$$

Similarly, the frequency noise spectral density is calculated from (8.115):

$$S_{\Delta\omega}(\Omega) = \frac{1}{\bar{A}^2} S_{F_a}(\Omega) \quad . \tag{8.128}$$

The instantaneous frequency $\omega(t) = \frac{d}{dt}\Delta\phi(t)$ has a white noise under the assumption that the two time correlation function of the noise operator is given by δ-functions such as (8.104)-(8.106).

8.4.3 Origin of the standard quantum limit

At far above threshold, the amplitude noise spectrum (8.121) is reduced to a simple Lorentzien,

$$S_{\Delta A}(\Omega) = \frac{\left(\frac{\omega}{Q}\right)}{\Omega^2 + \left(\frac{\omega}{Q}\right)^2} \quad . \tag{8.129}$$

Here the spontaneous emission noise $\frac{2\bar{N}_c}{\tau_{sp}}$ in (8.125) is negligible compared to the stimulated emission noise $E_{cv}\bar{n}$ in (8.125). However, the stimulated emisson noise in $S_{F_N}(\omega)$ and $S_{F_a}(\omega)$ cancel each other out due to their negative correlation (8.127). Consequently, only non-zero contributions to the amplitude noise spectrum at far above threshold are the pump noise $2P$ in (8.125) and the vacuum field fluctuation $2\left(\frac{\omega}{Q}\right)\bar{n}$ in (8.126).

Using Parseval theorem [4], the total amplitude noise power $\langle \Delta \hat{A}^2 \rangle$ is obtained as

$$\langle \Delta \hat{A}^2 \rangle = \int_0^\infty S_{\Delta A}(\Omega) \frac{d\Omega}{2\pi} = \frac{1}{4} \quad . \tag{8.130}$$

We now recover the identical result obtained from the density operator master equation and quantum mechanical Fokker-Planck equation, which suggests the laser field at far above threshold approaches to a coherent state with the Poisson photon statistics.

Also at far above threshold, the frequency noise spectrum (8.128) is reduced to the simple white noise:

$$S_{\Delta\omega}(\Omega) = \frac{\left(\frac{\omega}{Q}\right)}{\bar{A}^2} \quad . \tag{8.131}$$

The frequency noise spectrum is equally contributed by the vacuum field fluctuation $2\left(\frac{\omega}{Q}\right)\bar{n}$ and the stimulated emission noise $2E_{cv}\bar{n}$ in (8.126). If the atomic system is not perfectly inverted, it is straightforward to show the frequency noise is enhanced as

$$S_{\Delta\omega}(\Omega) = \frac{\left(\frac{\omega}{Q}\right) n_{sp}}{\bar{A}^2} \quad , \tag{8.132}$$

where $n_{sp} = \frac{N_e}{N_e - N_g}$ is the population inversion parameter which is a ratio of the spontaneous emission rate and net stimulated emission rate into a lasing mode.

8.4.4 Phase noise and spectral linewidth

The phase noise of a laser is a classic example of the Wiener-Levy process [4] which is the time integral of a white noise, i.e.

$$\Delta\phi(t) = \int_0^t \Delta\omega(t')dt' \quad , \tag{8.133}$$

and features a random walk diffusion. Such a statistically nonstationary process cannot be analyzed by the ordinary Fourier transform. We must define the gated function for the phase noise by

$$\Delta\phi_T(t) = \begin{cases} \int_0^t \Delta\omega(t')dt' & (0 \leq t \leq T) \\ 0 & \text{otherwise} \end{cases} \quad , \tag{8.134}$$

to use the Fourier analysis. The autocorrelation function of (8.134) is given by

$$\Phi_{\Delta\phi}(\tau, T) = \frac{1}{T} \int_0^{T-|\tau|} \langle \Delta\phi_T(t+\tau)\Delta\phi_T(t) \rangle dt \quad , \tag{8.135}$$

where τ is a small delay time ($\tau \ll T$) and

$$\langle \Delta\phi_T(t+\tau)\Delta\phi_T(t) \rangle = \int_0^{t+\tau} \int_0^t \langle \Delta\omega(t')\Delta\omega(t'') \rangle dt'dt'' \quad . \tag{8.136}$$

Since the frequency noise $\Delta\omega(t)$ is a statistically stationary process and satisfies the ergodic theorem [4] that the time average is equal to the ensemble average, we obtain

$$\begin{aligned}\langle \Delta\omega(t')\Delta\omega(t'')\rangle &= \Phi_{\Delta\omega}(t'-t'') \\ &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \Delta\omega(t+\tau)\Delta\omega(t)dt \\ &= \frac{1}{2\pi}\int_0^\infty S_{\Delta\omega}(\Omega)\cos(\Omega\tau)d\Omega \quad ,\end{aligned} \tag{8.137}$$

where $\tau = t' - t''$ and the Wiener-Khintchine theorem is used to derive the last line. Using (8.137) in (8.136), we have

$$\begin{aligned}\langle \Delta\phi_T(t+\tau)\Delta\phi_T(t)\rangle &= \frac{1}{2\pi}\int_0^\infty d\Omega S_{\Delta\omega}(\Omega)\int_0^{t+\tau}\int_0^t \cos\left[\Omega(t'-t'')\right]dt'dt'' \\ &= \frac{1}{2\pi}\int_0^\infty S_{\Delta\omega}(\Omega)\frac{1}{\Omega^2}\{1+\cos(\Omega\tau)-\cos(\Omega t) \\ &\quad -\cos\left[\Omega(t+\tau)\right]\}d\Omega \quad .\end{aligned} \tag{8.138}$$

If we set $\tau = 0$ in (8.138), the variance in phase noise is obtained as

$$\begin{aligned}\langle \Delta\phi_T(t)^2\rangle &= \frac{1}{\pi}\int_0^\infty S_{\Delta\omega}(\Omega)\frac{1}{\Omega^2}\left[1-\cos(\Omega t)\right]d\Omega \\ &= 2D_{\Delta\phi}t \quad ,\end{aligned} \tag{8.139}$$

where the diffuaion constant is given by

$$D_{\Delta\phi} = \frac{1}{4}S_{\Delta\omega}(\Omega) \quad . \tag{8.140}$$

For a non-zero τ, we split $\Delta\phi_T(t+\tau)$ into

$$\Delta\phi_T(t+\tau) = \Delta\phi_T(t) + \delta\left[\Delta\phi_T(t,\tau)\right] \quad . \tag{8.141}$$

Then we have

$$\begin{aligned}\langle \Delta\phi_T(t+\tau)\Delta\phi_T(t)\rangle &= \langle \Delta\phi_T(t)^2\rangle + \langle \Delta\phi_T(t)\ \delta\left[\Delta\phi_T(t,\tau)\right]\rangle \\ &= 2D_{\Delta\phi}t \quad .\end{aligned} \tag{8.142}$$

Here we use the fact that $\Delta\phi_T(t)$ and $\delta\left[\Delta\phi_T(t,\tau)\right]$ are uncorrelated and $\langle \delta\left[\Delta\phi_T(t,\tau)\right]\rangle = 0$. Using (8.142) in (8.135), we obtain

$$\begin{aligned}\Phi_{\Delta\phi}(\tau,t) &= \frac{1}{T}\int_0^{T-\tau} 2D_{\Delta\phi}tdt \\ &= D_{\Delta\phi}T\left(1-\frac{\tau}{T}\right)^2\end{aligned} \tag{8.143}$$

The Fourier transform of (8.143) provides the phase noise spectrum

$$\begin{aligned}S_{\Delta\phi}(\Omega,t) &= 4\int_0^\infty \Phi_{\Delta\phi}(\tau,t)\cos(\Omega\tau)d\tau \\ &= \frac{8D_{\Delta\phi}}{\Omega^2}\left[1-\frac{\sin(\Omega T)}{(\Omega T)}\right]\end{aligned} \tag{8.144}$$

The phase noise spectrum $S_{\Delta\phi}(\Omega, T)$ has a $1/\Omega^2$ dependence at a high frequency region $\Omega T \gg 1$. This is the characteristic of the Wiener-Levy process. At a low frequency region $\Omega T \ll 1$, the phase noise spectrum is flattened as

$$S_{\Delta\phi}\left(\Omega \ll 1/T\right) = 48 D_{\Delta\phi} T^2 \quad , \tag{8.145}$$

which is the artifact introduced by the finite gate time T.

The field power spectrum of a laser is defined as the Fourier transform of the two time correlation function,

$$I(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} \langle \hat{a}^{+}(t)\hat{a}(0)\rangle \quad , \tag{8.146}$$

where ω is an optical angular frequency. At well above threshold, the amplitude is well stabilized to its average value $\bar{A}$ but the phase goes through a random-walk diffusion. Thus, we have

$$\begin{aligned} \langle \hat{a}^{+}(0)\hat{a}(0)\rangle &\simeq \bar{A}^2 \langle \exp\left\{-i\left[\Delta\hat{\phi}(t) - \Delta\hat{\phi}(0)\right]\right\}\rangle \\ &= \bar{A}^2 \exp\left(-2D_{\Delta\phi} t\right) \quad . \end{aligned} \tag{8.147}$$

Substituting (8.147) into (8.146), we have the Lorentzian spectrum with the full width at half-maximum of

$$2D_{\Delta\phi} = \frac{\left(\frac{\omega}{Q}\right)}{2\bar{A}^2} n_{sp} \quad . \tag{8.148}$$

This is the Schawlow-Tounes linewidth obtained by the density operator master equation and the quantum mechanical Fokker-Planck equation

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