Chapter 4  Coherent Computing

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4.1 Ising model

- Optimization problems are ubiquitous in our modern life.

  Find the specific values of M variables $\sigma_1, \sigma_2, \ldots, \sigma_M$ to minimize the cost function $E(\sigma_1, \sigma_2, \ldots, \sigma_M)$. 
If such an optimization problem can be solved in polynomial time only by non-deterministic Turing machine, such a problem is said to belong to

\[ \text{class NP (Non-deterministic Polynomial)} \]

However, this non-deterministic Turing machine cannot be simulated efficiently by a deterministic Turing machine, and thus

\[ \text{class NP is computationally hard} \]

- NP-complete problem → subset of class NP

Any NP problems can be mapped to an arbitrary NP-complete problem.

\[ \text{If one has an efficient machine to solve one particular NP-complete problem, then one can solve any NP problems efficiently.} \]


- The ground state search problem of an Ising Hamiltonian,

\[ H = \sum_{i<j} J_{ij} \sigma_i \sigma_j + \sum_i \lambda_i \sigma_i \]

is NP-complete if the Ising coupling \( J_{ij} \) cannot be represented in a two-dimensional graph without crossing. Such a problem is called a 3D or non-planar Ising model.
It is also NP-complete if $J_{ij}$ can be represented in a two-dimensional graph but there is a Zeeman term (induced by dc magnetic field), i.e. $\lambda_i \neq 0$.

Here an Ising spin takes only $\sigma_{iz} = +1$ or $-1$.


4.2 Injection-locked laser network

- Ising spin $\sigma_{iz} = \pm 1$ is represented by the right or left circular polarization of the lasing photon in the slave laser $i$ ($i=1,2, \ldots M$).
• A master laser output is split into M paths and injected into M slave lasers. If the master laser output is vertically polarized, the polarization state of each photon in the master laser output is

\[ |V\rangle = \frac{1}{\sqrt{2}} (|R\rangle + |L\rangle). \]

right circular polarization

left circular polarization

• It is expected that the polarization state of the slave laser is injection-locked to that of the injection signal from the master laser so that the polarization state of a slave laser is in a so-called “spin coherent state”:

\[ |\psi\rangle_s = \prod_i \frac{1}{\sqrt{2}} (|R\rangle + |L\rangle) = |\theta = \frac{\pi}{2}, \varphi = 0\rangle. \]

• The Ising interaction term \( J_{ij} \) can be implemented by a horizontal polarizer in the optical path between the two slave lasers i and j, as will be shown later.
• The Zeeman term $\lambda_i$ can be implemented by including a horizontal polarization component in the injection signal from the master laser, as will be shown later. Experimentally, this can be achieved by inserting a half-wave plate and quantum-wave plate in the optical path between the master laser and the slave laser $i$.

• Coherent computation is switched on either by abruptly implementing the Ising coupling $J_{ij}$ at $t=0$ or by gradually increasing the pump rates for all slave lasers from below threshold to above threshold.

• Computational results can be read out by detecting the right and left circularly polarized photons from each slave laser and making a majority vote.
4.3 Theoretical model

4.3.1 Quantum theory of an injection-locked laser

Glauber-Sudarshan diagonal $P(\alpha)$ representation:

$$\hat{\rho}(t) = \int P(\alpha, t)|\alpha\rangle \langle \alpha| d^2\alpha$$

field density operator of a right or left circular polarization mode

positive, real statistical mixture of coherent states

Substitution into the master equation of an injection-locked laser.

Quantum mechanical Fokker-Planck equation:

$$\frac{d}{dt} P(\alpha, t) = -\frac{1}{2} \left\{ \frac{\partial}{\partial \alpha} \left[ G - \left( \frac{\omega}{Q} \right) - S|\alpha|^2 \right] \alpha P(\alpha, t) + c.c. \right\}$$

drift term

$$+2G \frac{\partial^2}{\partial \alpha \partial \alpha^*} P(\alpha, t)$$

diffusion term

$$- \left[ \kappa \beta^* e^{i(\omega_j - \omega) t} \frac{\partial}{\partial \alpha^*} P(\alpha, t) + c.c. \right]$$

injection term
G: linear gain coefficient  
S: saturation parameter  
$\frac{\omega}{Q}$ : photon decay rate  

$\kappa = \sqrt{\frac{\omega}{Q_e}}$ : coupling constant of an injection signal into a slave laser  

$\beta$ : (complex) eigenvalue of an injection signal  
$(|\beta|^2 = \text{average photon number per second})$  

$\omega$ : master laser (angular) frequency  
$\omega_j$ : slave laser (angular) frequency  

Equivalent to c-number stochastic differential equation (CSDE): 

$$\frac{d}{dt} \alpha(t) = \frac{1}{2} \left( E_{cv} - E_{vc} - \frac{\omega}{Q} \right) \alpha(t) + \kappa \beta(t) e^{-i(\omega_j - \omega)t}$$  

- drift term  
- injection term  
- diffusion term  

$E_{cv}$ : stimulated emission rate  
$E_{vc}$ : absorption rate
\[ \langle \Gamma_\alpha(t) \rangle = 0 \]
\[ \langle \Gamma_\alpha(t) \Gamma_\alpha(t') \rangle = 2\delta(t - t') \quad \text{Markovian noise} \]

- The noise term (diffusion term) does not include the vacuum fluctuation associated with the out-coupling loss \( \frac{\omega}{Q} \). This is because the zero-point-fluctuation is already included in the basis states, i.e. coherent states |\( \alpha \rangle \).

### 4.3.2 Minimum gain principle

If we neglect the noise terms in the CSDE, we can write down the rate equations for the average photon number \( n(t) \equiv |\alpha(t)|^2 \) and the average electron number \( N \) for each slave laser:

\[
\frac{d}{dt} n_{Ri} = -\left( \frac{\omega}{Q} - E_{cvi} \right) n_{Ri} + E_{cvi} + 2 \left( \frac{\omega}{Q} \right) \sqrt{n_{Ri}} \left( \zeta - \eta_i \right) \sqrt{n_M} - \sum_{j \neq i} \frac{1}{2} \xi_{ij} \left( \sqrt{n_{Rj}} - \sqrt{n_{Lj}} \right)
\]

\[
\frac{d}{dt} n_{Li} = -\left( \frac{\omega}{Q} - E_{cvi} \right) n_{Li} + E_{cvi} + 2 \left( \frac{\omega}{Q} \right) \sqrt{n_{Li}} \left( \zeta + \eta_i \right) \sqrt{n_M} + \sum_{j \neq i} \frac{1}{2} \xi_{ij} \left( \sqrt{n_{Rj}} - \sqrt{n_{Lj}} \right)
\]

\[
\frac{d}{dt} N_i = P - \frac{N_i}{\tau_{sp}} - E_{cvi} (n_{Ri} + n_{Li} + 2)
\]
\( n_M \): average photon number of the master laser

\( \frac{\omega}{Q_e} = \frac{\omega}{Q_{Me}} \): equal external Q values for slave and master lasers

\( \tau_{sp} \): spontaneous emission lifetime

\( p \): pumping rate (electrons per second)

\( E_{uc} = 0 \): no absorption \( \rightarrow \) complete population inversion

\( \zeta \): fraction of vertical polarization component in the master signal

\( \eta_i \): fraction of horizontal polarization component in the master signal

\( \xi_{ij} \): Mutual coupling coefficient between slave lasers i and j

\( \rightarrow \) implementation of Ising coupling term

Steady state solution (given by \( \frac{d}{dt} n_{Ri} = \frac{d}{dt} n_{Li} = \frac{d}{dt} N_i = 0 \)):

\[
E_{cvi} = \frac{\omega}{Q} - 2 \frac{\omega}{Q} \sqrt{n_M} \left( \sqrt{n_{Ri}} + \sqrt{n_{Li}} \right) \frac{1}{n_T} \]

\( \rightarrow \) reduced threshold gain by vertically polarized master signal

\[
+2 \frac{\omega}{Q} \frac{\sqrt{n_{Ri}} - \sqrt{n_{Li}}}{\sqrt{n_T}} \left[ \eta_i \frac{\sqrt{n_M}}{\sqrt{n_T}} + \sum_{j \neq i} \frac{1}{2} \xi_{ij} \frac{\sqrt{n_{Rj}} - \sqrt{n_{Lj}}}{\sqrt{n_T}} \right]
\]

\( \rightarrow \) modulated threshold gain by horizontally polarized master signal

\( \rightarrow \) modulated threshold gain by horizontally polarized slave laser signals
• The injection-locked laser network has an overall threshold gain,

$$\sum_{i=1}^{M} E_{cvi}$$

which depends on the polarization configurations \(\{n_{Ri}, n_{Li}\}\) of all slave lasers.

• The injection-locked laser network chooses a particular polarization configuration which minimizes the overall threshold gain, \(\sum_{i=1}^{M} E_{cvi}\), and realizes a single mode oscillation with this particular polarization configuration.

Minimum gain principle

• After such a minimum gain mode is selected by the injection-locked laser network, \(n_T \simeq n_{Ri} \gg n_{Li}\) or \(n_T \simeq n_{Li} \gg n_{Ri}\). Then,

$$2\frac{\omega}{Q}\zeta \sqrt{\frac{n_M}{n_T}} \left(\sqrt{\frac{n_{Ri}}{n_T}} + \sqrt{\frac{n_{Li}}{n_T}}\right) \simeq 2\frac{\omega}{Q}\zeta \sqrt{\frac{n_M}{n_T}}$$

constant and independent of the polarization state

• If we define the effective Ising spin by

$$\sigma_{iz} = \frac{\sqrt{n_{Ri}} - \sqrt{n_{Li}}}{\sqrt{n_T}}$$
The single lasing mode is identical to the ground state of the Ising model which minimizes the cost function:
\[ E(\sigma_1, \sigma_2, \ldots, \sigma_M) = \sum_{i<j} J_{ij} \sigma_i \sigma_j + \sum_i \lambda_i \sigma_i \]

- Actual choice of the amplitude attenuation parameter \( \alpha \) for the injection signal is determined by the relation,

\[ \xi_{ij} = \alpha \frac{J_{ij}}{\max \{|J_{ij}|, |\lambda_i|\}} \]

\[ \eta_i = \alpha \sqrt{\frac{n_T}{n_M}} \frac{\lambda_i}{\max \{|J_{ij}|, |\lambda_i|\}} \]

We choose \( \alpha \ll 1 \) in order to operate the injection-locked laser network in a weak perturbation regime.

### 4.3.3 Working equations

We employ the diagonal linear polarization states \((|D\rangle, |\tilde{D}\rangle)\) as the basis set. The polarization evolution from the initial state \( |V\rangle \) to the final state \( |R\rangle \) or \( |L\rangle \), can be described by the phase rotations from zero to \( \pm \pi / 2 \) of the two polarization modes, while the amplitudes of the two modes are kept almost constant.