$$|C(a,b)-C(a,b')| \le \int \left|A(a,\lambda)[B(b,\lambda)-B(b',\lambda)]\right|\underset{\uparrow}{\rho}(\lambda)d\lambda$$

positive value

$$\longleftarrow \ |A(a,\lambda)| = 1$$

$$= \int |B(b,\lambda)-B(b',\lambda)|\rho(\lambda)d\lambda \quad \text{———} \quad (1)$$

$$|C(a',b)+C(a',b')| \le \int \left|A(a',\lambda)[B(b,\lambda)+B(b',\lambda)]\right|\rho(\lambda)d\lambda$$

$$= \int |B(b,\lambda)+B(b',\lambda)|\rho(\lambda)d\lambda \quad \text{———} \quad (2)$$

(1) + (2) $\Longrightarrow$

$$|C(a,b)-C(a,b')| + |C(a',b)+C(a',b')|$$

$$\le \int \Big[\underbrace{|B(b,\lambda)-B(b',\lambda)| + |B(b,\lambda)+B(b',\lambda)|}\Big]\rho(\lambda)d\lambda$$

= 2 for all combinations of

$[A(a),\ B(b)] = (1,1),(1,-1),(-1,1),(-1,-1)$.

$$\int \rho(\lambda)d\lambda = 1 \ \Big\Downarrow$$

$$\boxed{|C(a,b)-C(a,b')| + |C(a',b)+C(a',b')| \le 2}$$

: Bell's inequality

This result contradicts QM prediction:

$$C(\underset{\theta_1}{a},\underset{\theta_2}{b}) = \langle A(a)B(b)\rangle \quad \longleftarrow \text{ quantum average}$$

$$= P(+,\theta_2;+,\theta_1)\times \underset{A(a)B(b)=(+1)\times(+1)}{(+1)} + P(-,\theta_2;-,\theta_1)\times \underset{A(a)B(b)=(-1)\times(-1)}{(+1)}$$

$$+P(+,\theta_2;-,\theta_1)\times \underset{A(a)B(b)=(-1)\times(+1)}{(-1)} + P(-,\theta_2;+,\theta_1)\times \underset{A(a)B(b)=(+1)\times(-1)}{(-1)}$$

$$= \alpha_1\alpha_2\left[\sin^2(\theta_1-\theta_2)-\cos^2(\theta_1-\theta_2)\right]$$

$$= -\alpha_1\alpha_2\cos\left[2(\theta_1-\theta_2)\right]$$

If $\alpha_1=\alpha_2=1$ and $\theta_1=0,\ \theta_2=\frac{3}{8}\pi,\ \theta_1'=-\frac{\pi}{4},\ \theta_2'=\frac{\pi}{8},$ we obtain

$$\left|C(\theta_1,\theta_2)-C(\theta_1,\theta_2')\right|+\left|C(\theta_1',\theta_2)+C(\theta_1',\theta_2')\right|$$

$$=\left|-\cos\left(\frac{-3}{4}\pi\right)+\cos\left(-\frac{\pi}{4}\right)\right|+\left|-\cos\left(-\frac{5}{4}\pi\right)-\cos\left(-\frac{3}{4}\pi\right)\right|$$

$$=2\sqrt{2}\quad\Longrightarrow$$ violates Bell's inequality!

This is a maximum violation point for Bell's inequality.

However, if $\alpha_1\alpha_2 \le \frac{1}{\sqrt{2}} \approx 0.71$, it is impossible to show the violation of Bell's inequality.

1.6.3 Clauser – Horne – Shimony - Holt (CHSH) inequality

no enhancement assumption:

$$P(\theta,\lambda)\le P(-,\lambda)$$

probability for a photon
to reach the detector
with a polarizer angle θ

probability for a photon
to reach the detector
without a polarizer

$$P_{12}(\theta_1,\theta_2)=\alpha_1\alpha_2\int P_1(\theta_1,\lambda)P_2(\theta_2,\lambda)\rho(\lambda)d\lambda$$

$$P_{12}(\theta_1,-)=\alpha_1\alpha_2\int P_1(\theta_1,\lambda)P_2(-,\lambda)\rho(\lambda)d\lambda$$

$$P_{12}(-,\theta_2)=\alpha_1\alpha_2\int P_1(-,\lambda)P_2(\theta_2,\lambda)\rho(\lambda)d\lambda$$

$$\begin{pmatrix} A(a,\lambda)\rightarrow P_1(\theta_1,\lambda) \\ B(b,\lambda)\rightarrow P_2(\theta_2,\lambda) \end{pmatrix}$$

normalized probability:

$$\begin{aligned} x &= P_1(\theta_1, \lambda)/P_1(-, \lambda) \\ x' &= P_1(\theta_1', \lambda)/P_1(-, \lambda) \\ y &= P_2(\theta_2, \lambda)/P_2(-, \lambda) \\ y' &= P_2(\theta_2', \lambda)/P_2(-, \lambda) \end{aligned} \quad \Rightarrow \quad 0 \le x, x', y, y' \le 1$$

$$\Downarrow$$

$$\boxed{-1 \le xy - xy' + x'y + x'y' - x' - y \le 0}$$

$$\int \underbrace{\alpha_1 \alpha_2 P_1(-,\lambda) P_2(-,\lambda) \rho(\lambda)}_{\text{positive}} \underbrace{[xy - xy' + x'y + x'y' - x' - y]}_{\text{negative}} d\lambda$$

$$\Downarrow$$

$$S = P_{12}(\theta_1, \theta_2) - P_{12}(\theta_1, \theta_2') + P_{12}(\theta_1', \theta_2) + P_{12}(\theta_1', \theta_2') - P_{12}(\theta_1', -) - P_{12}(-, \theta_2) \le 0$$

$\Downarrow$ every term is proportional to $\alpha_1 \alpha_2$

$$\frac{S}{\alpha_1 \alpha_2} \le 0$$:CHSH inequality

Since CHSH inequality is independent of $\alpha_1 \alpha_2$, a high-efficiency photo-detector is not necessary to demonstrate the violation of the inequality.

cf. QM prediction

$$\theta_1 = 0,\ \theta_1' = -\frac{\pi}{4},\ \theta_2 = \frac{3\pi}{8},\ \theta_2' = \frac{\pi}{8}$$

$$\Downarrow$$

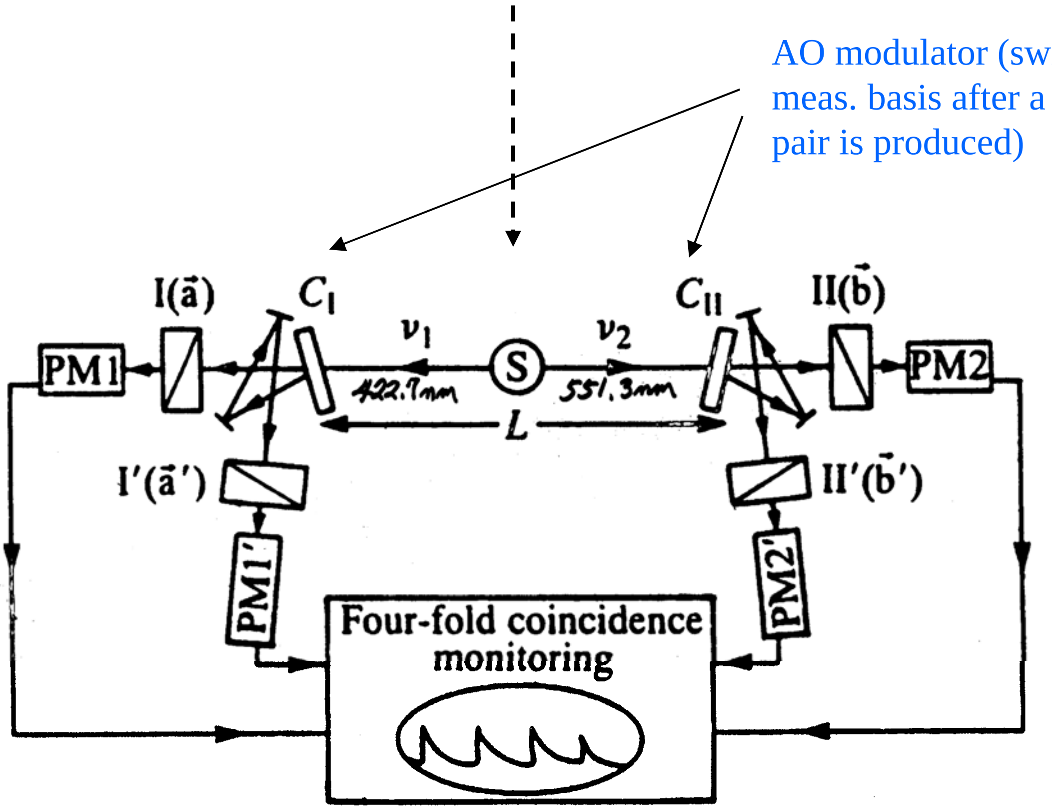
$$\frac{S}{\alpha_1 \alpha_2} = \frac{\sqrt{2} - 1}{2} > 0$$ (maximum violation of CHSH inequality)

Thus, QM predicts the violation of the CHSH inequality.

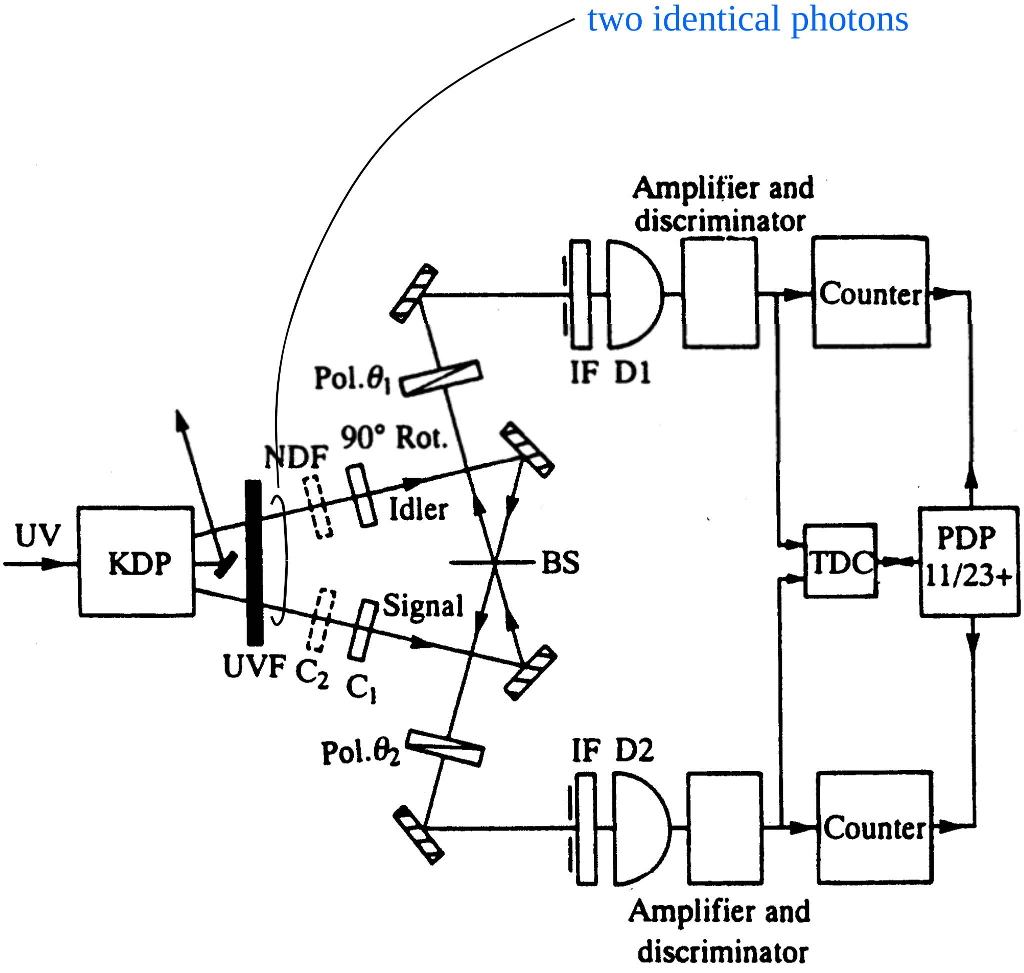
Ca atomic SPS cascade emission

(J = 0) → (J = 1) → (J = 0)

AO modulator (switch the meas. basis after a photon-pair is produced)

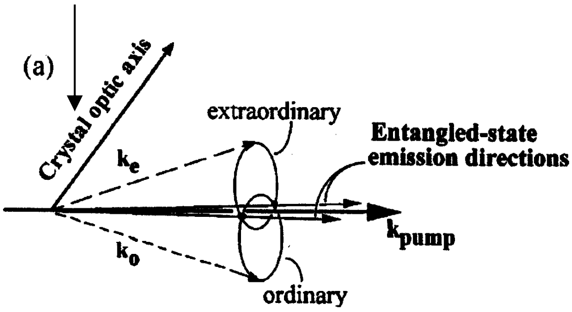


A. Aspect, J. Dalibard and G. Roger, *Phys. Rev. Lett*. **49**, 1804 (1982)



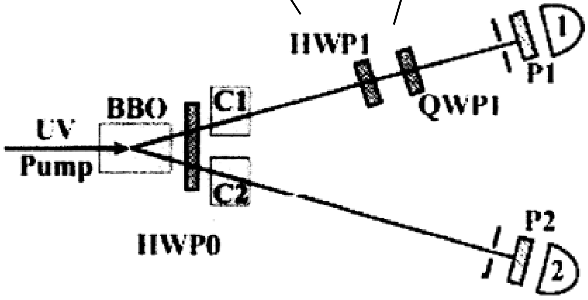
two identical photons

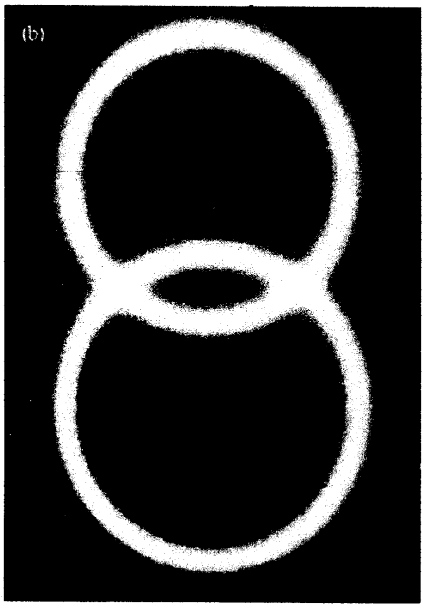
Z. Y. Ou and L. Mandel, *Phys. Rev. Lett.* **61**, 50 (1988) typeII (o, e)



polarization rotation ($V \rightleftarrows H$)

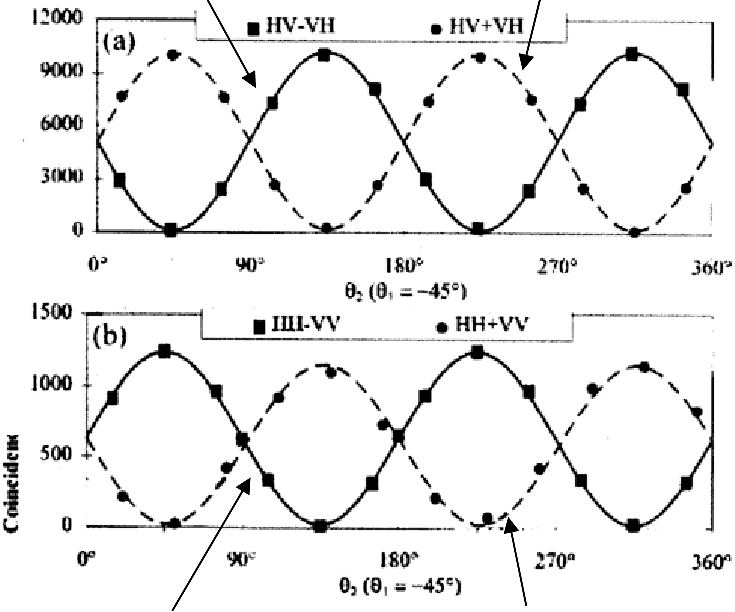
phase rotation (180°)





$\frac{1}{\sqrt{2}}\left[\left|H_1V_2\right\rangle-\left|V_1H_2\right\rangle\right]$ $\sin^2(\theta_1-\theta_2)$

$\frac{1}{\sqrt{2}}\left[\left|H_1V_2\right\rangle+\left|V_1H_2\right\rangle\right]$ $\sin^2(\theta_1+\theta_2)$

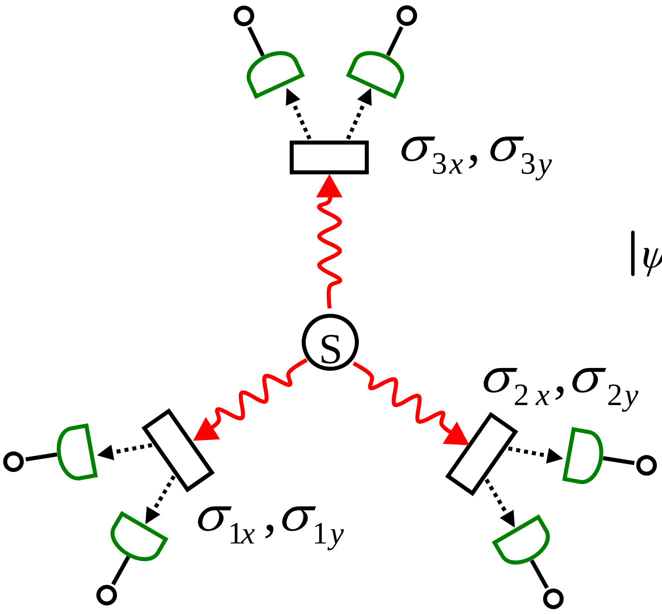


$\frac{1}{\sqrt{2}}\left[\left|H_1H_2\right\rangle-\left|V_1V_2\right\rangle\right]$ $\cos^2(\theta_1+\theta_2)$

$\frac{1}{\sqrt{2}}\left[\left|H_1H_2\right\rangle+\left|V_1V_2\right\rangle\right]$ $\cos^2(\theta_1-\theta_2)$

P. G. Kwiat, K. Mattle, H. Weinfurter and A. Zeilinger,
Phys. Rev. Lett. **75**, 4337 (1995)

1.6.4 Three Particle Entangled State (spin-1/2 particles)



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|u\rangle_1|u\rangle_2|u\rangle_3 - |v\rangle_1|v\rangle_2|v\rangle_3\right]$$

: GHZ state

Projective properties of Pauli spin operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x|u\rangle = |v\rangle \qquad \sigma_y|u\rangle = i|v\rangle \qquad \sigma_z|u\rangle = |u\rangle$$

$$\sigma_x|v\rangle = |u\rangle \qquad \sigma_y|v\rangle = -i|u\rangle \qquad \sigma_z|v\rangle = -|v\rangle$$

$|u\rangle \pm |v\rangle$ are the eigenstates of σ_x with eigenvalues of ±1

$|u\rangle \pm i|v\rangle$ are the eigenstates of σ_y with eigenvalues of ±1

Eigenvalue properties of $|\psi\rangle$:

$$\underbrace{\sigma_{1x}\sigma_{2y}\sigma_{3y}}|\psi\rangle = \frac{1}{\sqrt{2}}\left[(i)^2|v\rangle_1|v\rangle_2|v\rangle_3 - (-i)^2|u\rangle_1|u\rangle_2|u\rangle_3\right] = |\psi\rangle$$

projective meas. of 1*x*, 2*y*, 3*y* ⇨ (xyy) measurement

$$\sigma_{1y}\sigma_{2x}\sigma_{3y}|\psi\rangle = |\psi\rangle$$

$$\sigma_{1y}\sigma_{2y}\sigma_{3x}|\psi\rangle = |\psi\rangle$$

$$\sigma_{1x}\sigma_{2x}\sigma_{3x}|\psi\rangle = \frac{1}{\sqrt{2}}\left[|v\rangle_1|v\rangle_2|v\rangle_3 - |u\rangle_1|u\rangle_2|u\rangle_3\right] = -|\psi\rangle$$

Measurement results:

$m_{1x}m_{2x}m_{3x} = -1$ ⟵ If the spins of the particles #2 and #3 are measured along x-axis, the measurement result of the spin of the particle #1 along x-axis is predicted with certainty.

$$\text{4 different measurements} \left\{ \begin{array}{l} -1 \leftarrow (1, 1) \\ 1 \leftarrow (1, -1) \\ 1 \leftarrow (-1, 1) \\ -1 \leftarrow (-1, -1) \end{array} \right.$$

$m_{1x}m_{2y}m_{3y} = 1$ ⟵ If the spins of the particles #2 and #3 are measured along y-axis, the measurement result of the spin of the particle #1 along x-axis is predicted with certainty.

$$\begin{array}{l} 1 \leftarrow (1, 1) \\ -1 \leftarrow (1, -1) \\ -1 \leftarrow (-1, 1) \\ 1 \leftarrow (-1, -1) \end{array}$$

The above two m_{1x}'s are identical according to Einstein's local realism ⇦ An element of physical reality

$$m_{1y}m_{2x}m_{3y} = 1$$

$$m_{1y}m_{2y}m_{3x} = 1$$

⇩

$(m_{1x}m_{1y}m_{2x}m_{2y}m_{3x}m_{3y})^2 = -1$: contradiction

⇨ Violation of the local hidden variable theory without inequality

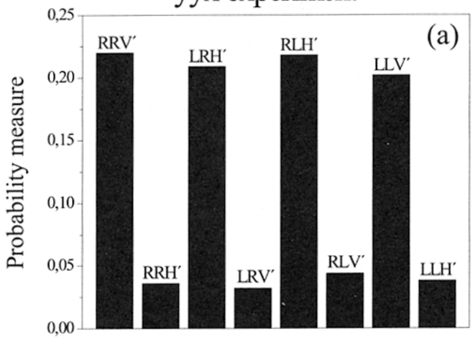
Experimental Test of Local Realism vs. Quantum Mechanics

GHZ state $$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}\left[|H\rangle_1|H\rangle_2|H\rangle_3 + |V\rangle_1|V\rangle_2|V\rangle_3\right]$$

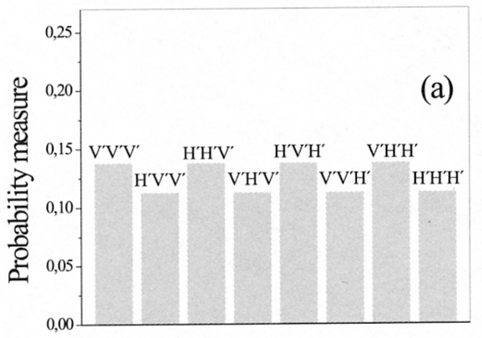
$$|R\rangle = \frac{1}{\sqrt{2}}\left[|H\rangle + i|V\rangle\right], \quad |L\rangle = \frac{1}{\sqrt{2}}\left[|H\rangle - i|V\rangle\right]$$

$$|H'\rangle = \frac{1}{\sqrt{2}}\left[|H\rangle + |V\rangle\right], \quad |V'\rangle = \frac{1}{\sqrt{2}}\left[|H\rangle - |V\rangle\right]$$

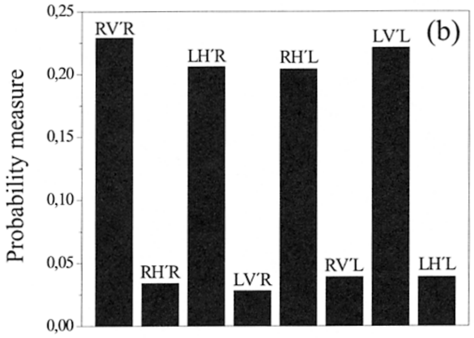
yyx experiment



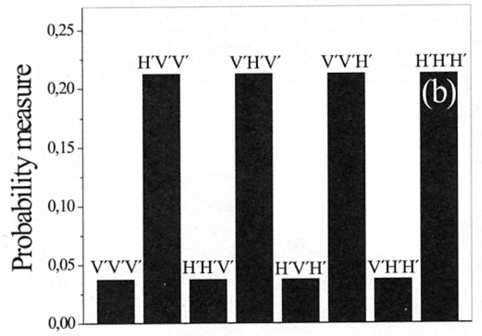
xxx local realstic prediction



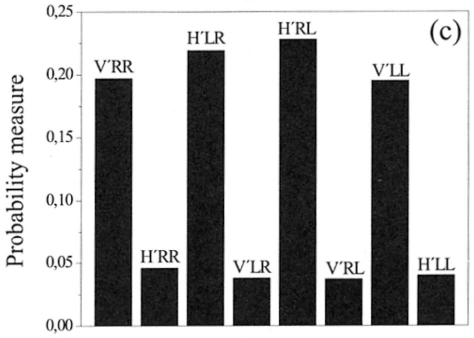
yxy experiment



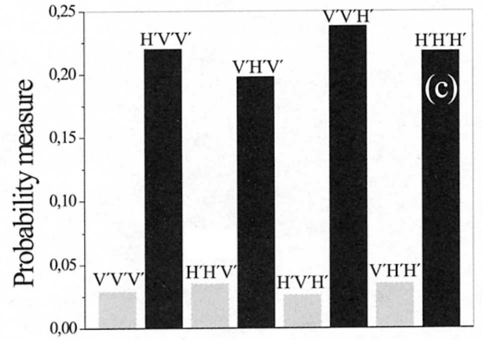
xxx quantum prediction



xyy experiment



xxx experimental results



Further reading for Chapter 1

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