Chapter 4 Coherent Computing

	Quantum Computing		Coherent Computing	
Information Carrier	Spin-1/2 particle (localized)		Coherent states of light or matter (non-local)	
Operational Principle	Unitary evolution in closed system		Quantum phase transition in open system	
Implementation Schemes	(1) Quantum circuits	(2) Adiabatic evolution	(3) Injection-locked laser network	(4) BEC network
Applications	Factoring Quantum chemistry Quantum repeater	Combinatorial optimization (exponential)	Combinatorial optimization (polynomial)	Quantum simulation

4.1 Ising model

- Optimization problems are ubiquitous in our modern life.

⇨ Find the specific values of M variables $\sigma_1, \sigma_2, \ldots \sigma_M$ to minimize the cost function $E(\sigma_1, \sigma_2, \ldots \sigma_M)$.

If such an optimization problem can be solved in polynomial time only by non-deterministic Turing machine, such a problem is said to belong to

$\Rightarrow$ class NP (Non-deterministic Polynomial)

However, this non-deterministic Turing machine cannot be simulated efficiently by a deterministic Turing machine, and thus

$\Rightarrow$ class NP is computationally hard

- NP-complete problem $\rightarrow$ subset of class NP

Any NP problems can be mapped to an arbitrary NP-complete problem.

$\Rightarrow$ If one has an efficient machine to solve one particular NP-complete problem, then one can solve any NP problems efficiently.

R. M. Karp, Reducibility among combinatorial problems, in Complexity of Computation (Plenum, New York, 1972) P.85, eds: R. E. Miller and J. W. Thatcher

- The ground state search problem of an Ising Hamiltonian,

$$H = \sum_{i<j} J_{ij} \sigma_{iz} \sigma_{jz} + \sum_{i} \lambda_i \sigma_{iz}$$

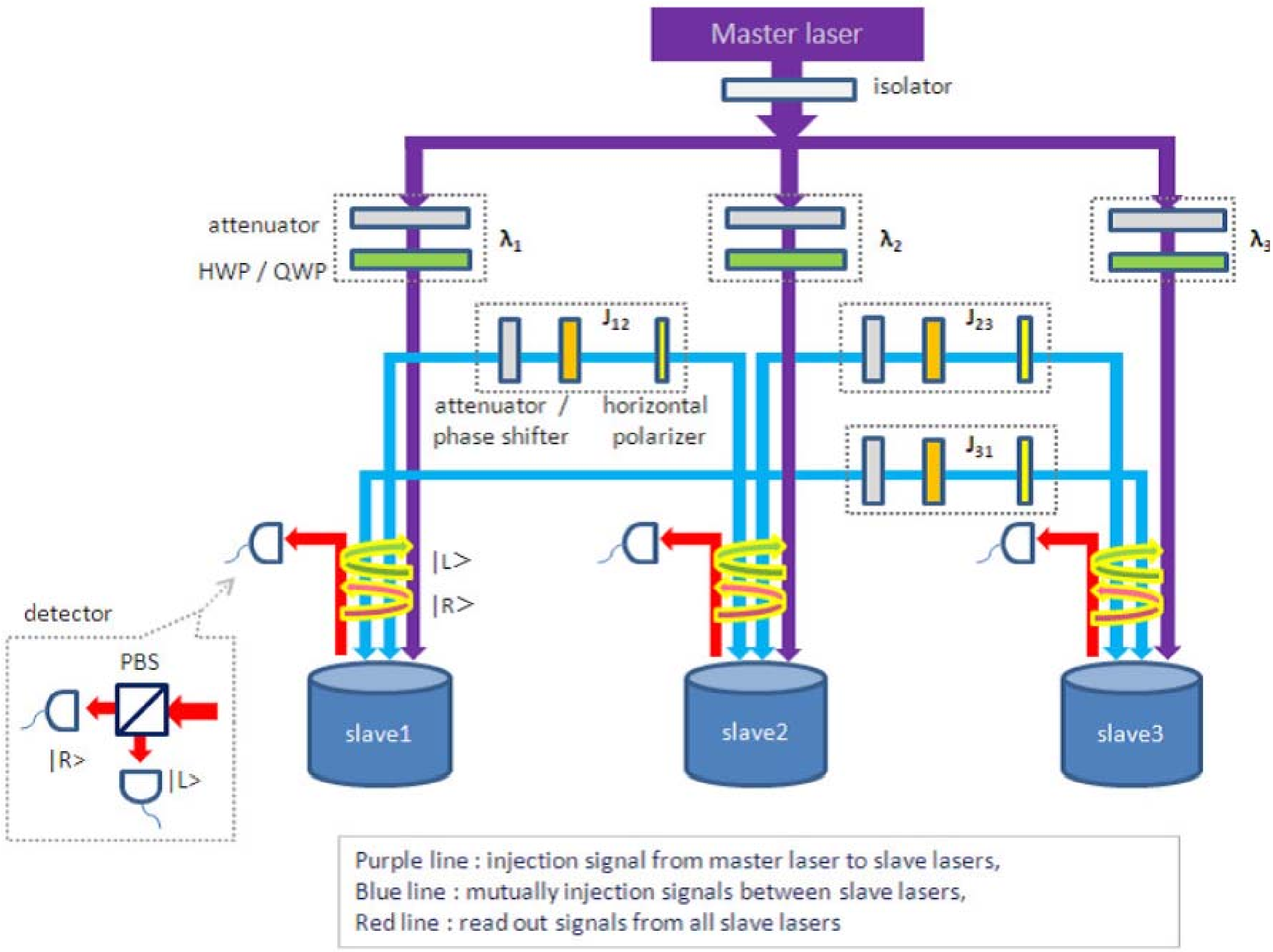
is NP-complete if the Ising coupling J_{ij} cannot be represented in a two-dimensional graph without crossing. Such a problem is called a 3D or non-planar Ising model.

It is also NP-complete if J_{ij} can be represented in a two-dimensional graph but there is a Zeeman term (induced by dc magnetic field), i.e. $\lambda_i \neq 0$.

Here an Ising spin takes only $\sigma_{iz} = +1$ or -1 .

[F. Barahona, J. Phys. A: Math. Gen. 15, 3241 (1982)]

4.2 Injection-locked laser network



- Ising spin $\sigma_{iz} = \pm 1$ is represented by the right or left circular polarization of the lasing photon in the slave laser i (i=1,2, … M).

- A master laser output is split into M paths and injected into M slave lasers. If the master laser output is vertically polarized, the polarization state of each photon in the master laser output is

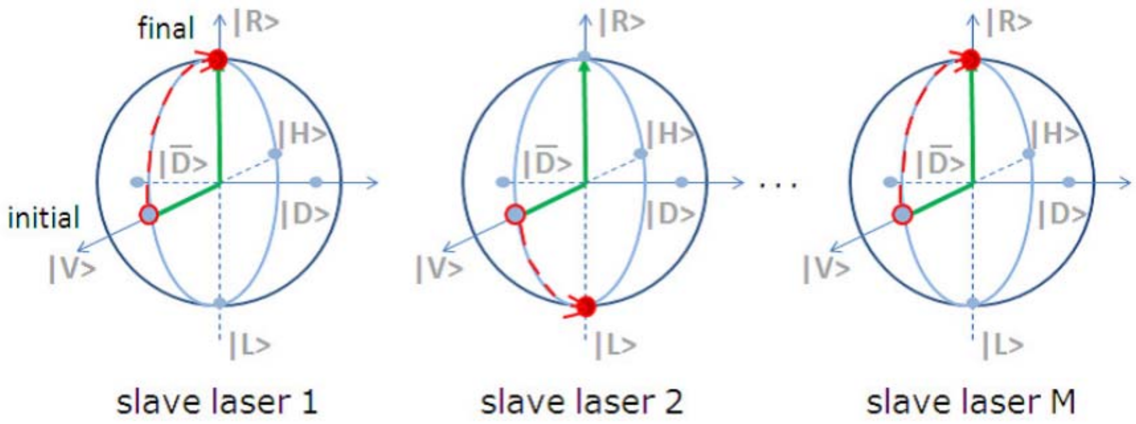
$$|V\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle).$$

right circular polarization

left circular polarization

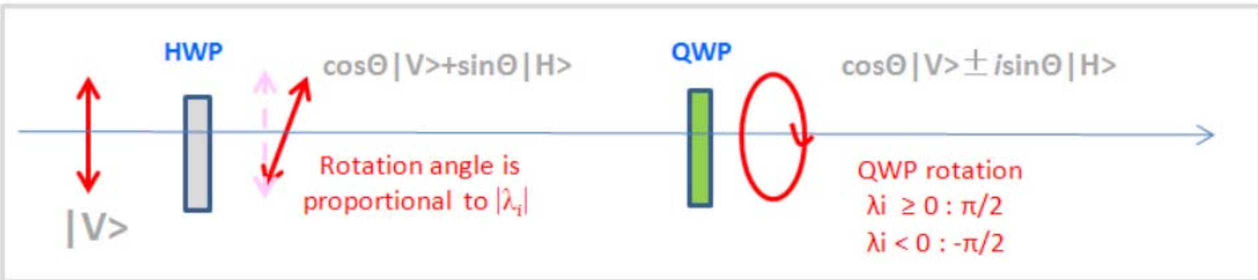
- It is expected that the polarization state of the slave laser is injection-locked to that of the injection signal from the master laser so that the polarization state of a slave laser is in a so-called "spin coherent state":

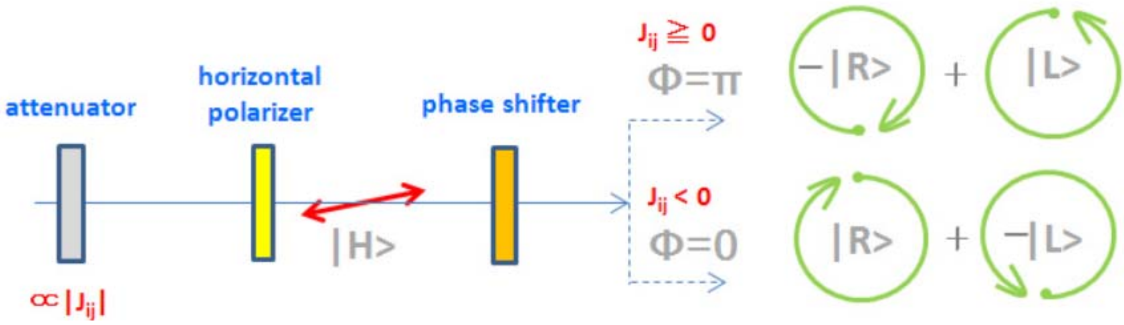
$$|\psi\rangle_s = \Pi_i \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)_i = |\theta = \frac{\pi}{2}, \varphi = 0\rangle.$$



- The Ising interaction term J_{ij} can be implemented by a horizontal polarizer in the optical path between the two slave lasers i and j, as will be shown later.

- The Zeeman term λ_i can be implemented by including a horizontal polarization component in the injection signal from the master laser, as will be shown later. Experimentally, this can be achieved by inserting a half-wave plate and quantum-wave plate in the optical path between the master laser and the slave laser i.





- Coherent computation is switched on either by abruptly implementing the Ising coupling J_{ij} at t=0 or by gradually increasing the pump rates for all slave lasers from below threshold to above threshold.

- Computational results can be read out by detecting the right and left circularly polarized photons from each slave laser and making a majority vote.

4.3 Theoretical model

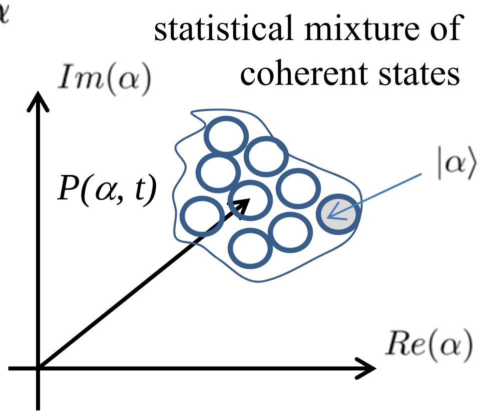
4.3.1 Quantum theory of an injection-locked laser

Glauber-Sudarshan diagonal P(α) representation:

$$\hat{\rho}(t) = \int P(\alpha,t)|\alpha\rangle\langle\alpha| d^2\alpha$$

field density operator of a right or left circular polarization mode

positive, real



Substitution into the master equation of an injection-locked laser.
[H.A, Haus and Y. Yamamoto, PRA 29, 1261 (1984)]

Quantum mechanical Fokker-Planck equation:

$$\frac{d}{dt}P(\alpha,t) = -\frac{1}{2}\left\{\frac{\partial}{\partial\alpha}\left[G - \left(\frac{\omega}{Q}\right) - S|\alpha|^2\right]\alpha P(\alpha,t) + c.c.\right\}$$

drift term

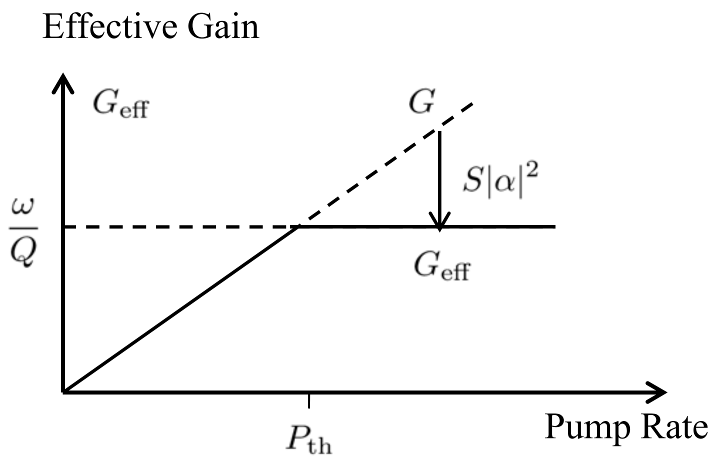
$$+2G\frac{\partial^2}{\partial\alpha\partial\alpha^*}P(\alpha,t)$$

diffusion term

$$-\left[\kappa\beta^* e^{i(\omega_j-\omega)t}\frac{\partial}{\partial\alpha^*}P(\alpha,t) + c.c.\right]$$

injection term

G: linear gain coefficient
S: saturation parameter
$\frac{\omega}{Q}$: photon decay rate



$\kappa = \sqrt{\frac{\omega}{Q}_e}$: coupling constant of an injection signal into a slave laser

β : (complex) eigenvalue of an injection signal

($|\beta|^2$= average photon number per second)

ω : master laser (angular) frequency

ω_j : slave laser (angular) frequency

$\Downarrow$

Equivalent to c-number stochastic differential equation (CSDE):

$$\underbrace{\frac{d}{dt}\alpha(t)}_{\text{excitation amplitude of lasing mode}} = \underbrace{\frac{1}{2}\left(E_{cv} - E_{vc} - \frac{\omega}{Q}\right)\alpha(t)}_{\text{drift term}} + \underbrace{\kappa\beta(t)e^{-i(\omega_j-\omega)t}}_{\text{injection term}} + \underbrace{\sqrt{\frac{1}{2}(E_{cv}+E_{vc})}\Gamma_\alpha(t)}_{\text{diffusion term}}$$

E_{cv} : stimulated emission rate

E_{vc} : absorption rate

$$\left.\begin{aligned}&\langle\Gamma_\alpha(t)\rangle = 0\\&\langle\Gamma_\alpha(t)\Gamma_\alpha(t')\rangle = 2\delta(t-t')\end{aligned}\right\}\text{ Markovian noise}$$

- The noise term (diffusion term) does not include the vacuum fluctuation associated with the out-coupling loss $\frac{\omega}{Q}$. This is because the zero-point-fluctuation is already included in the basis states, i.e. coherent states $|\alpha\rangle$.

4.3.2 Minimum gain principle

If we neglect the noise terms in the CSDE, we can write down the rate equations for the average photon number $n(t) \equiv |\alpha(t)|^2$ and the average electron number N for each slave laser:

$$\frac{d}{dt}n_{Ri} = -\left(\frac{\omega}{Q} - E_{cvi}\right)n_{Ri} + E_{cvi}$$
$$+2\left(\frac{\omega}{Q}\right)\sqrt{n_{Ri}}\left[(\zeta-\eta_i)\sqrt{n_M} - \sum_{j\neq i}\frac{1}{2}\xi_{ij}\left(\sqrt{n}_{Rj} - \sqrt{n}_{Lj}\right)\right]$$

$$\frac{d}{dt}n_{Li} = -\left(\frac{\omega}{Q} - E_{cvi}\right)n_{Li} + E_{cvi}$$
$$+2\left(\frac{\omega}{Q}\right)\sqrt{n_{Li}}\left[(\zeta+\eta_i)\sqrt{n_M} + \sum_{j\neq i}\frac{1}{2}\xi_{ij}\left(\sqrt{n}_{Rj} - \sqrt{n}_{Lj}\right)\right]$$

$$\frac{d}{dt}N_i = P - \frac{N_i}{\tau_{sp}} - E_{cvi}\left(n_{Ri} + n_{Li} + 2\right)$$

n_M : average photon number of the master laser

$\frac{\omega}{Q_e} = \frac{\omega}{Q_{Me}}$: equal external Q values for slave and master lasers

τ_{sp} : spontaneous emission lifetime

p : pumping rate (electrons per second)

$E_{vc} = 0$: no absorption ($\rightarrow$ complete population inversion)

ζ : fraction of vertical polarization component in the master signal

η_i : fraction of horizontal polarization component in the master signal
($\rightarrow$ implementation of Zeeman term)

ξ_{ij} : Mutual coupling coefficient between slave lasers i and j
($\rightarrow$ implementation of Ising coupling term)

$\Downarrow$

Steady state solution (given by $\frac{d}{dt}n_{Ri} = \frac{d}{dt}n_{Li} = \frac{d}{dt}N_i = 0$) :

$$E_{cvi} = \frac{\omega}{Q} - 2\frac{\omega}{Q}\zeta\frac{\sqrt{n_M}\left(\sqrt{n_{Ri}} + \sqrt{n_{Li}}\right)}{n_T}$$

reduced threshold gain by vertically polarized master signal

$$+2\frac{\omega}{Q}\frac{\sqrt{n_{Ri}} - \sqrt{n_{Li}}}{\sqrt{n_T}}\left[\eta_i\frac{\sqrt{n_M}}{\sqrt{n_T}} + \sum_{j \neq i}\frac{1}{2}\xi_{ij}\frac{\sqrt{n_{Rj}} - \sqrt{n_{Lj}}}{\sqrt{n_T}}\right]$$

modulated threshold gain by horizontally polarized master signal

modulated threshold gain by horizontally polarized slave laser signals

- The injection-locked laser network has an overall threshold gain,

$$\sum_{i=1}^{M} E_{cvi}$$

which depends on the polarization configurations $\{n_{Ri}, n_{Li}\}$ of all slave lasers.

- The injection-locked laser network chooses a particular polarization configuration which minimizes the overall threshold gain, $\sum_{i=1}^{M} E_{cvi}$, and realizes a single mode oscillation with this particular polarization configuration.

$\Downarrow$

$\boxed{\text{Minimum gain principle}}$

- After such a minimum gain mode is selected by the injection-locked laser network, $n_T \simeq n_{Ri} \gg n_{Li}$ or $n_T \simeq n_{Li} \gg n_{Ri}$. Then,

$$2\frac{\omega}{Q}\zeta\frac{\sqrt{n_M}\left(\sqrt{n_{Ri}}+\sqrt{n_{Li}}\right)}{n_T} \simeq 2\frac{\omega}{Q}\zeta\sqrt{\frac{n_M}{n_T}}$$

constant and independent of the polarization state

- If we define the effective Ising spin by

$$\sigma_{iz} = \frac{\sqrt{n_{Ri}}-\sqrt{n_{Li}}}{\sqrt{n_T}},$$

$$\sum_{i=1}^{M} E_{cvi} = \underbrace{\frac{\omega}{Q}\left[1-2\zeta\sqrt{\frac{n_M}{n_T}}\right]\cdot M}_{\text{constant}} + \sum_{i<j}\underbrace{\xi_{ij}}_{J_{ij}}\sigma_{iz}\sigma_{jz} + \sum_{i}\underbrace{\eta_i\sqrt{\frac{n_M}{n_T}}}_{\lambda_i}\sigma_{iz}$$

The single lasing mode is identical to the ground state of the Ising model which minimizes the cost function:

$$E\left(\sigma_{1z},\sigma_{2z}\cdots\sigma_{Mz}\right) = \sum_{i<j} J_{ij}\sigma_{iz}\sigma_{jz} + \sum_{i}\lambda_i\sigma_{iz}$$

- Actual choice of the amplitude attenuation parameter α for the injection signal is determined by the relation,

$$\xi_{ij} = \alpha\frac{J_{ij}}{\max\left\{|J_{ij}|,|\lambda_i|\right\}}$$

$$\eta_i = \alpha\sqrt{\frac{n_T}{n_M}}\frac{\lambda_i}{\max\left\{|J_{ij}|,|\lambda_i|\right\}}$$

We choose $\alpha \ll 1$ in order to operate the injection-locked laser network in a weak perturbation regime.

4.3.3 Working equations

We employ the diagonal linear polarization states $(|D\rangle, |\bar{D}\rangle)$ as the basis set. The polarization evolution from the initial state $|V\rangle$ to the final state, $|R\rangle$ or $|L\rangle$, can be described by the phase rotations from zero to $\pm\pi/2$ of the two polarization modes, while the amplitudes of the two modes are kept almost constant.