Chapter 3 Quantum Circuit Model of Quantum Computation

A standard way to implement the necessary unitary evolution in quantum algorithms is to use an universal gate set. In his chapter we will review this quantum circuit model of quantum computers.

3.1 Universal gate set

A. Barenco et al., Phys. Rev. A 52, 3457 (1995)

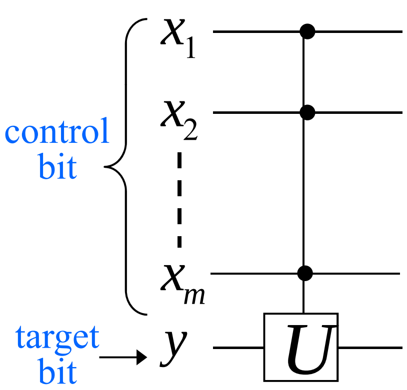
Arbitrary unitary operation on n-qubit system can be decomposed with

$$\hat{U}(2^n) \longrightarrow \hat{U}(2) + C - NOT$$

one-bit gate ($\hat{U}(2)$) two-bit gate ($C-NOT$) ⇒ universal gates

3.2 Various quantum gates

3.2.1 m -Controlled-U gate $\Lambda_m(U)$



$2^{m+1} \times 2^{m+1}$ matrix

$$\begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & 1 & & \\ & 0 & & & u_{00} & u_{01} \\ & & & & u_{10} & u_{11} \end{pmatrix} \begin{matrix} |00\cdots\cdots 0,0\rangle \\ |00\cdots\cdots 0,1\rangle \\ \vdots \\ \\ |11\cdots\cdots 1,0\rangle \\ |11\cdots\cdots 1,1\rangle \end{matrix}$$

control target

If $\prod_k x_k = 1, \quad \Lambda_m(U)|11\cdots 1,0\rangle = u_{00}|11\cdots 1,0\rangle + u_{10}|11\cdots 1,1\rangle$

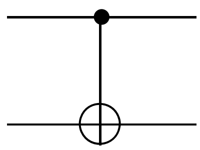
$\Lambda_m(U)|11\cdots 1,1\rangle = u_{01}|11\cdots 1,0\rangle + u_{11}|11\cdots 1,1\rangle$

If $\prod_k x_k = 0, \quad \Lambda_m(U)|x_1 x_2\cdots x_m, y\rangle = |x_1 x_2 \cdots x_m, y\rangle$

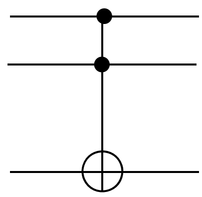
m -Controlled-*NOT* gate: C^{m}-*NOT*

$$\begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ in the above.

Example:



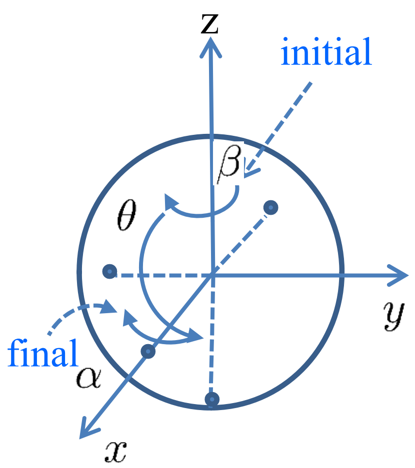
Controlled-*NOT*
(XOR)



Controlled- Controlled-*NOT*
(Toffoli)

3.2.2 one-bit gate: *U(2)* and *SU(2)*

arbitrary 2 × 2 unitary matrix $$U(2) = \Phi(\delta) R_z(\alpha) R_y(\theta) R_z(\beta)$$



$$\Phi(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix}$$: global phase shift

$$R_z(\alpha) = \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix}$$: rotation around *z* -axis by an angle of α

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$: rotation around *y* -axis by an angle of θ

special unitary matrix with det(M) = 1
$$SU(2) = \underbrace{R_z(\alpha) R_y(\theta) R_z(\beta)}_{W \text{ gate}}$$

3.2.3 Two bit gate

$$\left.\begin{aligned} A &= R_z(\alpha)R_y\left(\frac{\theta}{2}\right) \\ B &= R_y\left(-\frac{\theta}{2}\right)R_z\left(-\frac{\alpha+\beta}{2}\right) \\ C &= R_z\left(-\frac{\alpha-\beta}{2}\right) \end{aligned}\right\} ABC = \hat{I}$$

: identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A\sigma_x B\sigma_x C = R_z(\alpha)R_y\left(\frac{\theta}{2}\right)\sigma_x R_y\left(-\frac{\theta}{2}\right)R_z\left(-\frac{\alpha+\beta}{2}\right)\sigma_x R_z\left(-\frac{\alpha-\beta}{2}\right)$$

$$\hat{I} = \sigma_x\sigma_x$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ π-rotation about x-axis = NOT gate

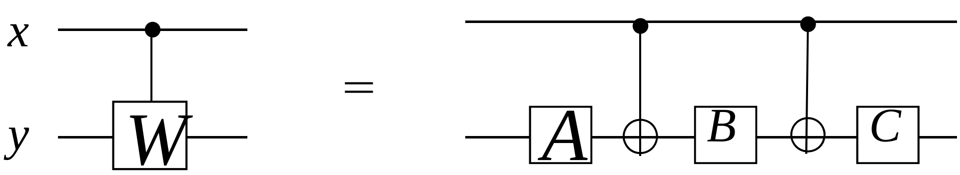
$$\sigma_x R_y(\theta)\sigma_x = R_y(-\theta)$$

$$\sigma_x R_z(\alpha)\sigma_x = R_z(-\alpha)$$

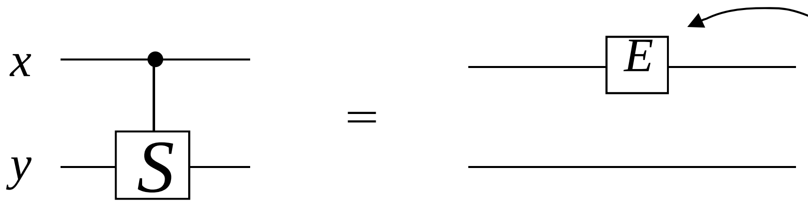
$$= R_z(\alpha)R_y\left(\frac{\theta}{2}\right)R_y\left(\frac{\theta}{2}\right)R_z\left(\frac{\alpha+\beta}{2}\right)R_z\left(-\frac{\alpha-\beta}{2}\right)$$

$$= R_z(\alpha)R_y(\theta)R_z(\beta)$$

$$= W$$



controlled-$SU(2)$ gate



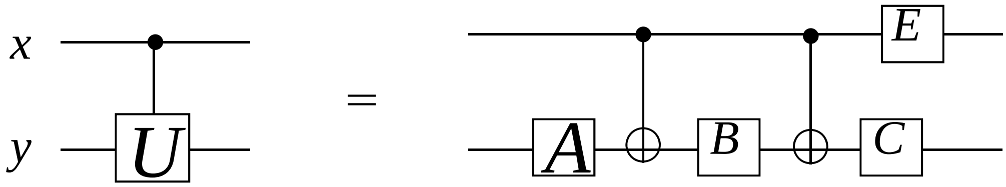
controlled-phase shift-gate

$$E = R_z(-\delta)\Phi\left(\frac{\delta}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{matrix} |0\rangle \\ |1\rangle \end{matrix}$$

control

$$\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & e^{i\delta} & \\ 0 & & & e^{i\delta} \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

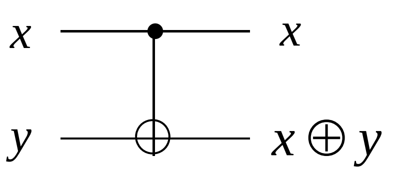
control
target



controlled-*U* gate

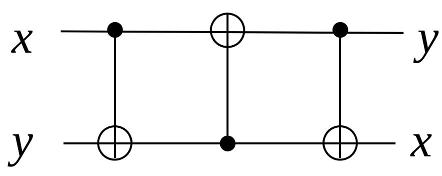
4 one-bit gates + 2 *C-NOT* gates

Applications of two-bit gate:



$$(\alpha|0\rangle + \beta|1\rangle)_x|0\rangle_y$$
$$\rightarrow \alpha|0\rangle_x|0\rangle_y + \beta|1\rangle_x|1\rangle_y$$

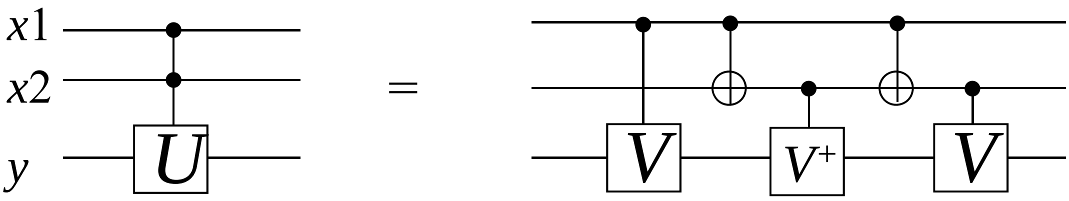
entangler



$$(\alpha|0\rangle + \beta|1\rangle)_x \otimes (\gamma|0\rangle + \delta|1\rangle)_y$$
$$\rightarrow (\gamma|0\rangle + \delta|1\rangle)_x \otimes (\alpha|0\rangle + \beta|1\rangle)_y$$

swapping

3.2.4 Three bit gate



$V^2 = U$

(1) V is applied to the target iff $x_1 = 1 \Rightarrow V^{x_1}$

(2) V^+ is applied to the target iff $x_1 \oplus x_2 = 1 \Rightarrow (V^+)^{x_1 \oplus x_2}$

(3) V is applied to the target iff $x_2 = 1 \Rightarrow V^{x_2}$

$$V^{x_1}\left(V^+\right)^{x_1 \oplus x_2} V^{x_2} = V^{x_1 + x_2 - x_1 \oplus x_2} = \left(V^2\right)^{x_1 \wedge x_2}$$

$$2x_1 \wedge x_2 = x_1 + x_2 - (x_1 \oplus x_2)$$

$\left(V^2\right) = U$

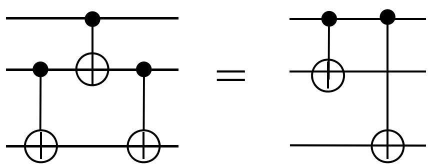
controlled-controlled-U gate

8 one-bit gates + 6 *C-NOT* gates

$12 - 2 \times 2$ $\qquad$ $2 + 1 + 2 + 1 + 2 - 2$

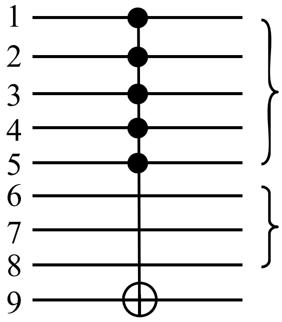
1. The last one bit gate (C) in $\Lambda_1(V)$ is cancelled with the first one bit gate (C^+) in $\Lambda_1(V^+)$.

2.



3.2.5 n-bit gate

$\Lambda_m(\sigma_x)$: *m*-controlled-NOT gate



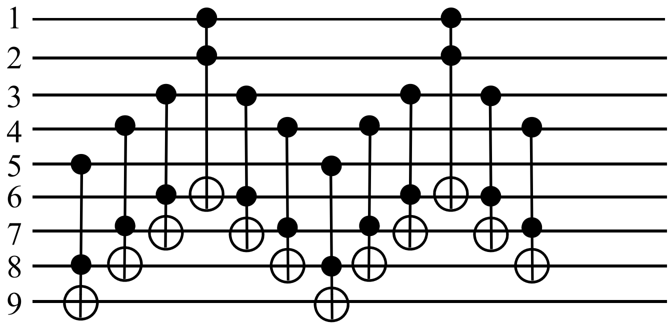
$n \geq 5, \quad m \in \{3, \cdots [n/2]\}$

⇓ constructed by

$4 \times (m-2)$ $\Lambda_2(\sigma_x)$:controlled-controlled-NOT gates

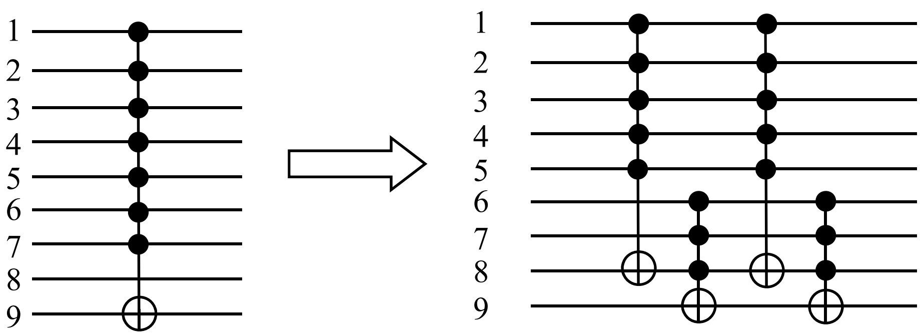
(example: n = 9, m = 5)

⇓



Iff $x_1 = x_2 = x_3 = x_4 = x_5 = 1$, one of the two 5-8-9 $\Lambda_2(\sigma_x)$ gates flipps the target 9.

(1) $\Lambda_{n-2}(\sigma_x)\{n > 5, m \in [2, \cdots n-3]\}$ can be constructed by $2\Lambda_m(\sigma_x)$ gates and $2\Lambda_{n-m-1}(\sigma_x)$ gates



(example: n = 9, m = 5)

$$2\Lambda_m(\sigma_x) \Rightarrow 2\times 4\times(m-2)\Lambda_2(\sigma_x)$$

$$2\Lambda_{n-m-1}(\sigma_x) \Rightarrow 2\times 4\times(n-m-3)\Lambda_2(\sigma_x)$$

$$\Bigg\} \; 8\times(n-5)\Lambda_2(\sigma_x)$$

$$\Downarrow \; \Lambda_2(\sigma_x) \leftarrow$$ 4 one-bit gate, 2 C-NOT gate

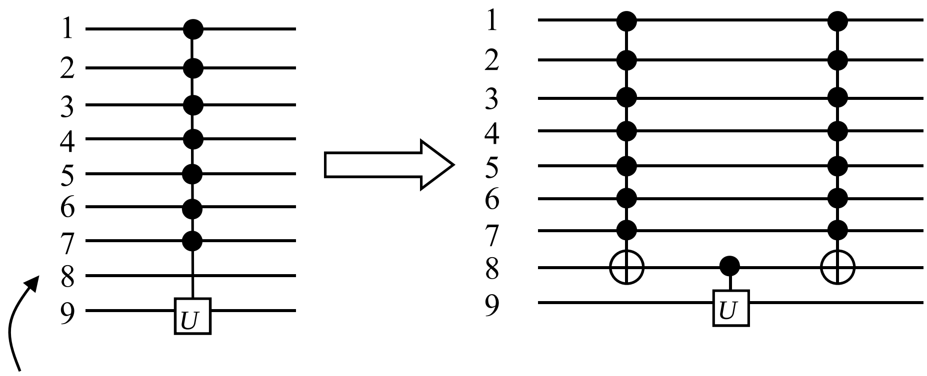
32(n - 5) one-bit gates
16 (n - 5) C-NOT gates

$\Downarrow$

scales ~ $O(n)$

One extra bit is required.

(2) $\Lambda_{n-2}(U)$



initialize to 0

$$2\Lambda_{n-2}(\sigma_x) \Rightarrow 16\times(n-5)\Lambda_2(\sigma_x)$$

$\Downarrow$

64(n - 5) one-bit gates
32 (n - 5) C-NOT gates

Controlled-U gate $\Rightarrow$ 4 one-bit gate, 2 C-NOT gate

$\Downarrow$

scales ~ $O(n)$

One initialized extra bit is required.

3.3 Implementation of Quantum Algorithms

3.3.1 Implementation of Deutsch-Josza algorithm

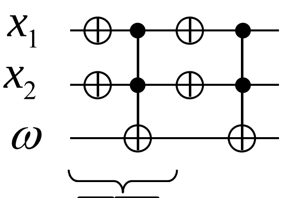
2-bit Boolean functions

x_1	x_2	f_c	$\overline{f_c}$	f_{x_1}	$\overline{f_{x_1}}$	f_{x_2}	$\overline{f_{x_2}}$	$f_{x_1 \oplus x_2}$	$\overline{f_{x_1 \oplus x_2}}$
0	0	0	1	0	1	0	1	0	1
0	1	0	1	0	1	1	0	1	0
1	0	0	1	1	0	0	1	1	0
1	1	0	1	1	0	1	0	0	1

$\Rightarrow f(x) = \overline{x_1}\,\overline{x_2} + x_1 x_2$

implementation of $\overline{f_{x_1 \oplus x_2}} = \overline{x_1}\,\overline{x_2} + x_1 x_2$

$$|x\rangle_c |0\rangle_\omega \xrightarrow{\hat{U}} |x\rangle_c |f(x)\rangle_\omega$$



$\overline{x_1}\,\overline{x_2}$ $\quad x_1 x_2$ realization

Canonical Sum of Product (CSOP) form of a Boolean function

$$f(x) = \{x_1 x_2, \overline{x_1} x_2, x_1 \overline{x_2}, \overline{x_1 x_2}\} : n = 2 \text{ bit}$$

$\Downarrow$

n-bit case $\qquad$ 2^n terms $\qquad$ example: $x_1\, \overline{x_2}\, \overline{x_3} \cdots x_n$

$$f(x) = \sum_{y=0}^{2^n - 1} a_y y, \quad a_y \in \{0, 1\}$$

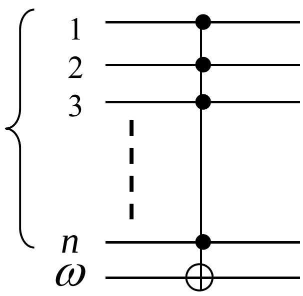
$f(x) = 0 \qquad f(x) = 1$

$\Downarrow$

Each term with $a_y = 1$ is implemented by $\Lambda_n(\sigma_x)$ gate.

control

target



Construction of one term in *f(x)* requires at most,

$2n$ NOT gates + one $\Lambda_n(\sigma_x)$ gate

↓

$32\,(n - 5)$ one-bit gates

$16\,(n - 5)$ C-NOT gates

⇓ There are 2^{n-1} terms in *f(x)* for balanced functions, which have $a_y = 1$ and so must be implemented. The other 2^{n-1} terms have $a_y = 0$, so need not be implemented.

$\sim O(n\ 2^{n-1})$ one-bit gates and C-NOT gates

⇓

This is a severe disadvantage of D-J algorithm.

3.3.2 Implementation of Grover algorithm

$$|x\rangle \rightarrow (-1)^{f(x)}|x\rangle$$

n-bit Boolean function $f(x) = y$

There is only one term instead of 2^{n-1}

Constructed by one $\Lambda_n(\sigma_x)$ gate

⇒ $\sim O(n)$ one-bit gates and C-NOT gates

⇓

However, this oracle must be repeated $\sim \sqrt{N}\left(=2^{n/2}\right)$ times.

example : $n = 3$ bit

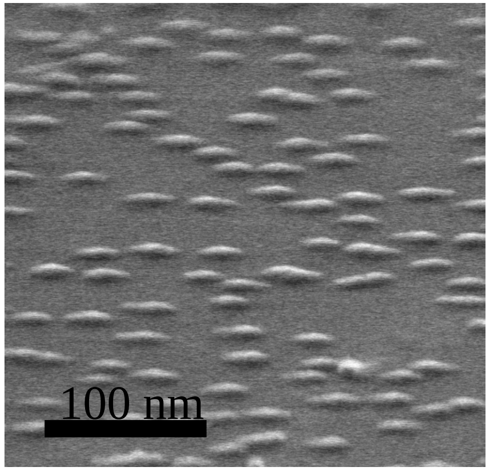
x_1	x_2	x_3	$f(x)$
0	0	0	0
	⋮		
1	0	0	1
	⋮		
1	1	1	0

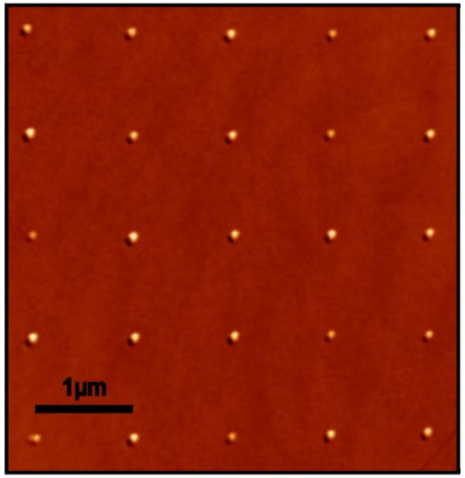
⇓

$$f(x) = x_1 \overline{x_2}\,\overline{x_3}$$

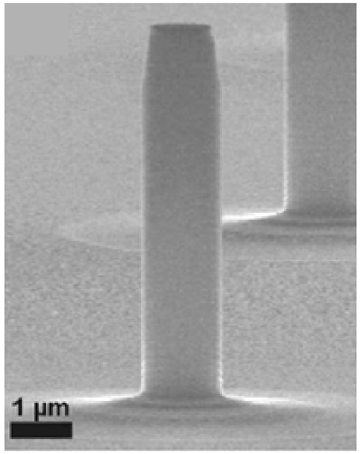
Physical Qubits

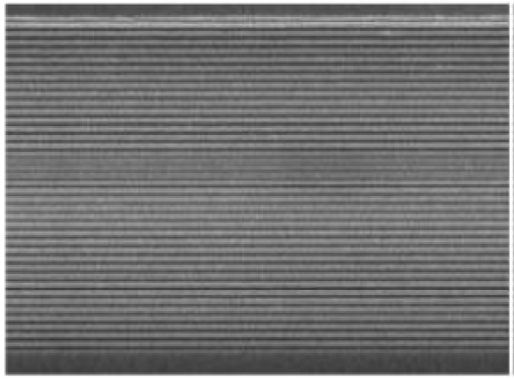
－ Cavity QED Systems with Single-Electron-Doped Quantum Dots －





“Random” → **“Scalable”**





A post-microcavity with top and bottom DBRs and self-assembled InGaAs QDs

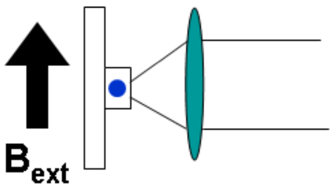
D. Press, S. Gotzinger, S. Reitzenstein, C. Hofmann, A. Löffler, M. Kamp, A. Forchel, and Y. Yamamoto, *PRL* 98, 117402 (2007)

A simple planar microcavity with 2D lattice of site-controlled QDs

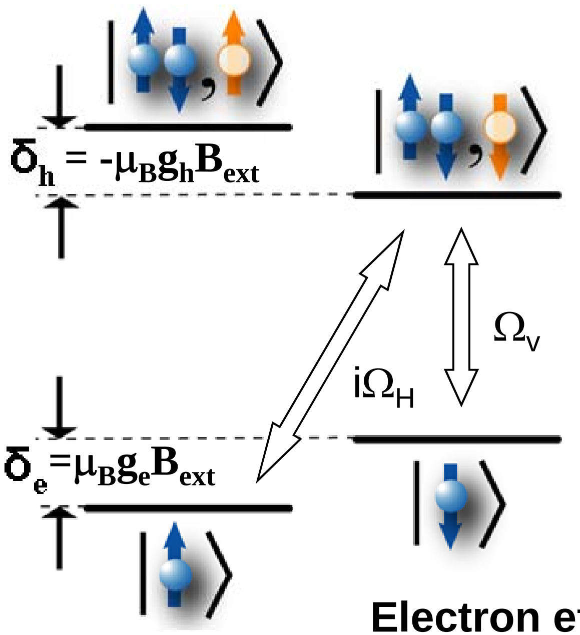
C. Schneider, M. Strauss, T. Sunner, A. Huggenberger, D. Wiener, S. Reitzenstein, M. Kamp, S. Höfling and A. Forchel *APL* 92, 183101 (2008)

Magnetic Spectrum of Charged Exciton (Trion) in InAs Quantum Dot
― Artificial Three-Level Atom in Lambda Configuration ―

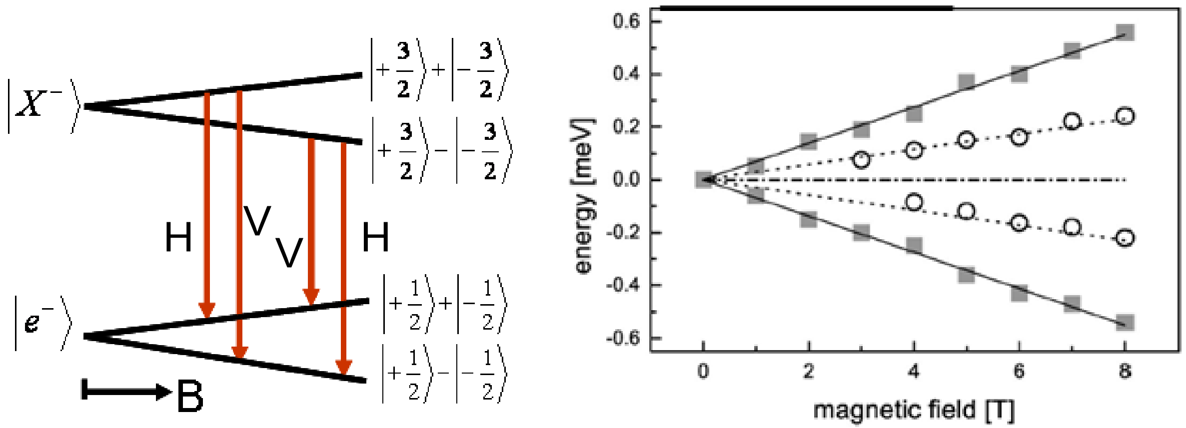
Magnetic field in Voigt geometry



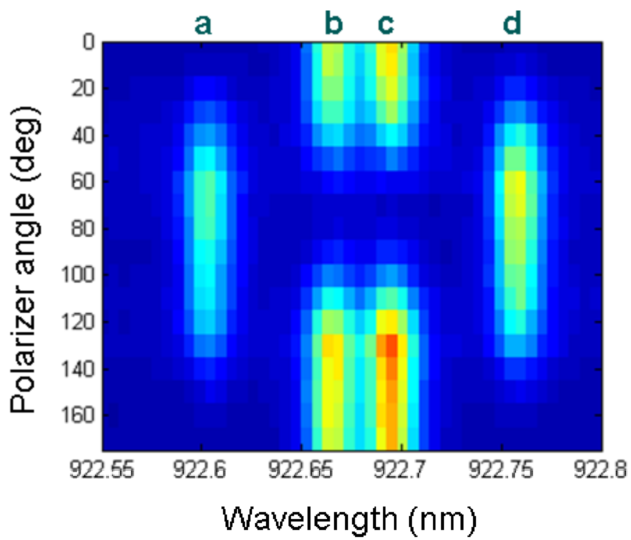
Trion X⁻ electron spins in singlet spin is governed by heavy hole



Electron e⁻



M. Bayer et al., Phys. Rev. B 65, 195305 (2002)

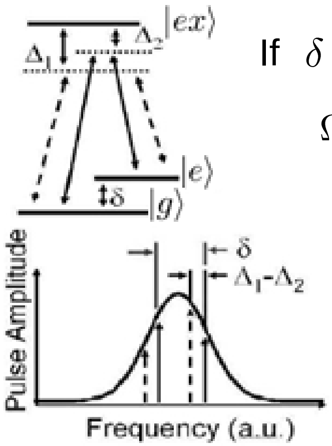


D. Press et al., Nature 456, 218 (2008)

Ultra-fast Spin Rotation with Single Optical Pulse

S. Clark et al., Phys. Rev. Lett. 99, 040501 (2007)

- A single broadband optical pulse can implement an arbitrary one-bit gate with fidelity of 0.999.



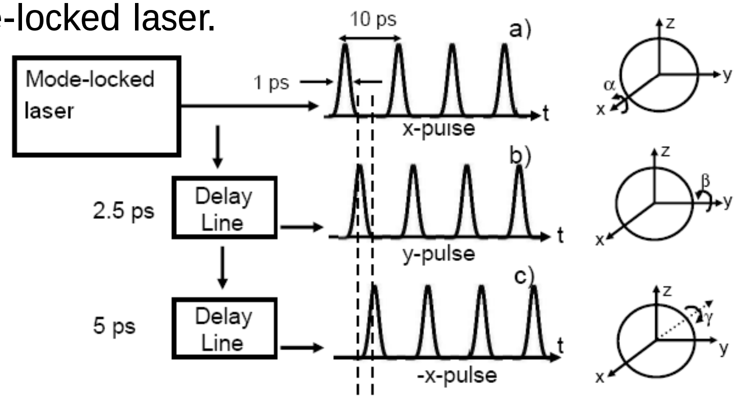
If $\delta \ll \Omega_0, \Omega_1 \ll \Delta$, an effective Rabi frequency

$$\Omega_{\text{eff}} = \frac{\Omega_0 \Omega_1^*}{2\Delta} \simeq \frac{|\Omega(t)|^2}{2\Delta}$$

$$\Omega(t) = \frac{\mu E(t)}{\hbar}$$

rotation angle $\int \Omega_{\text{eff}}\, dt$ is proportional to pulse energy

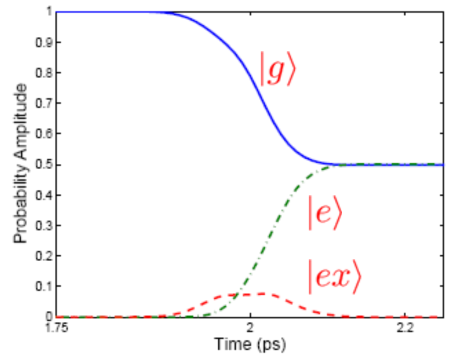
- A system clock is provided by the pulse arrival time from the mode-locked laser.



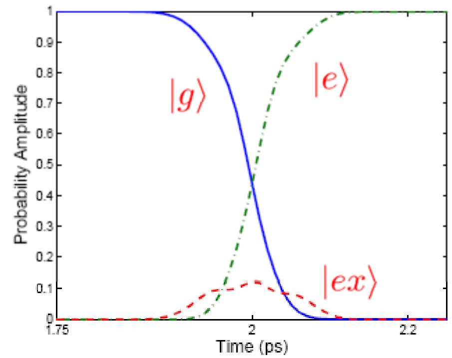
Numerical simulation based on the three-level master equation

$$\tau \ll T_1, T_2$$

100fs π/2 - pulse



100fs π - pulse



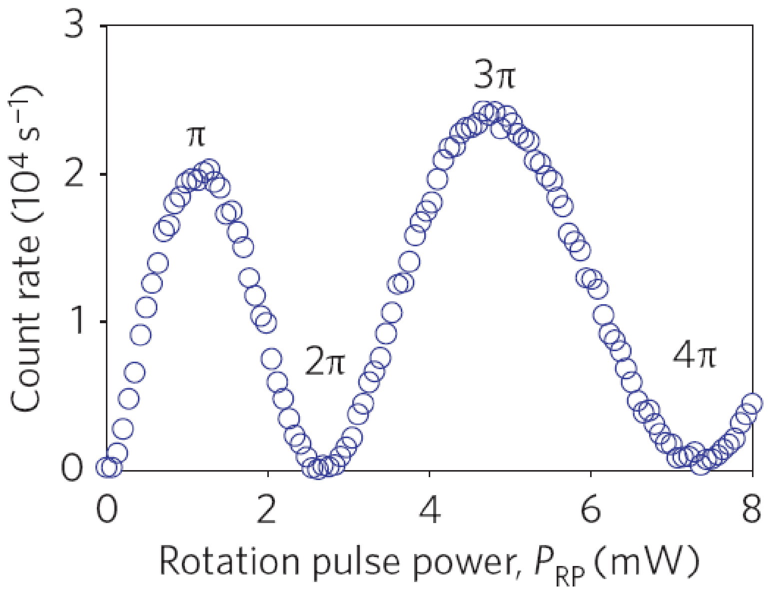
Arbitrary single qubit gates SU(2) can be implemented in one-half of Larmor oscillation period.

Experiment with an ensemble of donor spins : K.M. Fu et al., Nature Physics 4, 780 (2008)

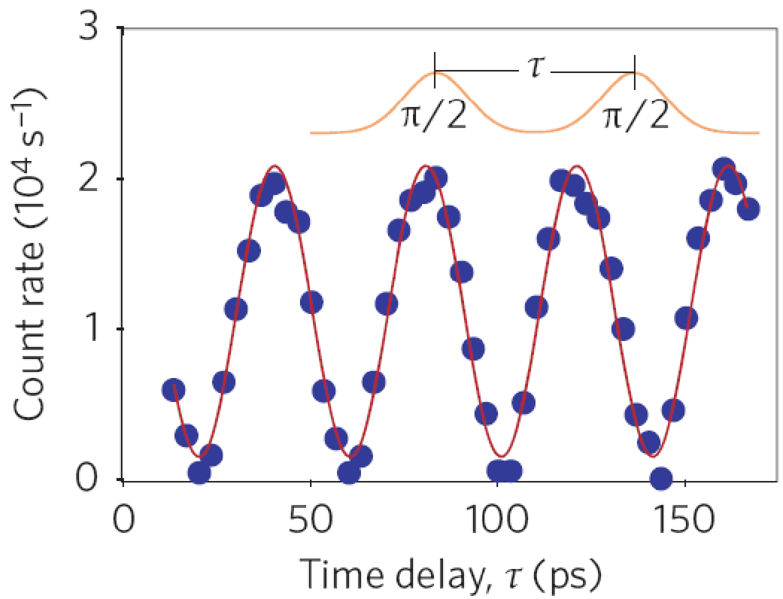
Improved Gate Fidelity for a Single Spin with a Microcavity
D. Press et al., Nature Photonics 4, 367 (2010)

Control optical pulse energy reduced by a cavity: 1/300
Optical pump pulse completely off during single qubit operation

Coherent Rabi oscillation experiment



Ramsey Interference experiment

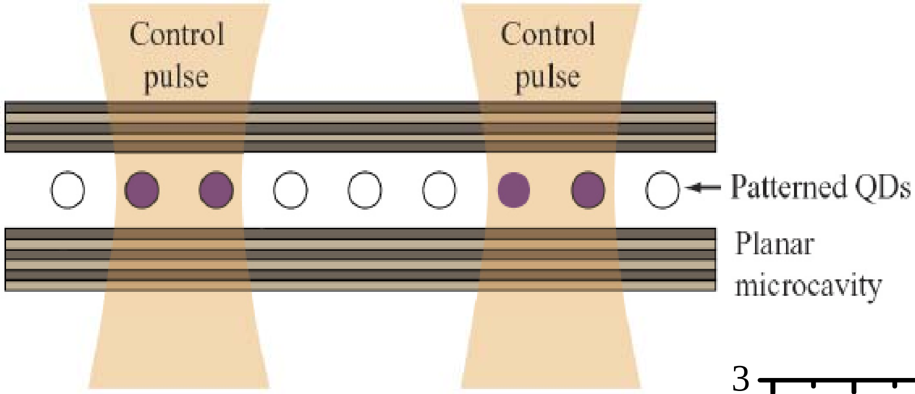


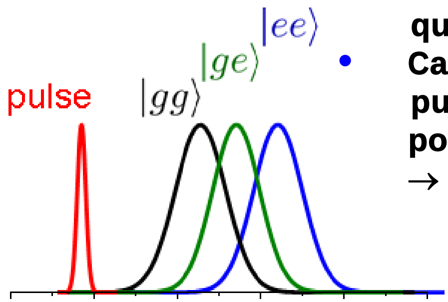
Spin rotation during control pulse
(Larmor period 40 psec vs. pulse duration 4 psec)

Single qubit gate fidelity: F=98~99%

Two Qubit Gate in Dissipative Planar Microcavity

T. Ladd et al., arXiv:0910.4988 (quant-ph)

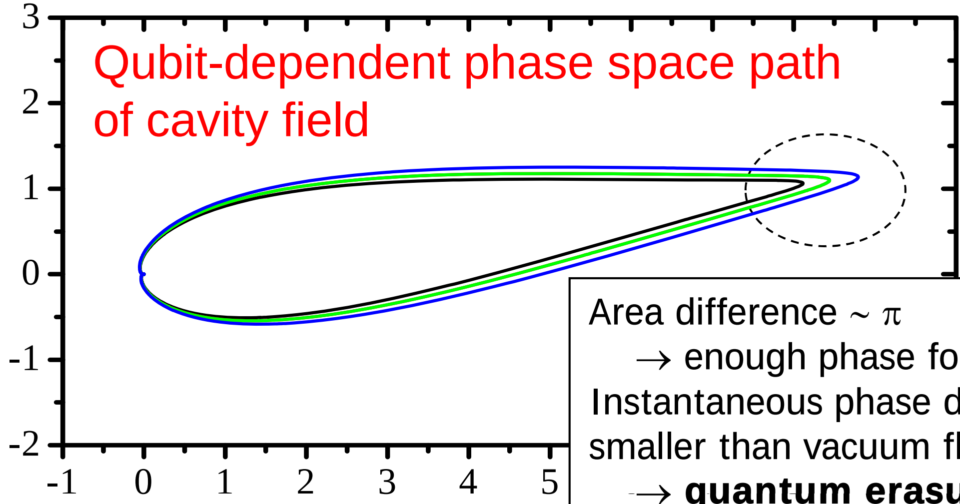




- **Cavity resonance depends on qubit states**
- **Cavity field amplitude by detuned pulse depends on resonance position**
  → Amplitude "path" of cavity internal field depends on qubit states

Unique mode spot size of
2D planar cavity

G. Björk et al., PRA 44, 669 (1991)



Area difference ~ π
→ enough phase for CZ gate
Instantaneous phase difference smaller than vacuum fluctuation
→ **quantum erasure**

→ CZ gate for surface code creation in a massive parallel operation
→ Master equation simulations indicate fidelity>99% with Q=10^5

τ~100nsec (purely optical), τ~100psec (polaritonic)

Primary Components of the Physical Layer