$$\frac{d}{dt}A_{Di} = -\frac{1}{2}\left(\frac{\omega}{Q} - E_{CVi}\right)A_{Di} + \frac{\omega}{Q}\sqrt{n_M}\sqrt{\zeta^2 + \eta_i^2 \cos}\,(\varsigma_i - \phi_{Di})$$

$$-\sum_{j\neq 1}\frac{1}{2}\xi_{ij}\frac{\omega}{Q}\left[A_{Di}\cos\left(\phi_{Dj} - \phi_{Di}\right) - A_{\bar{D}j}\cos\left(\phi_{\bar{D}j} - \phi_{Di}\right)\right] + F_{Di}(t)$$

$$\frac{d}{dt}\phi_{Di} = \frac{1}{A_{Di}}\frac{\omega}{Q}\sqrt{n_M}\sqrt{\zeta^2 + \eta_i^2}\,\sin\left(\varsigma_i - \phi_{Di}\right)$$

$$-\sum_{j\neq 1}\frac{1}{2}\xi_{ij}\frac{\omega}{Q}\left[A_{Dj}\sin\left(\phi_{Dj} - \phi_{Di}\right) - A_{\bar{D}j}\sin\left(\phi_{\bar{D}j} - \phi_{Di}\right)\right] + G_{Di}(t)$$

$$\frac{d}{dt}A_{\bar{D}i} = -\frac{1}{2}\left(\frac{\omega}{Q} - E_{CVi}\right)A_{\bar{D}i} + \frac{\omega}{Q}\sqrt{n_M}\sqrt{\zeta^2 + \eta_i^2}\cos\left(\varsigma_i - \phi_{\bar{D}i}\right)$$

$$-\sum_{j\neq 1}\frac{1}{2}\xi_{ij}\frac{\omega}{Q}\left[A_{Di}\cos\left(\phi_{\bar{D}j} - \phi_{Di}\right) - A_{\bar{D}j}\cos\left(\phi_{\bar{D}j} - \phi_{\bar{D}}\right)\right] + F_{\bar{D}i}(t)$$

$$\frac{d}{dt}\phi_{\bar{D}i} = \frac{1}{A_{\bar{D}i}}\frac{\omega}{Q}\sqrt{n_M}\sqrt{\zeta^2 + \eta_i^2}\sin\left(-\delta_i - \phi_{Di}\right)$$

$$-\sum_{j\neq 1}\frac{1}{2}\xi_{ij}\frac{\omega}{Q}\left[A_{Di}\sin\left(\phi_{Dj} - \phi_{Di}\right) - A_{\bar{D}j}\sin\left(\phi_{\bar{D}j} - \phi_{\bar{D}i}\right)\right] + G_{\bar{D}i}(t)$$

$$\frac{d}{dt}N_i = P - \frac{N_i}{\tau_{sp}}E_{CVi}\left(A_{Di}^2 + A_{\bar{D}i}^2 + 2\right) + F_{Ni}(t).$$

$$n_{Ri} = \left|\frac{1+i}{2}A_{Di}\exp(i\phi_{Di}) + \frac{1-i}{2}A_{\bar{D}i}\exp(i\phi_{\bar{D}i})\right|^2$$

$$n_{Li} = \left|\frac{1-i}{2}A_{Di}\exp(i\phi_{Di}) + \frac{1+i}{2}A_{\bar{D}i}\exp(i\phi_{\bar{D}i})\right|^2$$

4.4 Injection-locking bandwidth, phase shift and reflection dip

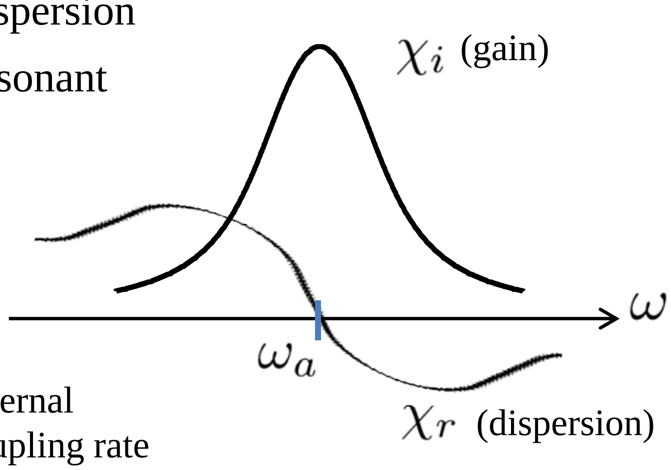
If we neglect the noise term but take into account the gain induced dispersion term in the previous working equation, the c number field amplitude obeys

$$\frac{d}{dt}A(t) = -i\omega_c A(t) - \frac{1}{2}\left[\frac{\omega}{Q} - \frac{\omega}{\mu^2}\left(\chi_i - i\chi_r\right)\right]A(t) + \sqrt{\frac{\omega}{Q_e}}F_0 e^{-i\omega t}$$

$\frac{\omega}{\mu^2}\chi_i = E_{cv} - E_{vc}$: net gain coefficient

$\frac{\omega}{\mu^2}\chi_r$: associated dispersion

ω_c : cold cavity resonant frequency



$$\frac{\omega}{Q} = \frac{\omega}{Q_i} + \frac{\omega}{Q_e}$$

total photon decay rate — internal decay rate — external coupling rate

$\omega_0 = \omega_c + \frac{\omega}{2\mu^2}\chi_r$: free-running laser frequency

$r(t) = -F_0 e^{-i\omega t} + \sqrt{\frac{\omega}{Q_e}}A(t)$: input-output relation

The total electron number obeys

$$\frac{d}{dt}N(t) = P - \frac{N(t)}{\tau_{sp}} - \left(E_{cv} - E_{vc}\right)n(t) - E_{cv}$$

$n(t) \equiv |A(t)|^2$: total photon number

If an injection signal $F_0 e^{-i\omega t}$ couples into a slave laser, the slave laser frequency is locked to that of the injection signal, the internal photon number *n(t)* increases and the electron number *N(t)* decreases. Also if $\omega \neq \omega_0$, the phase of the internal field is shifted with respect to that of the injection signal. Since the electron number changes, the gain and dispersion are also modulated by the injection signal.

$$N(t) = N_0 + \Delta N(t)$$

$$A(t) = [A_0 + \Delta A(t)]\, e^{-i(\omega t + \phi)}$$

$$n(t) = A_0^2 + 2A_0 \Delta A(t)$$

$$r(t) = [r_0 + \Delta r(t)]\, e^{-i(\omega t + \psi)}$$

$$\chi_i = \chi_{i0} + \frac{d\chi_{i0}}{dN_0}\Delta N(t)$$

$$\chi_r = \chi_{r0} + \frac{d\chi_{r0}}{dN_0}\Delta N(t)$$

$$\Downarrow$$

Steady state solutions, $\frac{d}{dt}A(t) = 0$, read

$$E_{cv} - E_{vc} = \frac{\omega}{Q} - 2\frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}}\cos\phi$$

$$\omega - \omega_0' = -\frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}}\sin\phi \qquad \Delta N = \frac{\chi_i - \chi_{i0}}{\frac{d\chi_{i0}}{dN_0}} = \frac{-2\frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}}\cos\phi}{\frac{\omega}{\mu^2}\cdot\frac{d\chi_{i0}}{dN_0}}$$

$$\omega_0' = \omega_c + \frac{\omega}{2\mu^2}\left[\chi_{r0} + \frac{d\chi_{r0}}{dN_0}\Delta N\right]$$

$$= \omega_0 - \left(\frac{d\chi_{r0}}{dN_0}\right)\frac{\frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}}\cos\phi}{\left(\frac{d\chi_{i0}}{dN_0}\right)}$$

$$\omega_0' = \omega_0 - \alpha \frac{F_0}{A_0} \sqrt{\frac{\omega}{Q_e}} \cos\phi$$

slave laser free-running frequency before injection-locking

$\alpha \equiv \left(\frac{d\chi_{r0}}{dN_0}\right) / \left(\frac{d\chi_{i0}}{dN_0}\right)$: linewidth enhancement factor

Frequency deturning parameter between slave laser and injection signal should be defined by

$$\begin{aligned}\Delta\omega &\equiv \omega - \omega_0 \\ &= -(\sin\phi + \alpha\cos\phi)\frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}}\end{aligned}$$

Since $E_{cv} - E_{vc} \le \frac{\omega}{Q}$ for an injection-locked laser, the phase shift is bounded by $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$

Case I: $\alpha = 0$

$$|\sin\phi| < 1 \longrightarrow -\frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}} < \Delta\omega < \frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}}$$

$\phi = \pi/2$ $\qquad\qquad$ $\phi = -\pi/2$

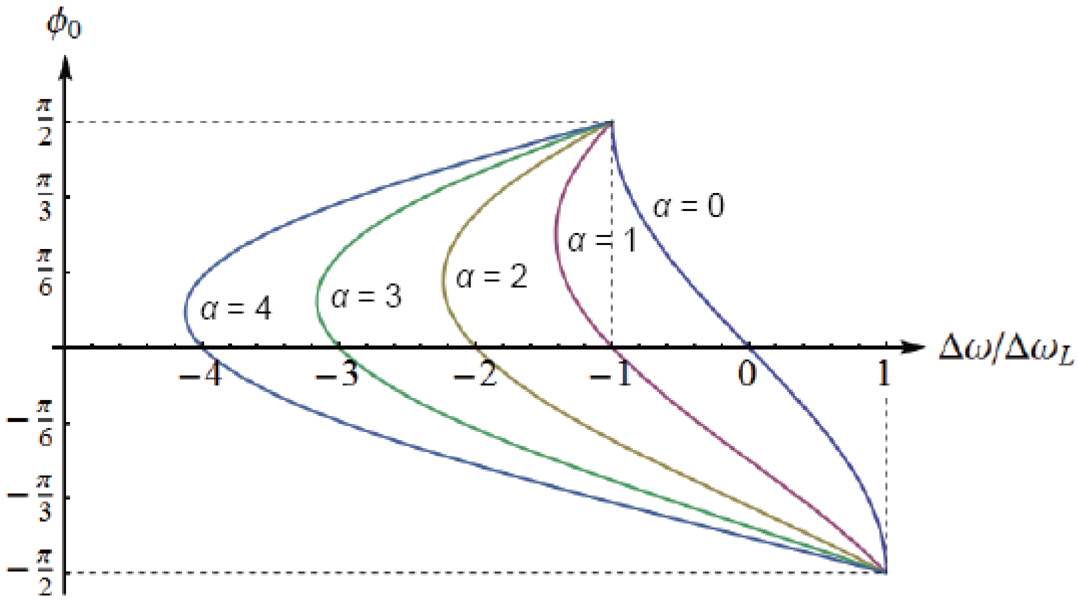
$$\Delta\omega_L \equiv \frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}} = \frac{\omega}{Q_e}\sqrt{\frac{p_{\text{in}}}{p_{\text{out}}}}$$: injection-locking bandwidth

Case II: $\alpha \neq 0$

$$-\sqrt{1+\alpha^2}\frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}} \le \Delta\omega < \frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}}$$ (when α >0)

$$\phi = 0 \quad \text{at} \quad \Delta\omega = -\alpha\frac{F_0}{A_0}\sqrt{\frac{\omega}{Q_e}}$$

The locking bandwidth becomes asymmetric.



When $\alpha > 0$, the locking bandwidth extends to a low-frequency side where the system shows a bi-state operation. When $\alpha < 0$, the locking bandwidth extends to a high-frequency side.

The output amplitude r and phase ψ of the slave laser are given by

$$r_0 = \left[\frac{\omega}{Q_e}A_0^2 + F_0^2 - 2F_0\sqrt{\frac{\omega}{Q_e}}A_0\cos\phi\right]^{1/2}$$

interference term

$$\sin\psi = \sqrt{\frac{\omega}{Q_e}}\frac{A_0}{r_0}\sin\phi$$

The output power is minimum, $r_0^2, \min = \left(\sqrt{\frac{\omega}{Q_e}}A_0 - F_0\right)^2$, and the phase is equal to ψ=0 at ϕ=0.

⇩

This is the point where we want to initialize the slave lasers before the computation starts.

4.5 Transient time of the injection-locked laser network

Mutual coupling via horizontal polarizer

Net loss difference between right answer (|R⟩ for instance) and wrong answer |L⟩:

$$2\xi_{ij} \sim 2\alpha\left(\frac{\omega}{Q}\right) \qquad \text{if} \quad n_M = n_{Ri} + n_{Li} = n_T$$

Monotonical increase in the right answer $n_R(t) \rightarrow \ \sim 2n_R(0)$

Exponential decrease in the wrong answer $n_L(t) \sim n_L(0)e^{-2\alpha\left(\frac{\omega}{Q}\right)t}$

Overall success probability

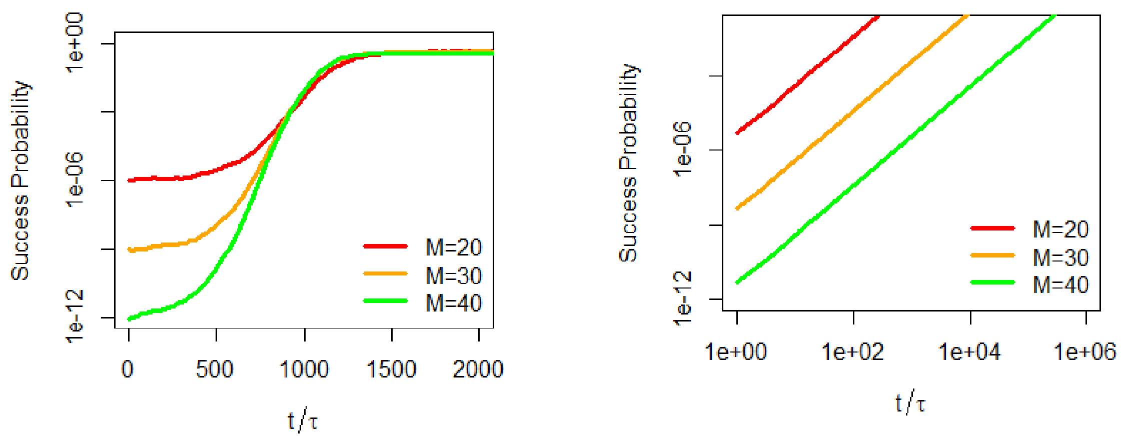
$$P_s(t) = \left[\frac{n_R(t)}{n_R(t) + n_L(t)}\right]^M = \begin{cases} 1/2^M \left(2\alpha\left(\frac{\omega}{Q}\right)t \ll 1\right) \\ 1 - \frac{M}{2}\exp\left[-2\alpha\left(\frac{\omega}{Q}\right)t\right] \left(2\alpha\left(\frac{\omega}{Q}\right)t \gg 1\right) \end{cases}$$

right answer　　wrong answer

$$P_s(t) \geq 1 - \delta(\delta \ll 1)$$ requires the computational time

$$t_c \gtrsim In\,(M\,/\,2\varsigma)\,/\,2\alpha\left(\frac{\omega}{Q}\right)$$

- The transient time has negligible dependence on the problem size (~ln (M))
- A strong coupling constant (α) and fast photon decay rate $\left(\frac{\omega}{Q}\right)$ speed up the transient time.



The overall success probability $P_S(t)$ of the injection locked laser network on the 1D anti-ferromagnetic Ising model with the problem size M vs. that of the quantum computer on the quantum counting algorithm.

$\tau = \left(\frac{\omega}{Q}\right)^{-1}$ ~1 psec for the laser network and τ ~1 msec for the quantum computer.