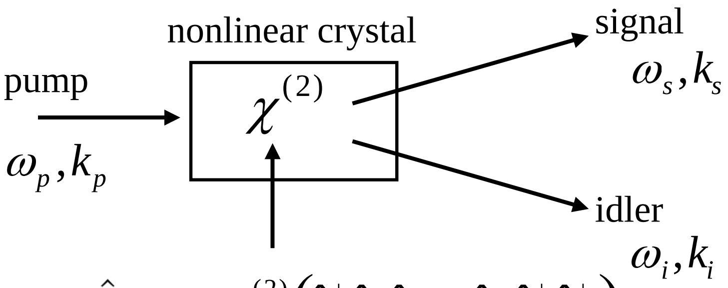
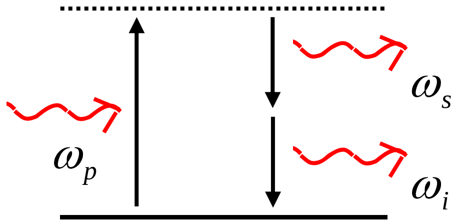
# 1.3 Two photon interference

## 1.3.1 Quantum theory of parametric down-conversion





$$\hat{\mathcal{H}}_I = \hbar\chi^{(2)}\left(\underset{\substack{\downarrow\\ \varepsilon_p^*}}{\hat{a}_p^+}\hat{a}_s\hat{a}_i + \underset{\substack{\downarrow\\ \varepsilon_p}}{\hat{a}_p}\hat{a}_s^+\hat{a}_i^+\right)$$

: strong pump wave (no pump depletion)

$$\left.\begin{array}{ll}\text{energy conservation:} & \omega_p = \omega_s + \omega_i \\ \text{momentum conservation:} & k_p = k_s + k_i\end{array}\right\} \Rightarrow$$

Imposes the correlation for the energy and momentum.
(propagation direction)

perturbation solution

$$|\psi(t)\rangle = \exp\left[\frac{1}{i\hbar}\int_0^t \hat{\mathcal{H}}_I(t')dt'\right]|\psi(0)\rangle$$

$|0\rangle_s|0\rangle_i$ : initial state

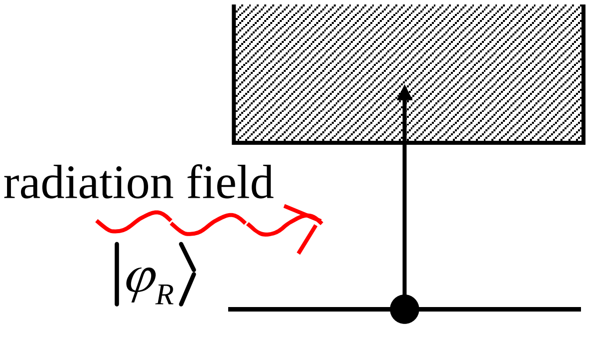
$$\approx c_0|0\rangle_s|0\rangle_i + c_1|1\rangle_s|1\rangle_i + \underbrace{\cdots}_{\text{negligible}}$$

$$c_1 \propto \varepsilon_p\chi^{(2)}t$$

c-number pump field amplitude (highly excited coherent state)
carries the phase information of the pump wave

$$|c_0|^2 + |c_1|^2 \approx 1$$ (normalization)

## 1.3.2. Quantum Theory of Photodetection



$|c\rangle$ : ionization continuum

atom bandwidth → $\Delta\omega_A \rangle\rangle \Delta\omega_R$ ← field bandwidth

(wideband photodetector)

$|a\rangle$ : ground state

Hamiltonian: $H = H_A + H_R + V$

$V = -d \cdot E$ (electric dipole interaction)

atomic dipole ↗ ↖ electric field

Interaction picture (with regard to $H_A + H_R$):

$$U_I(\Delta t) = 1 + \frac{1}{i\hbar}\int_0^{\Delta t} dt\, V_I(t)$$

$$V_I(t) = -d_I(t)E_I(t)$$

$$d_I(t) = e^{iH_A t/\hbar} d e^{-iH_A t/\hbar}$$

$$E_I(t) = e^{iH_R t/\hbar} E e^{-iH_R t/\hbar}$$

Initial state: $|\psi_I(0)\rangle = |a, \varphi_R\rangle$

Excitation probability: ↙ any orthonormal sets (ex. photon number eigenstates)

$$P_{exc}(\Delta t) = \sum_n \sum_{c \neq a} \left|\langle c, n|U_I(\Delta t)|a, \varphi_R\rangle\right|^2$$

← $\langle c|a\rangle = 0$, so $U_I(\Delta t) = 1$ does not contribute to $P_{\text{exc}}$.

$$= \sum_n \sum_c \frac{1}{\hbar^2}\int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' \langle\varphi_R|E_I(t')|n\rangle\langle n|E_I(t'')|\varphi_R\rangle$$

$$\times \langle a|d_I(t')|c\rangle\langle c|d_I(t'')|a\rangle$$

↗

We can include *a* because $\langle a|d_I|a\rangle = 0$

Closure relations:

$$\sum_n |n\rangle\langle n| = I$$

$$\sum_c |c\rangle\langle c| = I$$

⇩

$$P_{exc}(\Delta t) = \frac{1}{\hbar^2}\int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' G_A^*(t',t'') G_R(t',t'')$$

Atomic dipole correlation function:

$$\begin{aligned} G_A^*(t',t'') &= \langle a| d_I(t') d_I(t'') |a\rangle \\ &= \sum_c \langle a| d_I(t') |c\rangle\langle c| d_I(t'') |a\rangle \\ &= \sum_c |\langle a|d|c\rangle|^2 e^{-i\omega_{ca}(t'-t'')} \end{aligned}$$

$\omega_c - \omega_a$

Field correlation function:

$$E_I(t) = E_I^{(+)}(t) + E_I^{(-)}(t)$$

$$E_I^{(+)}(t) = i\sum_j \sqrt{\frac{\hbar\omega_j}{2\varepsilon_0 V}} a_j e^{-i\omega_j t}$$

$$E_I^{(-)}(t) = -i\sum_j \sqrt{\frac{\hbar\omega_j}{2\varepsilon_0 V}} a_j^+ e^{i\omega_j t}$$

oscillation canceled out

⇩

resonant

$$\begin{aligned} G_R(t',t'') = \langle \varphi_R | E_I(t') E_I(t'') | \varphi_R \rangle &= \langle \varphi_R | E_I^{(-)}(t') E^{(+)}(t'') | \varphi_R \rangle \rightarrow e^{i\omega_j(t'-t'')} \\ &\left.\begin{aligned} &+ \langle \varphi_R | E_I^{(+)}(t') E_I^{(-)}(t'') | \varphi_R \rangle \\ &+ \langle \varphi_R | E_I^{(+)}(t') E_I^{(+)}(t'') | \varphi_R \rangle \\ &+ \langle \varphi_R | E_I^{(-)}(t') E_I^{(-)}(t'') | \varphi_R \rangle \end{aligned}\right\} \text{off-resonant} \end{aligned}$$

Resonance condition: $\omega_{ca} \approx \omega_j$

$G_R(t',t'')$ varies much more slowly than $G_A^*(t',t'')$ since $\Delta\omega_A \rangle\rangle \Delta\omega_R$ (broadband photodetector).

⇩

$$P_{exc}(\Delta t) = \frac{1}{\hbar^2}\langle\varphi_R|E_I^{(-)}(0)E_I^{(+)}(0)|\varphi_R\rangle$$

$$\times \int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' G_A^*(t',t'')e^{i\overline{\omega}(t'-t'')}$$

$\overline{\omega}$ : center frequency of a radiation field

⇩

Photodetection rate:

$$\omega_{exc} = \frac{P_{exc}(\Delta t)}{\Delta t} = S\langle\varphi_R|E^{(-)}(0)E^{(+)}(0)|\varphi_R\rangle$$

$$S = \frac{2\pi}{\hbar^2}\sum_c |\langle a|d|c\rangle|^2 \delta(\overline{\omega} - \omega_{ca})$$ : Fermi's golden rule absorption rate

In general, k photon coincidence detection rate is given by

$$\omega(t_1,t_2,\cdots t_k) = S_1S_2\cdots S_k\langle\varphi_R|E_1^{(-)}(t_1)E_2^{(-)}(t_2)\cdots E_k^{(-)}(t_k)E_k^{(+)}(t_k)\cdots E_2^{(+)}(t_2)E_1^{(+)}(t_1)|\varphi_R\rangle$$

$S_1\cdots S_k$ : absorption rates of photodetectors $1 \sim k$

## 1.3.3 Two photon interferometer - Mach-Zehnder type -

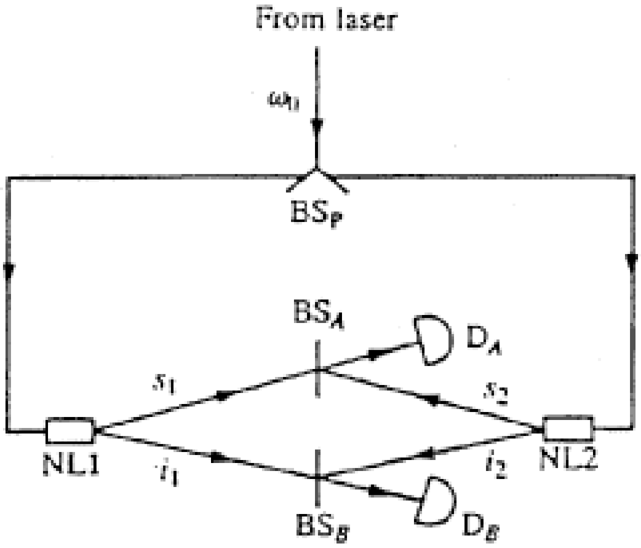
input states into beam splitter $BS_A$ and $BS_B$:

$$|\psi_1\rangle = c_{10}|0\rangle_{s1}|0\rangle_{i1} + c_{11}|1\rangle_{s1}|1\rangle_{i1}$$

$$|\psi_2\rangle = c_{20}|0\rangle_{s2}|0\rangle_{i2} + c_{21}|1\rangle_{s2}|1\rangle_{i2}$$

$$\Rightarrow \quad |\psi_{12}\rangle = |\psi_1\rangle|\psi_2\rangle$$

(product state)



Q1: If we shift a $BS_p$ laterally, does a single photon count rate at $D_A$ feature an oscillation?

Q2: If we shift a $BS_p$ laterally, does a double photon count rate at $D_A$ and $D_B$ feature an oscillation?

fields detected by $D_A$ and $D_B$ :

$$\hat{E}_A = \hat{E}_A^{(+)} + \hat{E}_A^{(-)}$$

negative frequency part

positive frequency part

$$\hat{E}_A^{(+)} = \underbrace{\sqrt{\frac{\hbar\omega_s}{2\varepsilon_0 V}}}_{\text{vacuum field amplitude}} \times \frac{1}{\sqrt{2}} \underbrace{(\hat{a}_{s1} + i\hat{a}_{s2})}_{\text{annihilation operators of the two signal outputs}} e^{-i\omega_s t} = \left[\hat{E}_A^{(-)}\right]^+$$

creation operators

$$\hat{E}_B = \hat{E}_B^{(+)} + \hat{E}_B^{(-)}$$

$$\hat{E}_B^{(+)} = \sqrt{\frac{\hbar\omega_i}{2\varepsilon_0 V}} \times \frac{1}{\sqrt{2}} \left(\hat{a}_{i1} + i\hat{a}_{i2}\right) e^{-i\omega_i t} = \left[\hat{E}_B^{(-)}\right]^+$$

## A. Single photon count rate

Probability that a detector $D_A$ detects a photon at $t$ :

$$\omega(r_A,t) = S_A \langle\psi_1|\langle\psi_2|\hat{E}_A^{(-)}(t) \quad \hat{E}_A^{(+)}(t) \quad \underbrace{|\psi_2\rangle|\psi_1\rangle}_{\text{Initial state}}$$

detector sensitivity

projection operator
that annihilates one photon

$$\hat{E}_A^{(+)}(t)|\psi_2\rangle|\psi_1\rangle$$

$$= K_A\left(\hat{a}_{s1} + i\hat{a}_{s2}\right)e^{-i\omega_s t}\{c_{10}c_{20}|0,0,0,0\rangle + c_{10}c_{21}|1,1,0,0\rangle + c_{11}c_{20}|0,0,1,1\rangle$$

$$+\cancel{c_{11}c_{21}|1,1,1,1\rangle}\}$$

$\sqrt{\frac{\hbar\omega_s}{2\varepsilon_0 V}} \times \frac{1}{\sqrt{2}}$

$s2,\ i2,\ s1,\ i1$

neglected

$\arg(\varepsilon_{p2})$ $\arg(\varepsilon_{p1})$

$$= K_A e^{-i\omega_s t}\{ic_{10}c_{21}|0,1,0,0\rangle + c_{11}c_{20}|0,0,0,1\rangle\}$$

orthogonal

Even though the phase information is kept, there is no interference.

$$\omega(r_A,t) = S_A K_A^2\left(|c_{10}c_{21}|^2 + |c_{11}c_{20}|^2\right)$$

Similarly,

$$\omega(r_B,t) = S_B K_B^2\left(|c_{10}c_{21}|^2 + |c_{11}c_{20}|^2\right)$$

a photon from
*NL*2

a photon from
*NL*1

No interference

(which-path information is leaked via the non-detected photon)

## B. Two photon coincidence count rate

Joint probability that a detector $D_A$ detects a photon at $t_1$ and a detector $D_B$ detects another photon at $t_2$:

initial state

$$\omega(r_A,t_1\,;r_B,t_2)=S_AS_B\langle\psi_{12}|\hat{E}_A^{(-)}(t_1)\hat{E}_B^{(-)}(t_2)\hat{E}_B^{(+)}(t_2)\hat{E}_A^{(+)}(t_1)|\psi_{12}\rangle$$

projection operator that annihilates one photon at $r_B$ and $t_2$.

projection operator that annihilates one photon at $r_A$ and $t_1$.

$$\hat{E}_B^{(+)}(t_2)\hat{E}_A^{(+)}(t_1)|\psi_{12}\rangle$$

$$=K_AK_B(\hat{a}_{i1}+i\hat{a}_{i2})(\hat{a}_{s1}+i\hat{a}_{s2})e^{-i\omega_it_2-i\omega_st_1}\{c_{10}c_{20}|0,0,0,0\rangle+c_{10}c_{21}|1,1,0,0\rangle$$

$$+c_{11}c_{20}|0,0,1,1\rangle\}$$

$$=K_AK_Be^{-i\omega_it_2-i\omega_st_1}(-\underbrace{c_{10}}_{\approx 1}\overset{\arg(\varepsilon_{p2})}{c_{21}}+\overset{\arg(\varepsilon_{p1})}{c_{11}}\underbrace{c_{20}}_{\approx 1})|0,0,0,0\rangle$$

projected onto identical state

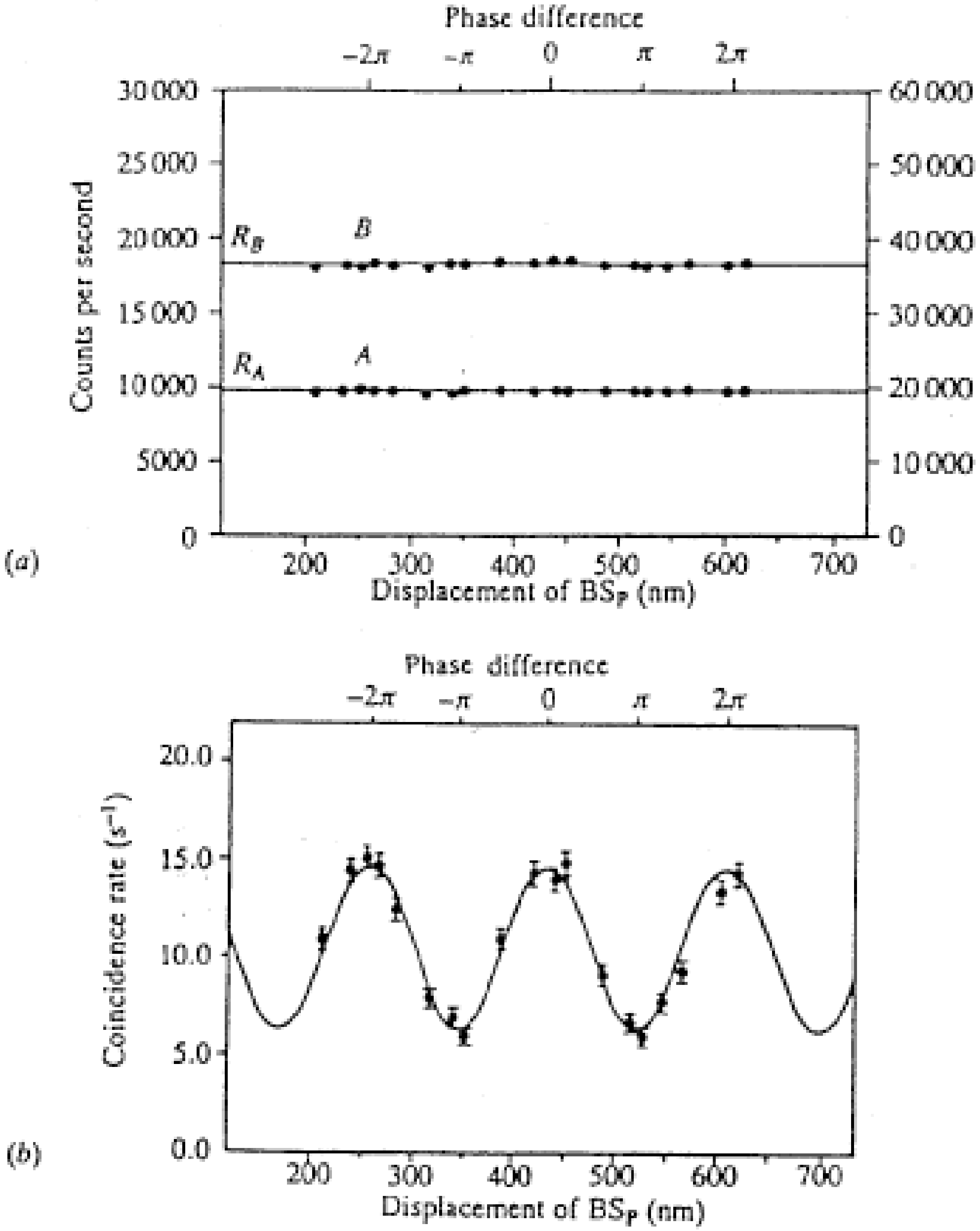
$$\omega(r_A,t_1\,;r_B,t_2)=S_AS_BK_A^2K_B^2|c_{11}c_{20}-c_{10}c_{21}|^2$$

$$=S_AS_BK_A^2K_B^2\left[|c_{11}c_{20}|^2+|c_{10}c_{21}|^2-2|c_{10}c_{20}c_{11}c_{21}|\cos\phi\right]$$

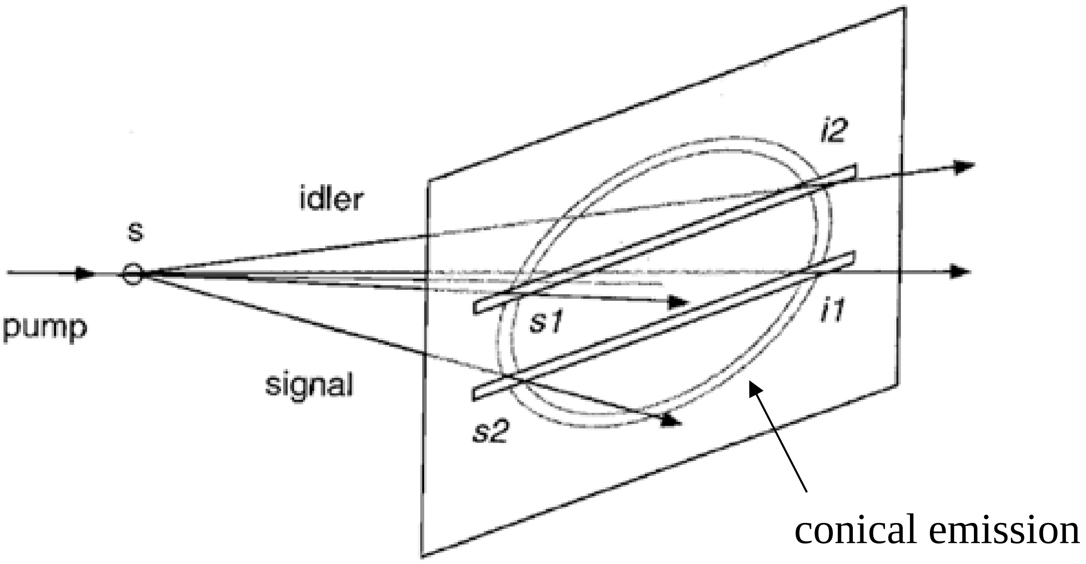
$$\phi=\arg(\varepsilon_{p1})-\arg(\varepsilon_{p2})+const.$$

interference term

The phase information is encoded between the probability amplitudes $c_{10}$ ($c_{20}$) ≅1 and $c_{11}$ ($c_{21}$).

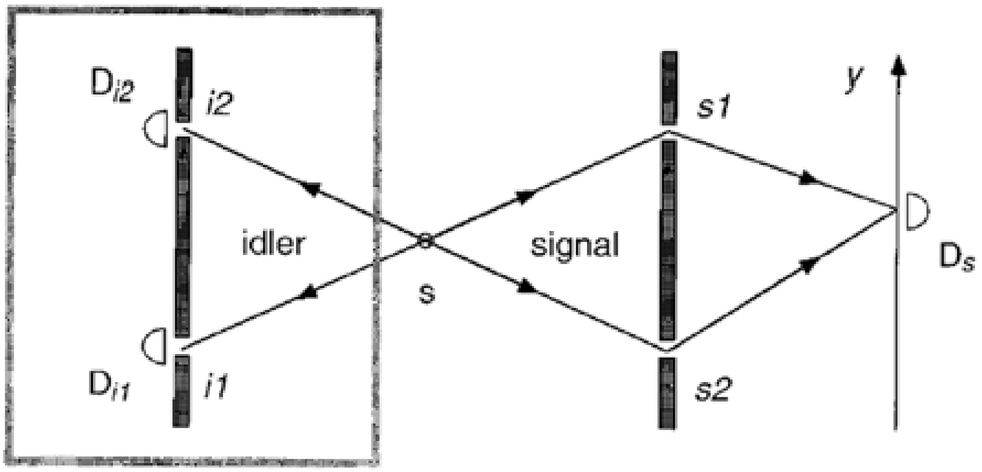

Phase difference
−2π
−π
0
π
2π
30 000
25 000
20 000
15 000
10 000
5000
0
60 000
50 000
40 000
30 000
20 000
10 000
0
Counts per second
$R_B$
B
$R_A$
A
(a)
200
300
400
500
600
700
Displacement of $BS_P$ (nm)
Phase difference
−2π
−π
0
π
2π
20.0
15.0
10.0
5.0
0.0
Coincidence rate ($s^{-1}$)
(b)
200
300
400
500
600
700
Displacement of $BS_P$ (nm)

## 1.3.4 Two photon interferometer - Young’s double-slit type-

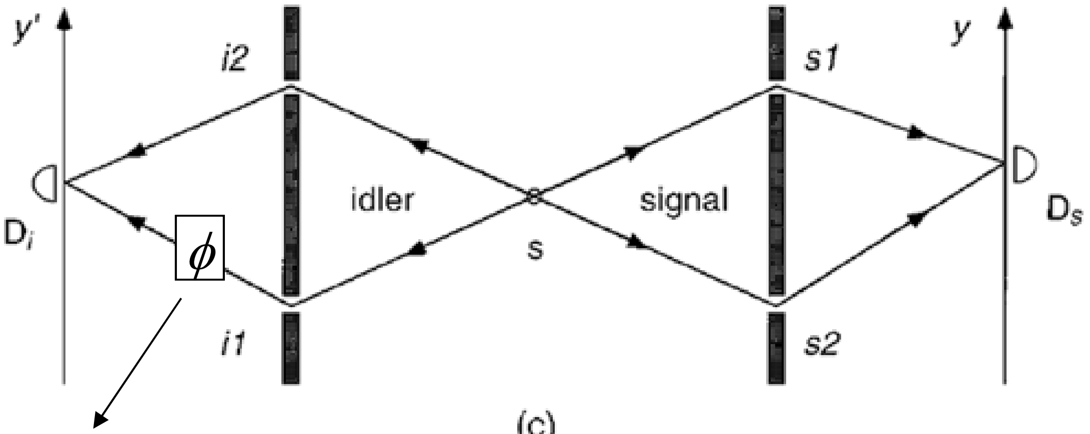


conical emission

(a)

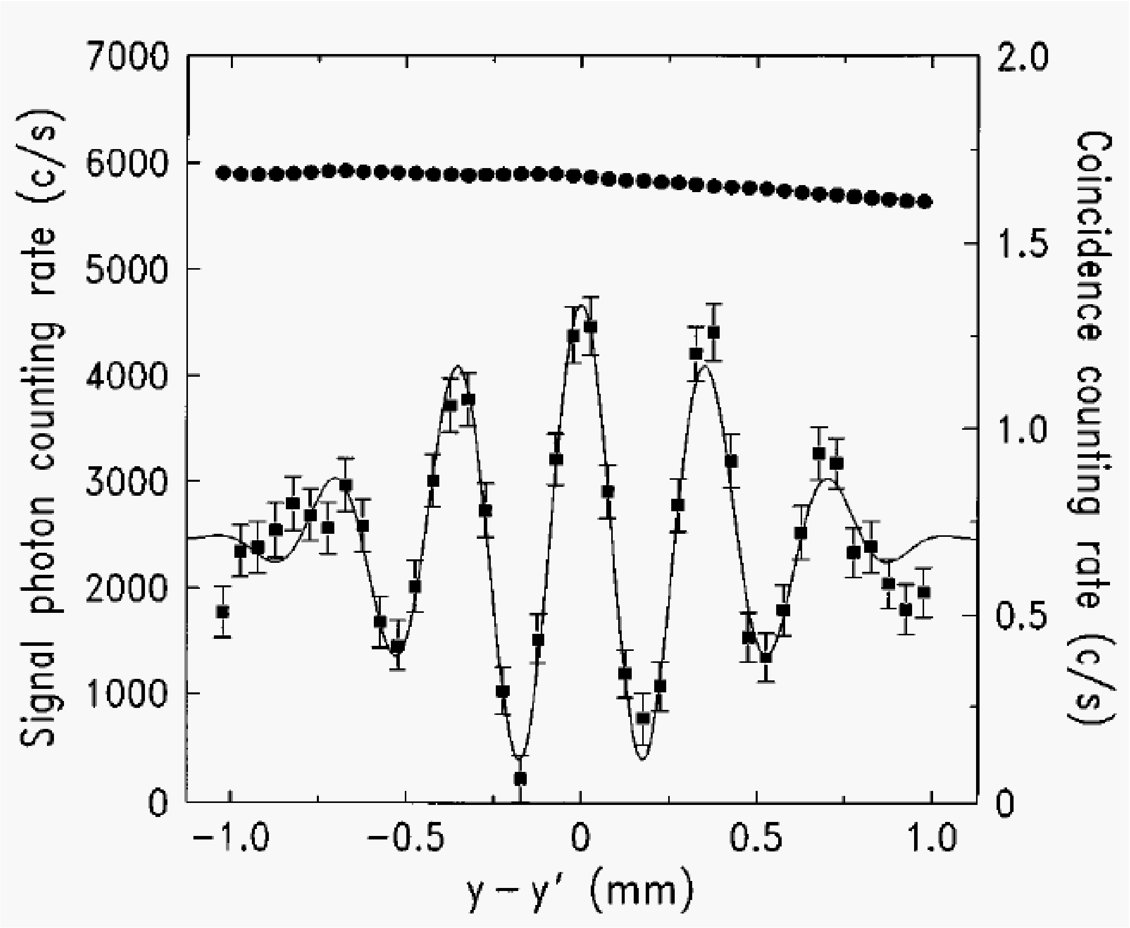


(b)



(c)

results in the shift of signal interference pattern



Measured signal photon-counting (●) and coincidence-counting (■) rates as functions of (*y* -*y* '). Error bars for the signal photon-counting rate are smaller than the dot size. The solid curve is the least-squares fit.

Physical interpretation: entanglement
which-path measurement
quantum erasure
nonlocality