Chapter 5

Quantization of the Spins

As pointed out already in chapter 3, the external degrees of freedom, position and momentum, of an ensemble of identical atoms is described by the Scödinger field operator. As for the quantization of the internal degrees of freedom of the same ensemble, we have an appropriate mathematical framework, which is the quantization of collective angular momentum operators. A single two-level atom is often represented by a (fermionic) Pauli spin operator, while an ensemble of two-level atoms is conveniently described by a (bosonic) collective angular momentum operator. In this section, we will present a formal theory of collective angular momentum algebra.

5.1 Quantization of the orbital angular momentum

5.1.1 Commutation relation

We do not need a new postulate to quantize an orbital angular momentum. A standard quantization postulate, $[\hat{q}, \hat{p}] = i\hbar$, automatically quantize an orbital angular momentum. An orbital angular momentum operator, $\mathbf{l} = \mathbf{q} \times \mathbf{p}$, is decomposed into three cartesian components:

$$\begin{aligned} \hat{l}_x &= \hat{q}_y \hat{p}_z - \hat{q}_z \hat{p}_y \\ \hat{l}_y &= \hat{q}_z \hat{p}_x - \hat{q}_x \hat{p}_z \\ \hat{l}_z &= \hat{q}_x \hat{p}_y - \hat{q}_y \hat{p}_z \quad . \end{aligned} \tag{5.1}$$

Using the commutation relations, $[\hat{q}_i, \hat{p}_j] = i\hbar\delta ij$, we can easily derive the commutation relation for $\hat{l}_x, \hat{l}_y$ and $\hat{l}_z$:

$$\begin{aligned} \left[\hat{l}_x, \hat{l}_y\right] &= [\hat{q}_y \hat{p}_z - \hat{q}_z \hat{p}_y, \hat{q}_z \hat{p}_x - \hat{q}_x \hat{p}_z] \\ &= \hat{q}_y [\hat{p}_z, \hat{q}_z] \hat{p}_x + \hat{q}_x [\hat{q}_z, \hat{p}_z] \hat{p}_y \\ &= i\hbar (\hat{q}_x \hat{p}_y - \hat{q}_y \hat{p}_x) \\ &= i\hbar \hat{l}_z, \\ \left[\hat{l}_y, \hat{l}_z\right] &= i\hbar \hat{l}_x, \\ \left[\hat{l}_z, \hat{l}_x\right] &= i\hbar \hat{l}_y \quad . \end{aligned} \tag{5.2}$$

If we define the total angular momentum operator by

$$\hat{l}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2 \quad , \tag{5.3}$$

we can show

$$\left[\hat{l}^2, \hat{l}_i\right] = 0 \quad (i = x, y, z) \quad . \tag{5.4}$$

The above result indicates the total angular momentum $\hat{l}^2$ and one of the three cartesian components, for instance $\hat{l}_z$, can be determined simultaneously without any quantum uncertainty. A post-measurement state after such a simultaneous measurement of the two observables, $\hat{l}^2$ and $\hat{l}_z$, is called an angular momentum eigenstate.

5.1.2 Angular momentum eigenstates

An angular momentum eigenstate is specified by the two eigenvalues $|\nu, m\rangle$ corresponding to the two observables $\hat{l}^2$ and $\hat{l}_z$:

$$\hat{l}^2 |\nu, m\rangle = \hbar^2 \nu |\nu, m\rangle \quad , \tag{5.5}$$

$$\hat{l}_z |\nu, m\rangle = \hbar m |\nu, m\rangle \quad . \tag{5.6}$$

In this case, z-direction is chosen as a quantization axis. We can alternatively choose x or y direction as a quantization axis.

Now let us introduce a non-Hamiltonian operator defined by

$$\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y \quad . \tag{5.7}$$

These operators do not commute mutually and also with $\hat{l}_z$:

$$\left[\hat{l}_+, \hat{l}_-\right] = 2\hbar \hat{l}_z \quad , \tag{5.8}$$

$$\left[\hat{l}_z, \hat{l}_\pm\right] = \pm\hbar \hat{l}_\pm \quad . \tag{5.9}$$

The total angular momentum is now expanded as

$$\hat{l}^2 = \hat{l}_z^2 + \frac{1}{2}\left(\hat{l}_+\hat{l}_- + \hat{l}_-\hat{l}_+\right) \quad . \tag{5.10}$$

From (5.8) and (5.10), we have the following relations

$$\hat{l}_+\hat{l}_- = \hat{l}^2 - \hat{l}_z^2 + \hbar\hat{l}_z \quad , \tag{5.11}$$

$$\hat{l}_-\hat{l}_+ = \hat{l}^2 - \hat{l}_z^2 - \hbar\hat{l}_z \quad . \tag{5.12}$$

If we create a new state by projecting $\hat{l}_+$ or $\hat{l}_-$ on the angular momentum eigenstate $|\nu, m\rangle$, the resulting new state still satisfies the original eigenvalue relation for $\hat{l}^2$:

$$\hat{l}^2 \left(\hat{l}_\pm |\nu, m\rangle\right) = \hat{l}_\pm \hat{l}^2 |\nu, m\rangle = \hbar^2 \nu \left(\hat{l}_\pm |\nu, m\rangle\right) \quad , \tag{5.13}$$

which suggests $|\nu, m\rangle, \hat{l}_+|\nu, m\rangle$ and $\hat{l}_-|\nu, m\rangle$ are the eigenstates of $\hat{l}^2$ with the identical eigenvalue ν. On the other hand, those two new states satisfy slightly different eigenvalue relations for $\hat{l}_z$:

$$\hat{l}_z \left(\hat{l}_\pm |\nu, m\rangle \right) = \hat{l}_\pm \hat{l}_z |\nu, m\rangle \pm \hbar \hat{l}_\pm |\nu, m\rangle = \hbar (m \pm 1) \left(\hat{l}_\pm |\nu, m\rangle \right) \quad , \tag{5.14}$$

which shows $|\nu, m\rangle, \hat{l}_+|\nu, m\rangle$ and $\hat{l}_-|\nu, m\rangle$ are the eigenstates of $\hat{l}_z$ with the different eigenvalues $\hbar m, \hbar(m+1)$ and $\hbar(m-1)$, respectively. Because of this property of shifting an eigenvalue by one, $\hat{l}_+$ and $\hat{l}_-$ are called a rasing and lowering operator.

The norm of a state $\hat{l}_+|\nu, m\rangle$ is calcurated by using (5.12)

$$\langle \nu, m | \hat{l}_- \hat{l}_+ | \nu, m\rangle = \hbar^2 \left(\nu - m^2 - m \right) \quad , \tag{5.15}$$

which must be non-negative so that the eigenvalue m is upper bounded by

$$m(m+1) \leq \nu \quad . \tag{5.16}$$

The maximum eigenvalue $m_{\max} = j$ is given by the relation, $\nu = j(j+1)$. Similarly, the norm of a state $\hat{l}_-|\nu, j-k\rangle$ is calculated by using (5.11)

$$\begin{aligned} \langle \nu, j-k | \hat{l}_+ \hat{l}_- | \nu, j-k\rangle &= \hbar^2 \left[j(j+1) - (j-k)^2 + (j-k) \right] \\ &= \hbar^2 (k+1)(2j-k) \quad , \end{aligned} \tag{5.17}$$

which must be also non-negative so that k is upper bounded by

$$k \leq 2j \quad . \tag{5.18}$$

Since the number of lowering operations k takes a positive integer, the maximum eigenvalue j of the observable $\hat{l}_z$ takes only $j = 1/2, 1, 3/2, \cdots$. We use j instead of $\nu = j(j+1)$ to represent the eigenvalue of $\hat{l}^2$ so that the angular momentum eigenstate is defined by

$$\hat{l}_z |j, m\rangle = \hbar m |j, m\rangle \quad , \tag{5.19}$$

$$\hat{l}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle \quad . \tag{5.20}$$

Three lowest angular momentum eigenstates are

$j = 1/2$	$\lvert\frac{1}{2}, -\frac{1}{2}\rangle$ $\lvert\frac{1}{2}, \frac{1}{2}\rangle$
$j = 1$	$\lvert 1, -1\rangle$ $\lvert 1, 0\rangle$ $\lvert 1, 1\rangle$
$j = 3/2$	$\lvert 3/2, -3/2\rangle$ $\lvert 3/2, -1/2\rangle$ $\lvert 3/2, 1/2\rangle$ $\lvert 3/2, 3/2\rangle$

5.1.3 Recursion relation

Let us next establish the projection property of the rasing and lowering operators. We write down a new state created by $\hat{l}_+$ as

$$\hat{l}_+ |j, m\rangle = \hbar \lambda_{jm} |j, m+1\rangle \quad , \tag{5.21}$$

where λ_{jm} is a c-number constant we want to determine. Projecting $\langle j, m+1|$ from the left of (5.21), we have

$$\langle j, m+1 | \hat{l}_+ | j, m\rangle = \hbar \lambda_{jm} \quad . \tag{5.22}$$

The adjoint of (5.22) is

$$\langle j,m|\hat{l}_-|j,m+1\rangle = \hbar\lambda_{jm}^* \quad , \tag{5.23}$$

which suggests

$$\hat{l}_-|j,m+1\rangle = \hbar\lambda_{jm}^*|j,m\rangle \quad . \tag{5.24}$$

Combining (5.21) and (5.24) together with (5.12), we have

$$\begin{aligned} \hat{l}_-\hat{l}_+|j,m\rangle &= \hbar^2|\lambda_{jm}|^2|j,m\rangle \\ &= \hbar^2\left[j(j+1)-m^2-m\right]|j,m\rangle \quad . \end{aligned} \tag{5.25}$$

If we neglect an irrelevant phase factor, a constant λ_{jm} is given by

$$\lambda_{jm} = \sqrt{j(j+1)-m(m+1)} \quad . \tag{5.26}$$

Using this result, we now establish the recursion relation for the raising and lowering operators:

$$\hat{l}_+|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}|j,m+1\rangle \quad , \tag{5.27}$$

$$\hat{l}_-|j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}|j,m-1\rangle \quad . \tag{5.28}$$

5.2 Connection of angular momentum algebra to an ensemble of two-level atoms

5.2.1 Pauli spin operator

A two-level atom with an excited state $|e\rangle$ and ground state $|g\rangle$ is described as a "fictitious" spin$-1/2$ particle with $j = 1/2$. An analogy we will use here is that an up-spin state corresponds to an excited state, $|e\rangle = |\frac{1}{2},\frac{1}{2}\rangle$, and a down-spin state corresponds to a ground state, $|g\rangle = |\frac{1}{2},-\frac{1}{2}\rangle$. We normally use a Pauli spin operator in place of an angular momentum operator defined by

$$\hat{\mathbf{l}} = \frac{\hbar}{2}\hat{\sigma} \quad . \tag{5.29}$$

The commutation relation (5.2) is now rewritten as

$$[\hat{\sigma}_i,\hat{\sigma}_j] = 2i\hat{\sigma}_k \quad (i,j,k = \text{cyclic permutation of x,y,z}) \quad . \tag{5.30}$$

The total angular momentum (5.3) and its commutator bracket with $\hat{l}_i$ are also rewritten as

$$\hat{\sigma}^2 = \hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_z^2 \quad , \tag{5.31}$$

$$\left[\hat{\sigma}^2,\hat{\sigma}_i\right] = 0 \quad . \tag{5.32}$$

In order to restrict the eigenvalues of $\hat{\sigma}_z$ to ± 1, which corresponds to the eigenvalues of $\hat{l}_z = \pm\hbar/2$, we introduce a new postulate:

$$[\hat{\sigma}_i,\hat{\sigma}_j]_+ = \hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{ij} \quad , \tag{5.33}$$

which is called an anti-commutation relation. From (5.30) and (5.33), we have

$$\hat{\sigma}_i^2 = 1 \quad , \tag{5.34}$$

$$\hat{\sigma}_i \hat{\sigma}_j = i\hat{\sigma}_k \quad (i \neq j) \quad . \tag{5.35}$$

The raising and lowering operators (5.7) can be re-defined as

$$\hat{\sigma}_{\pm} = \frac{1}{2}\left(\hat{\sigma}_x \pm i\hat{\sigma}_y\right) \quad , \tag{5.36}$$

which satisfy the anti-commutator bracket:

$$[\hat{\sigma}_+, \hat{\sigma}_-]_+ = 1 \quad . \tag{5.37}$$

From (5.33) and (5.34), $\hat{\sigma}_+$ and $\hat{\sigma}_-$ also satisfy

$$\hat{\sigma}_+^2 = \hat{\sigma}_-^2 = 0 \quad . \tag{5.38}$$

The above result indicates that we cannot raise or lower the eigenvalue of $\hat{\sigma}_z$ successively, which should be the case for a spin-1/2 particle (or two-level atom).

The matrix representation of the spin operators and eigenstates of $\hat{\sigma}_z$ are useful for later use and now summarized below:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{5.39}$$

$$\hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad . \tag{5.40}$$

$$|e\rangle = |\frac{1}{2}, \frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |g\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \tag{5.41}$$

5.2.2 Collective spin operators

An internal state of the ensemble of identical two-level atom is expanded by a set of 2^N orthogonal states such as $|e\rangle_1|g\rangle_2 \cdots |e\rangle_N$, where N is a total number of atoms. We can introduce collective spin operators to represent the internal state of the N atoms in a corresponding Hilbert space of 2^N dimensions:

$$\hat{J}_i = \frac{1}{2}\sum_{n=1}^{N} \sigma_{ni} \quad (i = x, y, z) \quad , \tag{5.42}$$

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \quad , \tag{5.43}$$

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hat{J}_k \quad (i, j, k = \text{cyclic permutation of x,y,z}) \quad , \tag{5.44}$$

$$\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y \quad , \tag{5.45}$$

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hat{J}_z \quad . \tag{5.46}$$

While the Pauli spin operator $\hat{\sigma}_{\pm}$ satisfy the anti-commutation relation (5.37) and a continuous excitation or de-exciation is inhibited as demonstrated by (5.38), the collective spin operator $\hat{J}_{\pm}$ does not have such a constraint. Because of this difference, the Pauli spin operator is said "fermionic" and the collective spin operator is called "bosonic". Of course, such a terminology is a simple analogy to fermionic and bosonic algebra and does not mean a particular atom of interest is either a fermion or boson.

5.3 Various quantum states of an ensemble of atoms

5.3.1 Angular momentum eigenstate

A simultaneous eigenstate of $\hat{J}^2$ and $\hat{J}_z$ is called an angular momentum eigenstate or Dicke state [1] and defined by

$$\hat{J}^2|J,M\rangle = J(J+1)|J,M\rangle \quad , \tag{5.47}$$

$$\hat{J}_z|J,M\rangle = M|J,M\rangle \quad , \tag{5.48}$$

where $J = N/2, M = -J, -J+1, \cdots J-1, J$ and N is a total number of atoms.

Using the recursion relation (5.27), the first, second and third excited Dicke states can be constructed from the ground state $|J,-J\rangle$:

$$\hat{J}_+|J,-J\rangle = \sqrt{2J}|J,-J+1\rangle \quad , \tag{5.49}$$

$$\hat{J}_+|J,-J+1\rangle = \sqrt{2(2J-1)}|J,-J+2\rangle \quad , \tag{5.50}$$

$$\hat{J}_+|J,-J+2\rangle = \sqrt{3(2J-2)}|J,-J+3\rangle \quad , \tag{5.51}$$

$$\vdots$$

$$\hat{J}_+|J,M-1\rangle = \sqrt{(J+M)(J-M+1)}|J,M\rangle \quad . \tag{5.52}$$

From (5.49) to (5.52), we have the mathematical construction of an arbitrary angular momentum eigenstate (Dicke state), which is analogous to the mathematical construction of a photon number eigenstate (Fock state):

$$|J,M\rangle = \frac{1}{(J+M)!}\begin{pmatrix} 2J \\ J+M \end{pmatrix}^{1/2} \hat{J}_+^{(M+J)}|J,-J\rangle \quad , \tag{5.53}$$

$$\updownarrow$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^+\right)^n|0\rangle \quad . \tag{5.54}$$

The highest excited state $|J,J\rangle$ and the ground state $|J,-J\rangle$ are defined by

$$\hat{J}_+|J,J\rangle = 0 \quad , \tag{5.55}$$

$$\hat{J}_-|J,-J\rangle = 0 \quad , \tag{5.56}$$

As shown in Fig. 5.1, there are $(N+1)$ mutually orthogonal states in this angular momentum eigenstates, $M = -J, -J+1, \cdots, J$, while the dimension of the Hilbert space for N two-level atoms is 2^N. Where are the missing states of a total number of $2^N - (N+1)$? We will come back to this question in the next section 5.4.

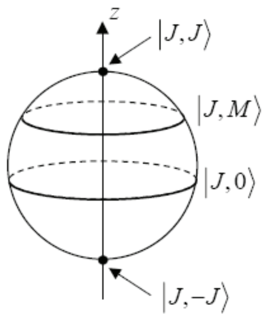

Figure 5.1: Angular momentum eigenstates (or Dicke states).

5.3.2 Spin coherent state

A state that can be created by the arbitrary rotation of the ground state $|J,-J\rangle$ in an extended Bloch sphere is called a spin coherent state or Bloch state [2]. We start with the introduction of the rotation operator (Fig. 5.2). If we rotate (x,y,z) axes by an angle φ around z-axis, the corresponding collective spin operators are transformed to

$$\begin{aligned} \hat{J}_z &= \hat{J}_z \\ \hat{J}_n &= \hat{J}_x \sin\varphi - \hat{J}_y \cos\varphi \\ \hat{J}_k &= \hat{J}_x \cos\varphi + \hat{J}_y \sin\varphi \quad . \end{aligned} \tag{5.57}$$

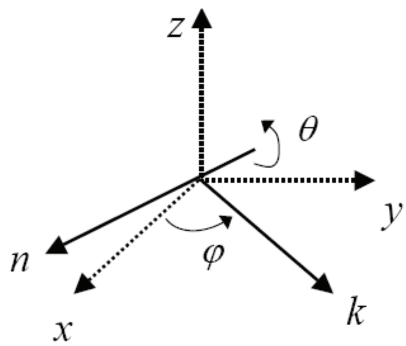

Figure 5.2: Definition of a rotation operator.

The rotation by an angle θ around a new axis n is thus given by

$$\begin{aligned} \hat{R}_{\theta\varphi} &= e^{-i\theta\hat{J}_n} \\ &= e^{-i\theta\left(\hat{J}_x \sin\varphi - \hat{J}_y \cos\varphi\right)} \\ &= \exp\left(\zeta\hat{J}_+ - \zeta^*\hat{J}_-\right) \quad , \end{aligned} \tag{5.58}$$

where $\zeta = \frac{\theta}{2}e^{-i\varphi}$. We used $\hat{J}_x = \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right)$ and $\hat{J}_y = \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right)$ to derive the third equality. A spin coherent state is mathematically constructed by the rotation of the ground state $|J,-J\rangle$ (see Fig 5.3), which is analogous to the mathematical construction of a coherent state via displacement of the vacuum state:

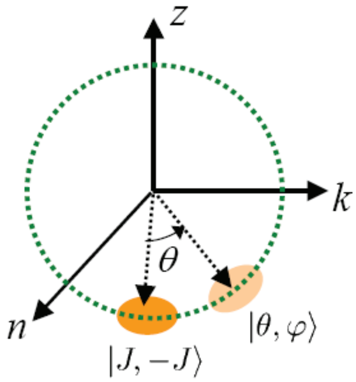

Figure 5.3: A spin coherent state (or Bloch state).

$$|\theta,\varphi\rangle = \hat{R}_{\theta\varphi}|J,-J\rangle \quad , \tag{5.59}$$

$$\updownarrow$$

$$|\alpha\rangle = \exp\left(\alpha\hat{a}^{+} - \alpha^{*}\hat{a}\right)|0\rangle \quad . \tag{5.60}$$

Next let us derive the eigenvalue equation for spin coherent states. Collective angular momentum operators $\hat{J}_k$ and $\hat{J}_n$ are expressed in terms of the rasing and lowering operators using (5.45) and (5.57),

$$\hat{J}_k = \frac{1}{2}\left(\hat{J}_{+}e^{-i\varphi} + \hat{J}_{-}e^{i\varphi}\right) \quad , \tag{5.61}$$

$$\hat{J}_n = \frac{i}{2}\left(\hat{J}_{+}e^{-i\varphi} - \hat{J}_{-}e^{i\varphi}\right) \quad . \tag{5.62}$$

Rotation around n-axis by an angle θ translates these operators into

$$\hat{R}_{\theta\varphi}\hat{J}_n\hat{R}_{\theta\varphi}^{-1} = \hat{J}_n \quad , \tag{5.63}$$

$$\hat{R}_{\theta\varphi}\hat{J}_k\hat{R}_{\theta\varphi}^{-1} = \hat{J}_k\cos\theta + \hat{J}_z\sin\theta \quad . \tag{5.64}$$

Thus, we can obtain the following operator expression:

$$\begin{aligned}\hat{R}_{\theta\varphi}\hat{J}_{-}\hat{R}_{\theta\varphi}^{-1} &= \hat{R}_{\theta\varphi}\left[\left(\hat{J}_k + i\hat{J}_n\right)e^{-i\varphi}\right]\hat{R}_{\theta\varphi}^{-1} \\ &= \left\{\hat{J}_z\sin\theta + e^{i\varphi}\cos^2\left(\frac{\theta}{2}\right)\hat{J}_{-} - e^{-i\varphi}\sin^2\left(\frac{\theta}{2}\right)\hat{J}_{+}\right\}e^{-i\varphi} \quad .\end{aligned} \tag{5.65}$$

If we project the rotation operator $\hat{R}_{\theta\varphi}$ from the left of (5.56) and use $\hat{I} = \hat{R}_{\theta\varphi}^{-1}\hat{R}_{\theta\varphi}$, we obtain

$$\hat{R}_{\theta\varphi}\hat{J}_{-}\left(\hat{R}_{\theta\varphi}^{-1}\hat{R}_{\theta\varphi}\right)|J,-J\rangle = \hat{R}_{\theta\varphi}\hat{J}_{-}\hat{R}_{\theta\varphi}^{-1}|\theta,\varphi\rangle = 0 \quad . \tag{5.66}$$

By substituting (5.65) into (5.66), we have the eigenvalue equation for spin coherent states, which is analogous to the eigenvalue equation for coherent states:

$$\left\{\hat{J}_z\sin\theta + e^{i\varphi}\cos^2\left(\frac{\theta}{2}\right)\hat{J}_{-} - e^{-i\varphi}\sin^2\left(\frac{\theta}{2}\right)\hat{J}_{+}\right\}|\theta,\varphi\rangle = 0 \quad , \tag{5.67}$$

$$\updownarrow$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad . \tag{5.68}$$

Since a spin coherent state $|\theta, \varphi\rangle$ is obtained by simple rotation of the ground state $|J, -J\rangle$, it is an eigenstate of the total angular momentum with the identical eigenvalue,

$$\hat{J}^2|\theta, \varphi\rangle = J(J+1)|\theta, \varphi\rangle \quad . \tag{5.69}$$

Therefore, it is possible to expand a spin coherent state $|\theta, \varphi\rangle$ in terms of angular momentum eigenstates $|J, M\rangle (M = -J, -J+1, \cdots, J)$. In order to derive this expansion, we rewrite (5.58) using Baker-Hausdorf relation [3],

$$\hat{R}_{\theta\varphi} = e^{\tau \hat{J}_+} e^{ln\left(1+|\tau|^2\right)\hat{J}_z} e^{-\tau^* \hat{J}_-} \quad , \tag{5.70}$$

where $\tau = e^{-i\varphi} \tan\frac{\theta}{2}$. Using (5.70) in (5.59), we have

$$\begin{aligned} |\theta, \varphi\rangle &= \frac{1}{\left(1+|\tau|^2\right)^J} e^{\tau \hat{J}_+}|J, -J\rangle \qquad (5.71)\\ &= \sum_{M=-J}^{J} \frac{1}{\left(1+|\tau|^2\right)^J} \frac{\tau^{M+J}}{(M+J)!} \hat{J}_+^{M+J}|J, -J\rangle \\ &= \sum_{M=-J}^{J} \frac{1}{\left(1+|\tau|^2\right)^J} \left(\begin{array}{c} 2J \\ M+J \end{array} \right)^{1/2} \tau^{M+J}|J, M\rangle \quad , \end{aligned}$$

$$\updownarrow$$

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{e^{-\frac{|\alpha|^2}{2}} \alpha^n}{\sqrt{n!}}|n\rangle \quad . \tag{5.72}$$

The inner product of two spin coherent states has an analogous non-orthogonality relation with two coherent states:

$$\langle\theta, \varphi|\theta', \varphi'\rangle = \cos^{4J}(\Phi/2) \quad , \tag{5.73}$$

where

$$\cos\Phi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi') \quad . \tag{5.74}$$

Since Φ is an angle between two vectors (θ, φ) and (θ', φ') in the extended Bloch sphere, a spin coherent state is only pair-wise orthogonal when $\Phi = \pi$. A set of spin coherent states, however, forms a complete set [2],

$$\begin{aligned} (2J+1)\int \frac{d\Omega}{4\pi}|\theta, \varphi\rangle\langle\theta, \varphi| &= \sum_{M=-J}^{J} |J, M\rangle\langle J, M| \qquad (5.75)\\ &= \hat{I} \quad , \end{aligned}$$

where $d\Omega$ is a differential solid angle. Therefore, spin coherent states from an overcomplete set just as coherent states.