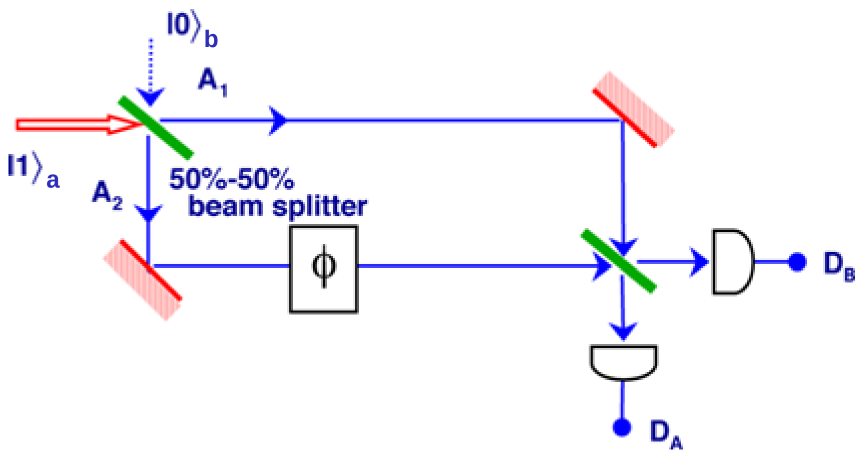
Chapter 1. Quantum interference

1.1 Single photon interference



Classical picture: Two "real physical waves" consisting of independent energy quanta (photons) are mutually coherent and so they interfere.

Quantum picture: Each individual photon simultaneously exists in the two arms with finite "probability amplitudes", which interfere.

input state: $|\psi_{in}\rangle = |1\rangle_a |0\rangle_b \Longrightarrow$ **ket vector**

Two states are uncorrelated.

after 1st B.S.: $|\psi_1\rangle = \hat{U}_{BS}|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_a|0\rangle_b + |0\rangle_a|1\rangle_b\right)$

B.S. unitary matrix $\hat{U}_{BS} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

basis vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle_a|0\rangle_b$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |0\rangle_a|1\rangle_b$

linear superposition state

Two mutually exclusive possibilities coexist.

after phase shifter: $|\psi_2\rangle = \hat{U}_{PS}|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_a|0\rangle_b + e^{i\phi}|0\rangle_a|1\rangle_b\right)$

$$\hat{U}_{PS} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{array}{l} |1\rangle_a|0\rangle_b \\ |0\rangle_a|1\rangle_b \end{array}$$

after 2nd B.S.: $|\psi_{out}\rangle = \hat{U}_{BS}^{+}|\psi_2\rangle = \frac{1}{2}\left[\left(1+e^{i\phi}\right)|1\rangle_a|0\rangle_b - \left(1-e^{i\phi}\right)|0\rangle_a|1\rangle_b\right]$

$$\hat{U}_{BS}^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

- Probability of obtaining a photon at the output port *a*

$$P_a = \langle\psi_{out}|\hat{n}_a|\psi_{out}\rangle = \frac{1}{4}\left|1+e^{i\phi}\right|^2 = \frac{1}{2}\left(1+\cos\phi\right)$$

$$\hat{n}_a \equiv \sum_n n|n\rangle_{a\,a}\langle n| \otimes \hat{I}_b$$

$\boxed{\text{self-adjoint operator}}$ $\hat{A} = \hat{A}^{+} \Rightarrow$ real eigenvalue

Hermitian operator represents a dynamical variable (observable).

- Probability of obtaining a photon at the output port *b*

$$P_b = \langle\psi_{out}|\hat{n}_b|\psi_{out}\rangle = \frac{1}{4}\left|1-e^{i\phi}\right|^2 = \frac{1}{2}\left(1-\cos\phi\right)$$

$$\frac{1}{2}\left[\left(1+e^{-i\phi}\right)_b\langle 0|_a\langle 1| - \left(1-e^{-i\phi}\right)_b\langle 1|_a\langle 0|\right] \Leftarrow \boxed{\text{bra bector}}$$

$$\langle\psi_{out}| = \left(|\psi_{out}\rangle\right)^{+}$$

$$\hat{n}_b \equiv \hat{I}_a \otimes \sum_n n|n\rangle_{b\,b}\langle n|$$

$\boxed{\text{Probability interpretation of QM}}$

$$P_a = \left|{}_a\langle n|_b\langle 0|\psi_{out}\rangle\right|^2$$

$$P_b = \left|{}_a\langle 0|_b\langle n|\psi_{out}\rangle\right|^2$$

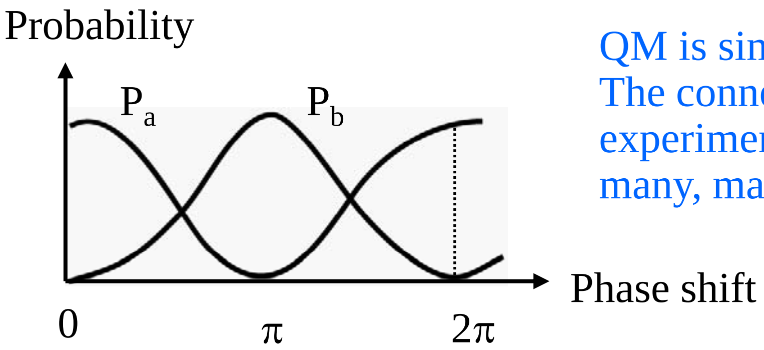
Schrödinger wavefunction

⇩

c-number

⇩

carries the amplitude & phase information simultaneously



QM is simply silent for an single event. The connection between the theory & experiment is only via statistics of many, many measurement events.

What interfere with each other are the two probability amplitudes of the linear superposition state, $|1\rangle_a|0\rangle_b$ and $|0\rangle_a|1\rangle_b$.

$\Downarrow$

Origin of interference is the lack of information for which path a photon takes before it is detected.

One photon interference does not distinguish a quantum picture from a classical picture based on "two real physical waves."

$\Downarrow$

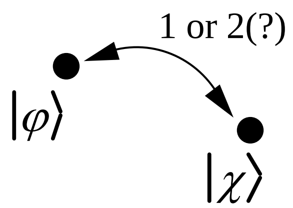
In order to see a truly quantum mechanical interference effect for which a classical picture fails, we need to study a multi-photon interference effect.

1.2 Symmetrization postulate and quantum indistinguishability

C. Cohen-Tannoudji et al., Quantum Mechanics (John Wiley and Sons, New York, 1977)

1.2.1 statement of the postulate

The physical state of a system including several identical quantum particles are completely symmetric or anti-symmetric with respect to permutation of these particles.



$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}\left[|\varphi\rangle_1|\chi\rangle_2 + \varepsilon|\chi\rangle_1|\varphi\rangle_2\right]$$

$$\varepsilon = \begin{cases} +1 & \text{: boson} \\ -1 & \text{: fermion} \end{cases}$$

Identical quantum particles are indistinguishable.

if $\langle\varphi|\chi\rangle = 0$ (orthogonal)

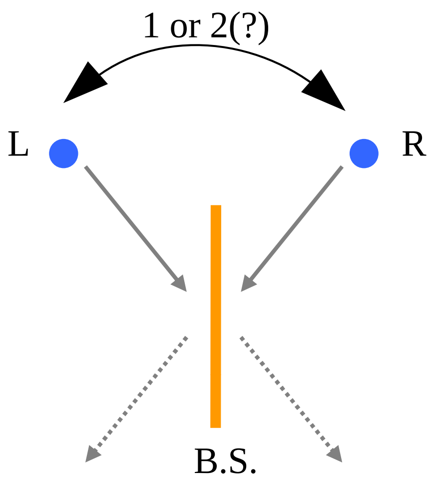
cf. $c_1|\varphi\rangle_1|\chi\rangle_2 + c_2|\chi\rangle_1|\varphi\rangle_2$

$$\left(|c_1|^2 + |c_2|^2 = 1\right)$$

does not correspond to a physical state even though it is a mathematically legal state.

1.2.2 Collision of two identical quantum particles

Spinless particles

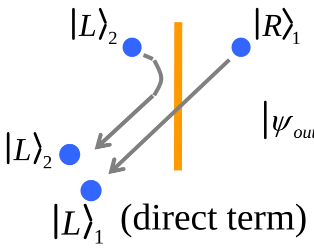


$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\left[|R\rangle_1|L\rangle_2 + \varepsilon|L\rangle_1|R\rangle_2\right]$$

$$\hat{U}_{BS} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

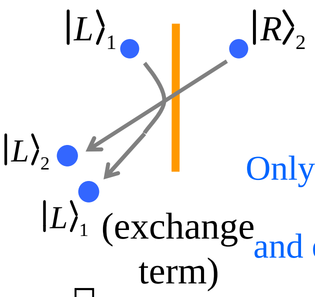
$$|R\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$$|L\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

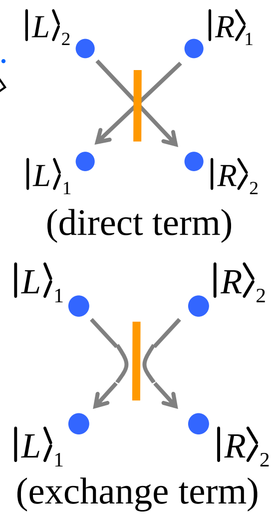


$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left\{-\left(\frac{1+\varepsilon}{2}\right)|R\rangle_1|R\rangle_2 + \left(\frac{1+\varepsilon}{2}\right)|L\rangle_1|L\rangle_2 + \left(\frac{1-\varepsilon}{2}\right)|R\rangle_1|L\rangle_2 - \left(\frac{1-\varepsilon}{2}\right)|L\rangle_1|R\rangle_2\right\}$$

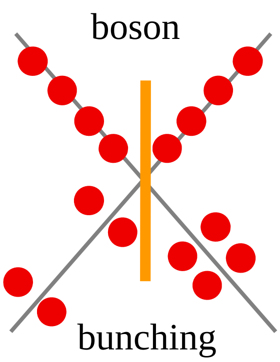
$$= \begin{cases}\frac{1}{\sqrt{2}}\left[|L\rangle_1|L\rangle_2 - |R\rangle_1|R\rangle_2\right] & : \text{boson}\\ \frac{1}{\sqrt{2}}\left[|R\rangle_1|L\rangle_2 - |L\rangle_1|R\rangle_2\right] & : \text{fermion}\end{cases}$$

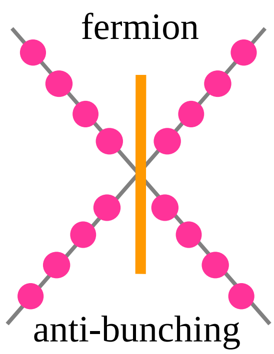


Only states with constructive interference between direct (ex. $|R\rangle_1|L\rangle_2 \rightarrow |L\rangle_1|L\rangle_2$) and exchange (ex. $|L\rangle_1|R\rangle_2 \rightarrow |L\rangle_1|L\rangle_2$) terms are selected out.



output particle flux





final state stimulation
Bose condensation,
superconductivity

Pauli exclusion principle

1.2.3 Collision of two non-identical particles

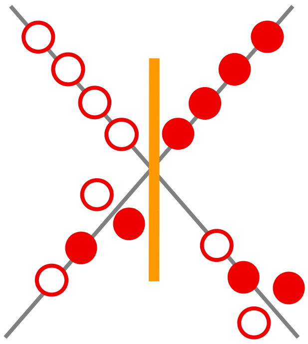
Particle 1 is a muon μ^-.
Particle 2 is an electron e^-.

The detector is only sensitive to the charge of the particles, giving no information about their masses.

initial state $\left|\psi_{in}\right\rangle = |R\rangle_1 |L\rangle_2$

final state $\left|\psi_f\right\rangle = \frac{1}{\sqrt{2}}\left(|R\rangle_1 + |L\rangle_1\right) \otimes \frac{1}{\sqrt{2}}\left(-|R\rangle_2 + |L\rangle_2\right)$

independent splitting (no interference)



$|R\rangle_1 |R\rangle_2$ (2, 0) 25%

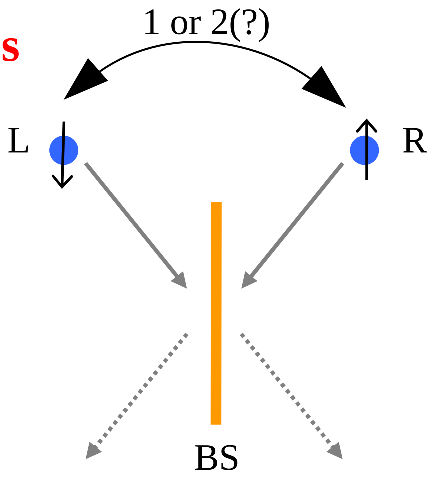
$|L\rangle_1 |L\rangle_2$ (0, 2) 25%

$|R\rangle_1 |L\rangle_2$ and $|L\rangle_1 |R\rangle_2$ (1, 1) 50%

Quantum interference disappears even if the actual detector cannot distinguish the two particles. The "theoretical possibility" of distinguishing the particle 1 and particle 2 is enough to eliminate the quantum interference effect.

1.2.4 Collision of two identical particles with spins (EPR-Bell state)

A. Spin singlet state



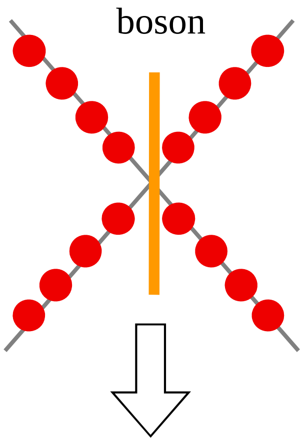
Spin part of the wavefunction (internal DOF)

$$\frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right] \equiv |\psi^-\rangle$$

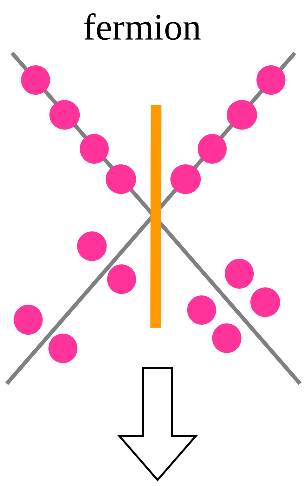
symmetrization postulate

$$|\psi_{12}\rangle = \begin{cases} \underbrace{\underbrace{\frac{1}{2}\left[|R\rangle_1|L\rangle_2 - |L\rangle_1|R\rangle_2\right]}_{\text{anti-symmetric orbital wavefunction}} \otimes \underbrace{\left[|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right]}_{\text{anti-symmetric spin wavefunction}}}_{\text{symmetric overall wavefunction: boson}} \\ \underbrace{\frac{1}{2}\left[|R\rangle_1|L\rangle_2 + |L\rangle_1|R\rangle_2\right] \otimes \left[|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right]}_{\text{anti-symmetric overall wavefunction: fermion}} \end{cases}$$

output particle flux



fermionic collision
(anti-bunching)



bosonic collision
(bunching)

B. Spin triplet states

spin part of the wavefunction (internal DOF) $\Rightarrow$

$$\frac{1}{\sqrt{2}}\Big[|\uparrow\rangle_1|\downarrow\rangle_2+|\downarrow\rangle_1|\uparrow\rangle_2\Big]\equiv|\psi^+\rangle$$

$$\frac{1}{\sqrt{2}}\Big[|\uparrow\rangle_1|\uparrow\rangle_2+|\downarrow\rangle_1|\downarrow\rangle_2\Big]\equiv|\phi^+\rangle$$

$$\frac{1}{\sqrt{2}}\Big[|\uparrow\rangle_1|\uparrow\rangle_2-|\downarrow\rangle_1|\downarrow\rangle_2\Big]\equiv|\phi^-\rangle$$

$\Downarrow$ symmetrization postulate

$$|\psi_{12}\rangle=\begin{cases}\underbrace{\underbrace{\frac{1}{2}\Big[|R\rangle_1|L\rangle_2+|L\rangle_1|R\rangle_2\Big]}_{\text{symmetric orbital wavefunction}}\otimes\underbrace{\begin{matrix}\Big[|\uparrow\rangle_1|\downarrow\rangle_2+|\downarrow\rangle_1|\uparrow\rangle_2\Big]\\ \Big[|\uparrow\rangle_1|\uparrow\rangle_2+|\downarrow\rangle_1|\downarrow\rangle_2\Big]\\ \Big[|\uparrow\rangle_1|\uparrow\rangle_2-|\downarrow\rangle_1|\downarrow\rangle_2\Big]\end{matrix}}_{\text{symmetric spin wavefunction}}}_{\text{symmetric overall wavefunction: boson}}\\[2ex] \underbrace{\underbrace{\frac{1}{2}\Big[|R\rangle_1|L\rangle_2-|L\rangle_1|R\rangle_2\Big]}_{\text{anti-symmetric orbital wavefunction}}\otimes\underbrace{\begin{matrix}\Big[|\uparrow\rangle_1|\downarrow\rangle_2+|\downarrow\rangle_1|\uparrow\rangle_2\Big]\\ \Big[|\uparrow\rangle_1|\uparrow\rangle_2+|\downarrow\rangle_1|\downarrow\rangle_2\Big]\\ \Big[|\uparrow\rangle_1|\uparrow\rangle_2-|\downarrow\rangle_1|\downarrow\rangle_2\Big]\end{matrix}}_{\text{symmetric spin wavefunction}}}_{\text{anti-symmetric overall wavefunction: fermion}}\end{cases}$$

$\boxed{\text{output particle flux is identical to that of spinless particles.}}$

boson $\Rightarrow$ bosonic collision

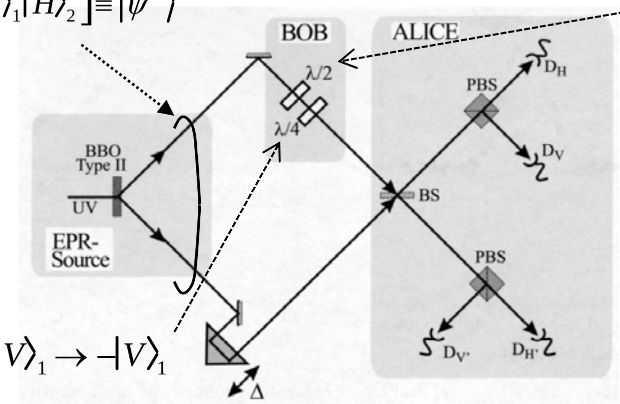
fermion $\Rightarrow$ fermion collision

1.2.5 Bell state analysis

A. Linear optics Bell state analyzer

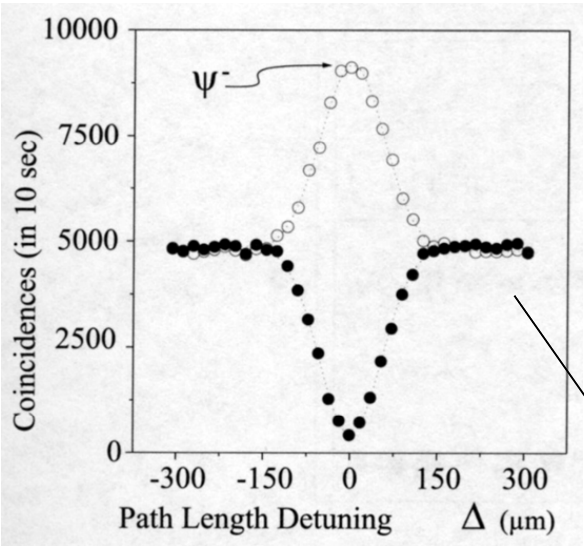
$\frac{1}{\sqrt{2}}\left[|H\rangle_1|V\rangle_2 + |V\rangle_1|H\rangle_2\right] \equiv |\psi^+\rangle$

$|H\rangle_1 \to |V\rangle_1$
$|V\rangle_1 \to |H\rangle_1$



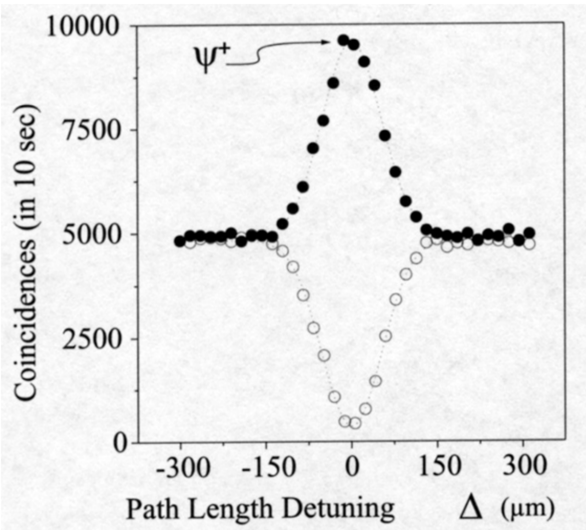
$|V\rangle_1 \to -|V\rangle_1$

Experimental set-up for collision of two identical particles



Coincidence rates C_{HV} (●) and $C_{HV'}$ (○) depending on the path length difference Δ, for transmission of the state $|\psi^-\rangle$
The constructive interference for the rate $C_{HV'}$ enables one to read the information associated with that state (bosonic singlet).

Symmetrization/anti-symmetrization is not required if the two wavepackets do not overlap. ⇨ “distinguishable from detection time”

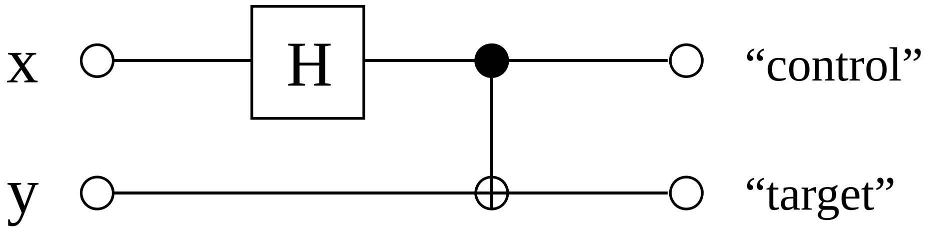


Coincidence rates C_{HV} (●) and $C_{HV'}$ (○) as functions of the path length difference Δ when the state $|\psi^+\rangle$ is transmitted.
For perfect timing (Δ=0), constructive interference occurs for C_{HV}, allowing identification of the state sent (bosonic triplet).

⇩

Linear optics EPR-Bell state analyzer (cannot distinguish ϕ^+ and ϕ^- states)

B. Full Bell State Analyzer (nonlinear quantum circuit)



Hadamard gate $H \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad |0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\quad |1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Controlled-NOT gate $U_{CNOT} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\quad |00\rangle \to |00\rangle$, $|01\rangle \to |01\rangle$, $|10\rangle \to |11\rangle$, $|11\rangle \to |10\rangle$

IN (out)	OUT (in)
$\|00\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle) \equiv \|\phi^+\rangle$
$\|01\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle) \equiv \|\psi^+\rangle$
$\|10\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle) \equiv \|\phi^-\rangle$
$\|11\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle) \equiv \|\psi^-\rangle$