

Chapter 4

Macroscopic Conductors

There are two types of intrinsic noise in every physical system: thermal noise and quantum noise. These two types of noise cannot be eliminated even when a device or system is perfectly constructed and operated. Thermal noise is a dominant noise source at high temperatures and/or low frequencies, while quantum noise is dominant at low temperatures and/or high frequencies. A conductor with a finite electrical resistance is a simplest system which manifests these two types of intrinsic noise. The intrinsic noise of a macroscopic conductor will be discussed in this chapter, and that of a mesoscopic conductor will be discussed in the next chapter.

A conductor in thermal equilibrium with its surroundings (heat reservoir) shows, at its terminals, an open-circuit voltage or short-circuit current fluctuation, as shown in Fig. 4.1. Thermal equilibrium noise was first experimentally discovered by J. B. Johnson in 1927[1].

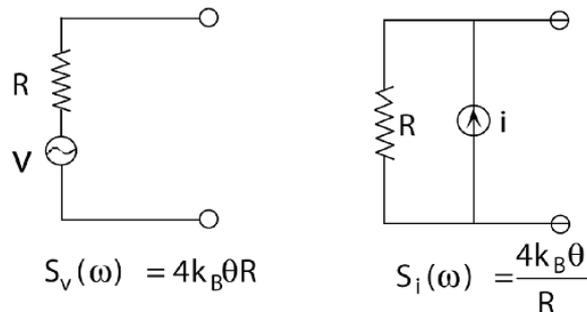


Figure 4.1: Open circuit voltage noise source and short circuit current noise source.

He discovered that the open circuit voltage noise power spectral density is independent of the material a conductor is made of and the measurement frequency, and is determined only by the temperature and electrical resistance: $S_v(\omega) = 4k_B\theta R$. The corresponding short-circuit current noise spectral density is $S_i(\omega) = 4k_B\theta/R$. This noise is referred to as thermal noise and is the most fundamental noise.

The physical origin of the thermal noise in a macroscopic conductor is a “random-walk” of thermally-fluctuated charged carriers (electrons, holes or ions). An electron in a metallic

conductor undergoes a Brownian motion via collisions with the lattices of a conductor. The fundamental properties of a Brownian particle were first studied by A. Einstein[2] and then elegantly formulated by M. P. Langevin[3] twenty years before Johnson's observation of thermal noise. The electrons in a conductor are thermally energetic via collisions with the lattice and travel randomly. The electron velocity fluctuation is a statistically-stationary process. However, the mean-square displacement of an electron increases in proportion to the observation time. The electron position fluctuation is a statistically-nonstationary process. Such a microscopic approach can indeed explain Johnson's observation.

Nyquist employed a completely different approach to the problem. He introduced the concept of "electromagnetic field modes" as a degree-of-freedom (DOF) of the system by assuming a transmission line terminated by two conductors. He then applied the equipartition theorem of statistical mechanics to the transmission line modes. In this way he could explain Johnson's observation without going into the specific details of a microscopic electron transport process. Nyquist's approach is very general and is easily extended to include quantum noise[4]. The microscopic theory of quantum noise in a macroscopic conductor was established later by Callen and Welton[6].

Johnson-Nyquist thermal noise is the intrinsic property of a conductor at thermal equilibrium, when there is no applied voltage and no net current (energy flow) in the system. However, the Johnson-Nyquist thermal noise formula is experimentally known to be valid even when there is a finite current flow across the conductor. This puzzling fact has been fully understood only recently by the noise study of a mesoscopic system. The current noise in a mesoscopic conductor with a finite dc current is often dominated by shot noise, while shot noise is absent in a macroscopic conductor. This interesting issue will be discussed in the next chapter in the context of a transition from mesoscopic conductor to macroscopic conductor.

4.1 Brownian Particle Model of Thermal Noise

4.1.1 Mean Free-Time and Mobility

Consider a one-dimensional conductor under an applied dc voltage. We assume that an electron is accelerated by the uniform electric field between collisions with the lattice and that an electron velocity returns to zero at every collision event. This is not true in a real collision process in a conductor, but the conclusion one obtains using this assumption is essentially the same as that obtained by more realistic collision models. Since the electron drift velocity is given by $u(t) = \alpha t = \frac{qE}{m}t$ during the time between collisions τ_f , the displacement between two collisions is

$$x(\tau_f) = \frac{\alpha}{2}\tau_f^2 = \frac{qE}{2m}\tau_f^2 \quad . \quad (4.1)$$

After K collisions with the lattice, the total displacement is $\frac{qE}{2m}\overline{\tau_f^2}K$. In this chapter we use a bar to indicate an ensemble average instead of a time average. Therefore, the mean drift velocity \bar{u} is given by

$$\bar{u} \equiv \frac{\text{total displacement}}{\text{total time}} = \frac{\left(\frac{qE}{2m}K\right)\overline{\tau_f^2}}{K\overline{\tau_f}} = \frac{q\overline{\tau_f^2}}{2m\overline{\tau_f}}E \quad . \quad (4.2)$$

Here, $\bar{\tau}_f$ and $\overline{\tau_f^2}$ are the mean free time and mean-square free time. The mobility μ is defined by $\bar{u} = \mu E$,

$$\mu = \frac{q\overline{\tau_f^2}}{2m\bar{\tau}_f} . \quad (4.3)$$

τ_f is randomly distributed with a mean value $\bar{\tau}_f$. The probability $p_i(m, \tau)$ that the i^{th} electron experiences exactly m collisions in a time interval $[0, \tau]$ obeys a Poisson distribution if the probability of electron collision with the lattice is independent of the electron drift velocity, which is a reasonable assumption for a weak dc field. Strictly speaking, the electron collision with a lattice obeys a Bernoulli process with a very small collision (success) probability, which is well approximated by a Poisson distribution, as discussed in Chapter 1. Thus, one obtains

$$p_i(m, \tau) = \frac{(\nu_i \tau)^m}{m!} e^{-\nu_i \tau} , \quad (4.4)$$

where ν_i is the mean rate for collision per second. The probability $q(\tau_{fi})d\tau_{fi}$ for a free time τ_{fi} lying between τ_{fi} and $\tau_{fi} + d\tau_{fi}$ is equal to the joint probability of zero collisions in a time interval $[0, \tau_{fi}]$ and one collision in a time interval $[\tau_{fi}, \tau_{fi} + d\tau_{fi}]$. Thus,

$$\begin{aligned} q(\tau_{fi})d\tau_{fi} &= p_i(0, \tau_{fi}) \times p_i(1, d\tau_{fi}) \\ &= \nu_i e^{-\nu_i \tau_{fi}} d\tau_{fi} , \end{aligned} \quad (4.5)$$

where $e^{-\nu_i d\tau_{fi}} \simeq 1$ is assumed and $\nu_i e^{-\nu_i \tau_{fi}}$ is considered a probability density function by which we can calculate the mean free time and mean-square free time,

$$\bar{\tau}_{fi} = \nu_i \int_0^\infty \tau_{fi} e^{-\nu_i \tau_{fi}} d\tau_{fi} = \frac{1}{\nu_i} , \quad (4.6)$$

$$\overline{\tau_{fi}^2} = \nu_i \int_0^\infty \tau_{fi}^2 e^{-\nu_i \tau_{fi}} d\tau_{fi} = \frac{2}{\nu_i^2} = 2\bar{\tau}_{fi}^2 . \quad (4.7)$$

As seen from (4.6) and (4.7), τ_{fi} does not obey a Poisson distribution, but rather it obeys a geometrical distribution. Figure 4.2 compares $p_i(m, \tau)$ and $q_i(\tau_{fi})$. If N electrons behave in a similar but independent way, the addition theorem of a Poisson process (see Chapter 1) is applied and the total probability of collision $p(m, \tau)$ still obeys a Poisson distribution

with the mean rate $\nu = \sum_{i=1}^N \nu_i = \frac{N}{\bar{\tau}_f}$. The (collective) mean free time $\bar{\tau}_f$ is defined by

$$\bar{\tau}_f = \frac{N}{\nu} .$$

Using (4.7) in (4.3), the mobility is uniquely related to the mean free time,

$$\mu = \frac{q\bar{\tau}_f}{m} . \quad (4.8)$$

According to this collision model, any departure of the drift velocity $\Delta u(t) = u(t) - \bar{u}$ from its mean value decays with a time constant $\bar{\tau}_f = \frac{m\mu}{q}$.

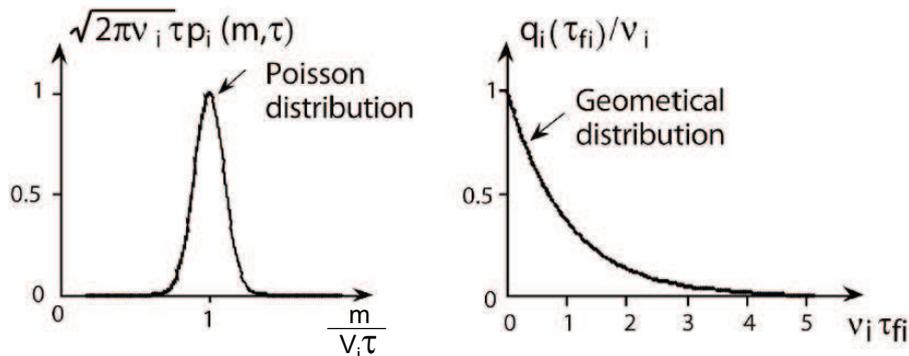


Figure 4.2: The two distribution functions $p_i(m, \tau)$ and $q_i(\tau_{fi})$.

4.1.2 Langevin Equation for a Brownian Motion

A sufficiently small particle immersed in a liquid exhibits a random motion. This phenomenon is called “Brownian motion” and reveals very clearly the statistical fluctuations which occur in a system in thermal equilibrium.

There are a variety of important situations which are basically described by the Brownian motion. Examples are the random motion of the mirror mounted on the suspension fiber in a sensitive galvanometer and gravitational wave detection laser interferometer, and the random motions of charged particle carriers in an electric resistor. Thus Brownian motion can serve as a prototype problem whose analysis provides considerable insight into the mechanisms responsible for the existence of fluctuations associated with the dissipation of energy. This problem is also of great practical interest because such fluctuations imposes limitations on the possible accuracy of delicate physical measurements.

For the sake of simplicity we shall treat the problem of Brownian motion in *one* dimension. We consider a particle of mass m whose center-of-mass coordinate at time t is designated by $x(t)$ and whose corresponding velocity is $v \equiv dx/dt$. This particle is immersed in a heat reservoir at the absolute temperature θ . It would be a hopelessly complex task to describe in detail the interaction of the center-of-mass coordinate x with all the many degrees of freedom of the heat reservoir (for instance, those describing the motions of the lattice of a resistor). These other degrees of freedom can be regarded as constituting a heat reservoir at some equilibrium temperature θ , and their interaction with x can be lumped into some net force $F(t)$ effective in determining the time dependence of x . In addition, the particle may also interact with some external systems, such as dc electric field, gravity or electromagnetic field, through an external force denoted by $\mathcal{F}(t)$. The velocity v of the particle may, in general, be appreciably different from its mean value in equilibrium.

Focusing attention on the center-of-mass coordinate x , Newton’s second law of motion can then be written in the form

$$m \frac{dv}{dt} = \mathcal{F}(t) + F(t) \quad . \quad (4.9)$$

Here very little is known about the force $F(t)$ which describes the interaction of the system with the many degrees of freedom of the reservoir. Basically, $F(t)$ must depend on the positions of very many atoms which are in constant motion. Thus $F(t)$ is some rapidly fluctuating function of the time t and varies in a highly irregular fashion. Indeed, one cannot specify the precise functional dependence of F on t . To make progress, one has to formulate the problem in statistical terms. One must, therefore, envisage an ensemble of very many similarly prepared systems, each of them consisting of a particle and the surrounding heat reservoir. For each of these, the force $F(t)$ is some random function of t . One can then attempt to make statistical statements about this ensemble.

The rate at which $F(t)$ varies can be characterized by some ‘‘correlation time’’ τ^* which measures roughly the mean time between two successive maxima (or minima) of the fluctuation function $F(t)$. This time τ^* is quite small on a *macroscopic* scale. (It ought to be roughly of the order of a mean intermolecular separation divided by a mean molecular velocity, e.g., about 10^{-13} sec if $F(t)$ describes interactions with molecules of a typical liquid.) Furthermore, if one contemplates a situation where the particle is not drifted to a certain direction, there is no preferred direction in space; then $F(t)$ must be as often positive as negative so that the ensemble average $\bar{F}(t)$ vanishes.

Equation (4.9) holds for each member of the ensemble, and our aim is to deduce from it statistical statements about v . Since $F(t)$ is a rapidly fluctuating function of time, it follows that v also fluctuates in time. Moreover, superimposed upon these fluctuations, the time dependence of v may also exhibit a more slowly varying trend. For example, one can focus attention on the ensemble average \bar{v} of the velocity, which is a much more slowly varying function of the time than v itself, and write

$$v = \bar{v} + v' \quad , \quad (4.10)$$

where v' denotes the part of v which fluctuates rapidly [although less rapidly than $F(t)$, since the mass m is appreciable] and whose mean value vanishes. The slowly varying part \bar{v} is of crucial importance (even if it is small) because it is of primary significance in determining the behavior of the particle over long periods of time. To investigate its time dependence, let us integrate (4.9) over some time interval τ which is small on a *macroscopic* scale, but large in the sense that $\tau \gg \tau^*$. Then one gets

$$m[v(t + \tau) - v(t)] = \mathcal{F}(t)\tau + \int_t^{t+\tau} F(t')dt' \quad , \quad (4.11)$$

where we have assumed that the external force \mathcal{F} is varying slowly enough that it changes by a negligible amount during a time τ . The last integral in (4.11) ought to be very small since $F(t)$ changes sign many times in the time τ . Hence one might expect that any slowly varying part of v should be due only to the external force \mathcal{F} :

$$m \frac{d\bar{v}}{dt} = \mathcal{F} \quad . \quad (4.12)$$

However, this order of approximation is too crude to describe the physical situation. Indeed, the interaction with the environment expressed by $F(t)$ must be such that it always tends to restore the particle to the equilibrium situation. Suppose, for example, that the external force $\mathcal{F} = 0$. The interaction expressed by F must then be such that,

if $\bar{v} \neq 0$ at some initial time, it causes \bar{v} to approach gradually its ultimate equilibrium value $\bar{v} = 0$. Equation (4.12) fails to predict this kind of trend of \bar{v} toward its equilibrium value. We did not consider the fact that the interaction force F must actually be affected by the motion of the particle in such a way that F itself also contains a slowly varying part \bar{F} tending to restore the particle to equilibrium. Hence we shall write, analogously to (4.10)

$$F = \bar{F} + F' \quad , \quad (4.13)$$

where F' is the rapidly fluctuating part of F whose average value vanishes. The slowly varying part \bar{F} must be some function of \bar{v} which is such that $\bar{F}(\bar{v}) = 0$ in equilibrium when $\bar{v} = 0$. If \bar{v} is not too large, $\bar{F}(\bar{v})$ can be expanded in a power series in \bar{v} whose first nonvanishing term must then be linear in \bar{v} . Thus \bar{F} must have the general form

$$\bar{F} = -\alpha\bar{v} \quad , \quad (4.14)$$

where α is some positive constant (called the “friction constant”) and the minus sign indicates explicitly that the force \bar{F} acts in such a direction that it tends to reduce \bar{v} to zero as time increases. We can surmise that α must in some way be expressible in terms of F itself, since the frictional restoring force is also caused by the interactions described by $F(t)$.

In the general case the slowly varying part of (4.9) becomes then

$$m \frac{d\bar{v}}{dt} = \mathcal{F} + \bar{F} = \mathcal{F} - \alpha\bar{v} \quad . \quad (4.15)$$

If one includes the rapidly fluctuating parts v' and F' of (4.10) and (4.13), (4.9) can be written

$$m \frac{dv}{dt} = \mathcal{F} - \alpha v + F'(t) \quad , \quad (4.16)$$

where we have put $\alpha\bar{v} \approx \alpha v$ with negligible error [since the rapidly fluctuating contribution $\alpha v'$ can be neglected compared to the predominant fluctuating term $F'(t)$]. Equation (4.16) is called the “Langevin equation.” It differs from the original (4.9) by explicitly decomposing the force $F(t)$ into a slowly varying part $-\alpha v$ and into a rapidly fluctuating part $F'(t)$ which is “purely random,” i.e., such that its mean value always vanishes irrespective of the velocity or position of the particle. The Langevin equation (4.16) describes in this way the behavior of the particle at all later times if its initial conditions are specified.

Since the Langevin equation contains the frictional force $-\alpha v$, it implies the existence of processes whereby the energy associated with the coordinate x of the particle is dissipated in the course of time to the other degrees of freedom (e.g., to the molecules of the liquid surrounding the particle). Consider a system A in contact with some large system B . The *microscopic* equations governing the motion of the combined system ($A+B$) do *not* involve any frictional forces. The total energy is conserved, and the motion is reversible. (That is, if the sign of the time t were reversed, the equations of motion would be essentially unchanged and all particles would retrace their paths in time.) But if one focuses attention on A , its interaction with the heat reservoir B can be adequately described by equations of motion involving frictional forces. There is thus dissipation of energy from A to the heat reservoir B and the motion of A is not reversible. The question is to understand in detail

how this situation comes about, what conditions must be satisfied for this description to be approximately valid, and how the modified equations of motion for A are derivable from the microscopic equations. The interested reader may consult with excellent texts on statistical mechanics[6], [7].

Our interest is the problem of an electrical conductor with self-inductance L carrying a current I . Let the applied voltage be $\mathcal{V}(t)$. The current I carried by the electrons is affected by the interactions with the lattice of the conductor. The net effect of these interactions on the current I can be represented by some effective fluctuating voltage $V(t)$. The latter can be decomposed into a slowly varying part $-RI$ (where R is some positive constant) and into a rapidly fluctuating part $V'(t)$ whose mean value vanishes. The analogue of the Langevin equation (4.16) then becomes

$$L \frac{dI}{dt} = \mathcal{V} - RI + V'(t) \quad . \quad (4.17)$$

The friction constant R is here simply the electrical resistance of the conductor.

4.1.3 Mean-Square Displacement

Let us assume the validity of Langevin equation as an adequate phenomenological description of Brownian motion and illustrate how it can be applied to the calculation of quantities of physical interest. In the absence of external forces (4.16) becomes

$$m \frac{dv}{dt} = -\alpha v + F'(t) \quad . \quad (4.18)$$

Consider the situation of thermal equilibrium. Clearly the mean displacement \bar{x} of the particle vanishes (i.e., $\bar{x} = 0$) by symmetry, since there is no preferred direction in space. To calculate the magnitude of the fluctuations, we now use (4.18) to calculate the mean-square displacement $\overline{x^2}$ of the particle in a time interval t . Equation (4.18) contains the quantities $v = \dot{x}$ and $dv/dt = d\dot{x}/dt$. Multiplying both sides of (4.18) by x , one thus gets

$$mx \frac{d\dot{x}}{dt} = m \left[\frac{d}{dt}(x\dot{x}) - \dot{x}^2 \right] = -\alpha x\dot{x} + xF'(t) \quad . \quad (4.19)$$

One can now take the ensemble average of both sides of (4.19). As pointed out in connection with the Langevin (4.16), the mean value of the fluctuating force F' always vanishes, irrespective of the value of v or x . Hence $\overline{xF'} = \overline{x\dot{x}'} = 0$. Furthermore, the equipartition theorem yields $\frac{1}{2}m\overline{\dot{x}^2} = \frac{1}{2}k_B\theta$. Thus (4.19) becomes

$$m \overline{\frac{d}{dt}x\dot{x}} = m \overline{\frac{d}{dt}x\dot{x}} = k_B\theta - \alpha \overline{x\dot{x}} \quad . \quad (4.20)$$

The relation (4.20) is a simple differential equation which can immediately be solved for the quantity $\overline{x\dot{x}} = \frac{1}{2}(d\overline{x^2}/dt)$. Thus one obtains

$$\overline{x\dot{x}} = Ce^{-\gamma t} + \frac{k_B\theta}{\alpha} \quad , \quad (4.21)$$

where C is a constant of integration and

$$\gamma \equiv \frac{\alpha}{m} \quad , \quad (4.22)$$

so that γ^{-1} denotes a characteristic time constant of the system. Assuming that each particle in the ensemble starts out at $t = 0$ at the position $x = 0$, so that x measures the displacement from the initial position. The constant C in (4.21) must be such that $0 = C + k_B\theta/\alpha$. Hence (4.21) becomes

$$\overline{x\dot{x}} = \frac{1}{2} \frac{d}{dt} \overline{x^2} = \frac{k_B\theta}{\alpha} (1 - e^{-\gamma t}) \quad . \quad (4.23)$$

Integrating once more one obtains the final result

$$\overline{x^2} = \frac{2k_B\theta}{\alpha} [t - \gamma^{-1}(1 - e^{-\gamma t})] \quad . \quad (4.24)$$

Note two interesting limiting cases. If $t \ll \gamma^{-1}$, then

$$e^{-\gamma t} = 1 - \gamma t + \frac{1}{2}\gamma^2 t^2 - \dots \quad .$$

Thus, for $t \ll \gamma^{-1}$,

$$\overline{x^2} = \frac{k_B\theta}{m} t^2 \quad . \quad (4.25)$$

The particle behaves during a short initial time interval as though it were a free particle moving with the constant thermal velocity $v = (k_B\theta/m)^{1/2}$.

On the other hand, if $t \gg \gamma^{-1}$, $e^{-\gamma t} \rightarrow 0$. Thus (4.24) becomes simply

$$\overline{x^2} = \frac{2k_B\theta}{\alpha} t \quad . \quad (4.26)$$

The particle then behaves like a diffusing particle executing a random walk. Indeed, since the diffusion equation leads to the relation $\overline{x^2} = 2Dt$, comparison with (4.26) shows the corresponding diffusion coefficient to be given by

$$D = \frac{k_B\theta}{\alpha} \quad . \quad (4.27)$$

4.1.4 Einstein relation

If the particle carries an electric charge q and is placed in a uniform electric field \mathcal{E} , the Langevin equation (4.16) becomes

$$m \frac{dv}{dt} = q\mathcal{E} - \alpha v + F'(t) \quad . \quad (4.28)$$

Taking the mean values of both sides and considering the steady-state situation where $d\bar{v}/dt = 0$ this yields

$$e\mathcal{E} - \alpha\bar{v} = 0 \quad . \quad (4.29)$$

This shows that $\bar{v} \propto \mathcal{E}$. The ‘‘mobility’’ $\mu \equiv \bar{v}/\mathcal{E}$ is then given by

$$\mu \equiv \frac{\bar{v}}{\mathcal{E}} = \frac{q}{\alpha} \quad . \quad (4.30)$$

Thus the mobility μ and the diffusion coefficient D in (4.27) are both expressible in terms of α . There exists therefore an intimate connection between these two coefficients, namely,

$$\frac{\mu}{D} = \frac{q}{k_B\theta} \quad . \quad (4.31)$$

This is known as the ‘‘Einstein relation’’ between the mobility and the diffusion coefficient.

4.1.5 Particle Velocity and Current Fluctuations

In a thermal equilibrium condition (zero applied voltage), the mean drift velocity of an electron is zero, $\bar{v} = 0$. However, an electron acquires a non-zero momentum when it collides with a lattice which decays toward the mean value of $\bar{v} = 0$ with a time constant $\bar{\tau}_f = \frac{m}{\alpha}$. The equation of motion is then

$$\frac{dv(t)}{dt} = -\frac{v(t)}{\bar{\tau}_f} + \frac{F'(t)}{m} \quad . \quad (4.32)$$

The drift velocity $v(t)$ is kicked randomly by the Langevin noise source [second part of the RHS of (4.32)] and is simultaneously damped by the friction term [first part of the RHS of (4.32)]. The Fourier analysis of (4.32) requires the introduction of a gated function $v_T(t) = v(t)$ for $-\frac{T}{2} \leq t \leq \frac{T}{2}$ and 0 for otherwise, because $v(t)$ is a statistically-stationary process. [see Chapter 1 for the detailed discussion on the Fourier analysis of a statistically stationary process.] The Fourier transform of this gated function leads to

$$V_T(i\omega) = \frac{\frac{\mu}{q} F'_T(i\omega)}{1 + i\omega\bar{\tau}_f} \quad . \quad (4.33)$$

From the Carson theorem, the power spectral density of the velocity fluctuation is given by

$$S_v(\omega) = 2\nu \overline{|F_T(i\omega)|^2} \frac{(\mu/q)^2}{1 + \omega^2\bar{\tau}_f^2} \quad , \quad (4.34)$$

where $\nu = \frac{1}{\bar{\tau}_f}$ is the mean rate of collisions per second, and $\overline{|F_T(i\omega)|^2}$ is the mean square of the Fourier transformed Langevin noise force.

In order to determine $\overline{|F_T(i\omega)|^2}$, we need to invoke the equipartition theorem again. For a thermal equilibrium condition, the mean square of the electron momentum is determined by the equipartition theorem:

$$\begin{aligned} \frac{\overline{p^2}}{2m} &= \frac{m}{2} \int_0^\infty S_v(\omega) \frac{d\omega}{2\pi} \quad (\text{Parseval Theorem}) \\ &= \frac{1}{2} k_B \theta \quad (\text{one - dimensional case}) \quad . \end{aligned} \quad (4.35)$$

From (4.34) and (4.35), one obtains $\overline{|F_T(i\omega)|^2} = 2k_B\theta m$ and thus (4.34) becomes

$$S_v(\omega) = \frac{4k_B\theta\mu/q}{1 + \omega^2\bar{\tau}_f^2} \quad , \quad (4.36)$$

where (4.8) is used.

From the Wiener-Khintchine theorem [see Chapter 1 for detail], one can calculate the autocorrelation function

$$\begin{aligned} \phi_v(\tau) &= \frac{1}{2\pi} \int_0^\infty S_v(\omega) \cos \omega\tau \, d\omega \\ &= \frac{k_B\theta}{m} \exp\left(-\frac{|\tau|}{\bar{\tau}_f}\right) \quad . \end{aligned} \quad (4.37)$$

Therefore, the mean square value of $v(t)$ is given by $\phi_v(0) = \frac{k_B\theta}{m}$. Figures 4.3 (a) and (b) show the normalized $S_v(\omega)$ and $\phi_v(\tau)$.

According to the Ramo theorem introduced in Chapter 1, the velocity fluctuation $v(t)$ of the electron produces a short-circuited current fluctuation $i(t) = qv(t)/L$ in an external circuit, where L is the length of the conductor. Since the current fluctuation of each electron is additive, the total current noise power spectral density is given by

$$S_i(\omega) = S_v(\omega) \frac{q^2}{L^2} ALn \quad , \quad (4.38)$$

where A is a cross-sectional area of the conductor and n is the electron density. If one uses (4.36) and the expression for electrical resistance, $R = \rho \frac{L}{A} = \frac{L}{nq\mu A}$, where ρ is a resistivity, in (4.38), one obtains

$$S_i(\omega) = \frac{4k_B\theta/R}{1 + \omega^2\bar{\tau}_f^2} \quad . \quad (4.39)$$

Therefore, the short-circuited current fluctuation power spectral density at low frequency is equal to $4k_B\theta/R$. This is exactly what Johnson observed experimentally[1].

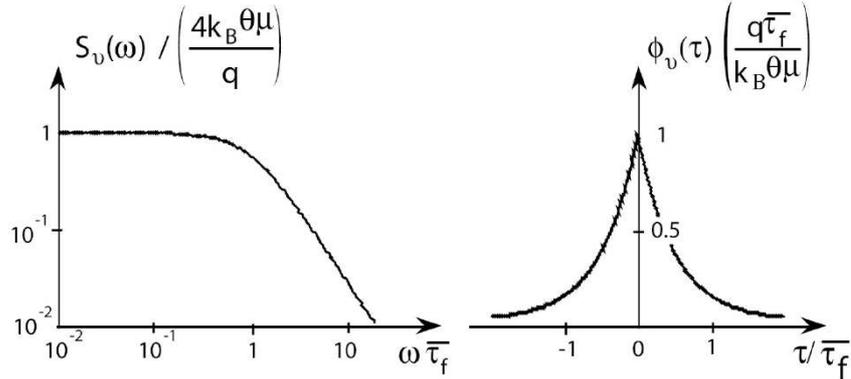


Figure 4.3: The power spectrum and the autocorrelation function of an electron velocity $v(t)$ in a resistor at thermal equilibrium.

4.1.6 Particle Position and Charge Fluctuations

The integral of a statistically-stationary process with a white power spectral density is a statistically non-stationary process and is called a Wiener-Levy process[8]. The surface charge induced at two terminal electrodes by the fluctuating short-circuited current across a conductor is

$$q(t) = \int_0^t i(t')dt' \quad . \quad (4.40)$$

Consider again the one-dimensional conductor. An electron transit over a free path ℓ_f between collisions gives rise to a surface charge on the electrode equal to $q \left(\frac{\ell_f}{L} \right)$. If there

are $m = \nu t$ independent electron transit events in a time duration t , then the mean-square value of the surface charge $q(t)$ is calculated as

$$\overline{q(t)^2} = q^2(\nu t) \frac{\overline{\ell_f^2}}{L^2} = \frac{2k_B\theta}{R} t \quad , \quad (4.41)$$

where $\nu = nAL/\bar{\tau}_f$ is the mean rate of electron collision with lattice, $\overline{\ell_f^2} = \overline{v_T^2} \overline{\tau_f^2} = \frac{k_B T}{m} \cdot 2\bar{\tau}_f^2 = 2\mu k_B\theta \bar{\tau}_f/q$ is the mean-square free path, and $\overline{v_T^2} = k_B\theta/m$ is the mean-square thermal velocity; therefore, the charge fluctuation is not stationary, but, rather, diffuses with time. The diffusion coefficient, defined by $\overline{q(t)^2} = 2D_q t$, is given by

$$D_q = \frac{k_B\theta}{R} = \frac{1}{4} S_i(\omega \approx 0) \quad . \quad (4.42)$$

Since $q(t)$ is a cumulative process, one can write

$$q(t + \tau) = q(t) + \Delta q(t, \tau) \quad , \quad (4.43)$$

where $\Delta q(t, \tau)$ is the charge fluctuation added between times t and $(t + \tau)$. Since the correlation time of the additive fluctuations is very short ($\bar{\tau}_f \ll \tau$), the two processes in the RHS of Eq. (4.43) are uncorrelated; therefore, the covariance function is equal to the mean square value:

$$\overline{q(t + \tau)q(t)} = \overline{q(t)^2} + \overline{q(t)\Delta q(t, \tau)} \simeq \overline{q(t)^2} \quad . \quad (4.44)$$

The autocorrelation function must be redefined in such a manner that the observation time remains finite rather than infinite and the ordinate is set to zero outside the observation time interval. Thus, the autocorrelation function is a function of the time delay τ and the observation time T :

$$\begin{aligned} \phi_q(\tau, T) &= \frac{1}{T} \int_0^{T-|\tau|} \overline{q(t + \tau)q(t)} dt \\ &= \frac{1}{T} \int_0^{T-|\tau|} \frac{2k_B\theta}{R} t dt \\ &= \frac{k_B\theta}{R} T \left(1 - \frac{|\tau|}{T}\right)^2 \quad . \end{aligned} \quad (4.45)$$

The power spectral density is obtained by the Wiener-Khintchine theorem as:

$$\begin{aligned} S_q(\omega, T) &= 4 \int_0^T \phi_q(\tau, T) \cos \omega\tau d\tau \\ &= \frac{8k_B\theta}{\omega^2 R} \left[1 - \frac{\sin(\omega T)}{\omega T}\right] \quad . \end{aligned} \quad (4.46)$$

When $\omega T \gg 1$, the power spectral density is proportional to ω^{-2} , which is the characteristic feature of a Wiener-Levy process. Figure 4.4 (a)–(c) show the real-time function, autocorrelation function, and power spectral density of the surface charge $q(t)$.

The integral of an electron velocity $v(t)$ is an electron displacement $x(t)$, which is also a Wiener-Levy process, as shown in the previous section. By performing the similar calculation described above, one obtains the (position) diffusion coefficient,

$$D_x = \frac{k_B\theta\bar{\tau}_f}{m} = \frac{k_B\theta\mu}{q} = \frac{1}{4} S_v(\omega \simeq 0) \quad . \quad (4.47)$$

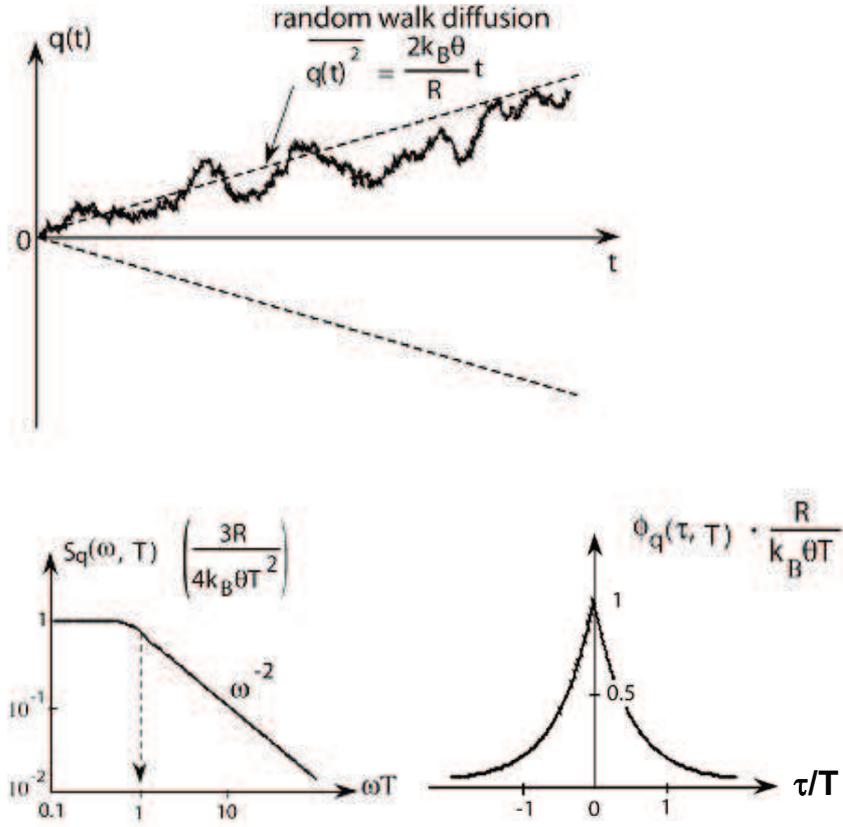


Figure 4.4: The real time (sample) function $q(t)$, the power spectrum $S_q(\omega, T)$, and the autocorrelation function $\phi_q(\tau, T)$ of a surface charge.

4.2 Fluctuation-Dissipation Theorem

If we assume $\varepsilon = 0$ (zero external force) in (4.27), the equation of motion for the particle velocity $v(t)$ is rewritten as

$$\frac{d}{dt}e^{\gamma t}v(t) = \frac{e^{\gamma t}}{m}F'(t) \quad . \quad (4.48)$$

The formal integral of (4.48) from $t = -\infty$ to $t = 0$ results in

$$[e^{\gamma t}v(t)]_{-\infty}^0 = v(0) = \frac{1}{m} \int_{-\infty}^0 e^{\gamma t}F'(t)dt \quad . \quad (4.49)$$

Therefore the mean square of $v(0)$ is given by

$$\langle v(0)^2 \rangle = \frac{1}{m^2} \int_{-\infty}^0 \int_{-\infty}^0 dt dt' e^{\gamma(t+t')} \langle F'(t)F'(t') \rangle \quad . \quad (4.50)$$

Since $F'(t)$ is a statistically stationary process, the two time correlation function

$\langle F'(t)F'(t') \rangle$ can be expressed in terms of $s' = t - t'$:

$$\begin{aligned}
K(s') &= \langle F'(t)F'(t-s') \rangle \\
&= \langle F'(t+s')F'(t) \rangle \\
&= \langle F'(t)F'(t+s') \rangle \\
&= K(-s') \quad .
\end{aligned} \tag{4.51}$$

$K(s') = K(-s')$ is non-zero only in a small range of $|s'| \lesssim \tau^*$, where τ^* is the correlation time of the heat reservoir. We further introduce $s = t + t'$. Then, (4.52) is rewritten in terms of new time coordinates s and s' :

$$\langle v(0)^2 \rangle = \frac{1}{m^2} \int_{-\infty}^0 ds e^{\gamma s} \int_A ds' K(s') \quad , \tag{4.52}$$

where the integral range A is a small range of $|s'|$ mentioned above. See Fig. 4.5 for the integral range. We can substitute,

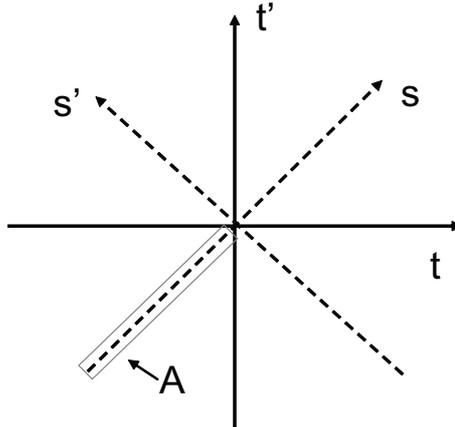


Figure 4.5: The integral range of (4.52).

$$\int_{-\infty}^0 ds e^{\gamma s} = \frac{1}{\gamma} \quad , \tag{4.53}$$

$$\int_A ds' K(s') = \frac{1}{2} \int_{-\infty}^{\infty} \langle F'(s')F'(0) \rangle ds' \quad , \tag{4.54}$$

$$\langle v(0)^2 \rangle = \frac{k_B T}{m} \quad , \tag{4.55}$$

into (4.80) to finally obtain

$$\gamma = \frac{1}{2k_B \theta m} \int_{-\infty}^{\infty} \langle F'(s')F'(0) \rangle ds' \quad . \tag{4.56}$$

This is the fluctuation-dissipation theorem. If a particle with a mass m is at thermal equilibrium, the damping rate γ for the velocity deviation $v(t)$ from an equilibrium value $\langle v(t) \rangle = 0$ is uniquely related to the correlation function $\langle F'(s)F'(0) \rangle$ of the reservoir fluctuating force.

4.3 Transmission Line Model of Thermal Noise

4.3.1 Nyquist model

Nyquist's treatment of thermal noise[4] appeared very soon after Johnson's observation[1] and thus thermal noise is often referred to as Johnson-Nyquist noise. Nyquist model initially considered two electrical resistors, R_1 and R_2 , connected in parallel, as shown in Fig. 4.6. The open-circuit voltage noise source v_1 associated with resistance R_1 produces a current fluctuation in the circuit, leading to an absorbed power $R_2 \overline{v_1^2} / (R_1 + R_2)^2$ by resistance R_2 . A similar flow of absorbed power, $R_1 \overline{v_2^2} / (R_1 + R_2)^2$, exists from R_2 to R_1 . Since the two conductors are at the same temperature, the power flow in each direction must be exactly the same and cancel each other out; otherwise, the second law of thermodynamics would be violated.

The second law of thermodynamics[6], [7] states that it is impossible to take heat from one reservoir and put it to another reservoir at an equilibrium temperature. That is, the entropy of the whole system does not decrease spontaneously. In order to satisfy this exact cancellation of power flow from R_1 to R_2 , and vice versa, the open circuit voltage noise $\overline{v_i^2}$ should be proportional to the electrical resistance R_i . This exact cancellation of power flow must hold not only for the total power, but also for the power exchanged in any frequency band; otherwise, the second law of thermodynamics would be violated simply by inserting a frequency filter between the resistors. In other words, the power spectrum $S_v(\omega)$ of the voltage fluctuation should be independent of the detailed structure and material of the conductor and should be a universal function of R , θ , and (angular) frequency ω . If the conductors R_1 and R_2 are at different temperatures, θ_1 and θ_2 , the net heat flow should be proportional to the temperature difference $\theta_1 - \theta_2$ and thus the power spectrum $S_v(\omega)$ is proportional to the temperature θ .

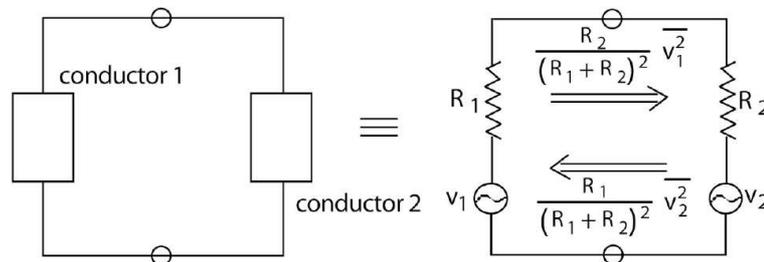


Figure 4.6: Two parallel resistors and the balance of power flow at thermal equilibrium.

Nyquist model then extended the above thought experiment to a long, lossless trans-

mission line terminated at either end by resistors with electrical resistance R_1 and R_2 (Fig. 4.7). This lossless LC transmission line has a characteristic impedance $Z_0 = \sqrt{\frac{L}{C}}$ which is made equal to the end terminal resistance $R_1 = R_2 = R$. The electromagnetic wave propagates with a velocity $v = \frac{1}{\sqrt{LC}}$, where L and C are the inductance and capacitance of the transmission line per unit length. The power delivered to the transmission line from R_1 or R_2 in a frequency interval $d\omega/2\pi$ is

$$dP = \frac{1}{4R} S_v(\omega) \frac{d\omega}{2\pi} \quad . \quad (4.57)$$

A time duration in which this noise power travels in the transmission line is $t = \frac{\ell}{v}$, where ℓ is the length of the transmission line. Hence, the total energy stored in the transmission line in the same frequency band is

$$dE = dP \times t \times 2 = \frac{\ell}{2Rv} S_v(\omega) \frac{d\omega}{2\pi} \quad . \quad (4.58)$$

Suppose the transmission line is suddenly short-circuited by closing the two switches

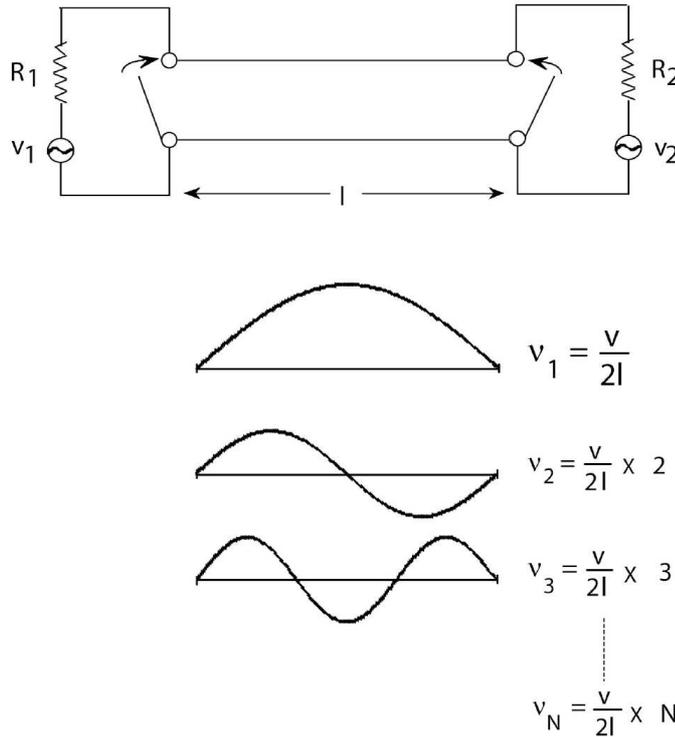


Figure 4.7: Two resistors separated by a transmission line with a switch. After closing the two switches, the thermal excitation from two resistors is trapped as resonant modes of a transmission line cavity.

across R_1 and R_2 . Then, the energy density (4.58) on the transmission line is “trapped”

as standing waves of a transmission line cavity with perfect reflectors. The resonant frequencies of these standing waves are $\nu_N = \frac{v}{2\ell} N$, where N is a positive integer. The number of standing modes in the frequency interval $\frac{d\omega}{2\pi}$ is given by

$$m = \left(\frac{d\omega}{2\pi} \right) / \left(\frac{v}{2\ell} \right) = \frac{d\omega \cdot \ell}{\pi v} \quad . \quad (4.59)$$

As the transmission line length ℓ increases, the number of degrees of freedom (DOF) of the system (given by the number of the standing modes) also increases and the energy separation becomes much smaller than $k_B\theta$. Moreover, the electric field energy and the magnetic field energy of each mode are proportional to the square of the electric and magnetic field amplitudes, respectively. Therefore, it is permissible to invoke the equipartition theorem to determine the total energy in the transmission line at thermal equilibrium. The average energy per mode (with two DOFs: electric and magnetic fields) is $k_B\theta$ and thus the stored energy in the frequency band $\frac{d\omega}{2\pi}$ is given by

$$dE = m k_B\theta = \frac{d\omega \ell}{\pi v} k_B\theta \quad . \quad (4.60)$$

From (4.58) and (4.60), the power spectrum is obtained as

$$S_v(\omega) = 4k_B\theta R \quad . \quad (4.61)$$

As expected from a thermodynamic argument, $S_v(\omega)$ is indeed proportional to R and θ .

If an angular frequency ω becomes greater than $k_B\theta/\hbar$, the quantized (photon) energy of the standing mode becomes larger than $k_B\theta$. In such a case, the average thermal energy per mode is calculated as

$$\hbar\omega \bar{n}_{th} = \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{k_B\theta}\right) - 1} \quad , \quad (4.62)$$

where \bar{n}_{th} is the average photon number of a mode at frequency ω and temperature θ [6], [7]. The photon number distribution function P_n obeys the photon statistics

$$P_n = \frac{\bar{n}_{th}^n}{(1 + \bar{n}_{th})^{n+1}} = \exp\left(-\frac{n\hbar\omega}{k_B\theta}\right) \left[1 - \exp\left(-\frac{\hbar\omega}{k_B\theta}\right) \right] \quad , \quad (4.63)$$

and thus the mean thermal photon number is

$$\bar{n}_{th} \equiv \sum_{n=0}^{\infty} n P_n = \frac{1}{\exp\left(\frac{\hbar\omega}{k_B\theta}\right) - 1} \quad . \quad (4.64)$$

Therefore, the total energy in the frequency band $\frac{d\omega}{2\pi}$ is

$$dE = m \hbar\omega \bar{n}_{th} = \frac{d\omega \ell}{\pi v} \cdot \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{k_B\theta}\right) - 1} \quad . \quad (4.65)$$

Comparing this result with (4.58), the power spectral density of the voltage noise is found to be[4]

$$S_v(\omega) = \frac{4\hbar\omega R}{\exp\left(\frac{\hbar\omega}{k_B\theta}\right) - 1} \quad . \quad (4.66)$$

In (4.66), the quantized energy $\hbar\omega$ of a transmission line cavity mode is correctly taken into consideration and the spectral density $S_v(\omega)$ is reduced to $4k_BTR$ when $\hbar\omega \ll k_B T$. However, the full quantum mechanical analysis of the problem suggests that (4.66) is still insufficient since the zero-point fluctuation is not included.

4.3.2 Quantum Noise

Let us consider a lossless LC circuit. The resonant frequency of such an LC circuit, $\omega_0 = \frac{1}{\sqrt{LC}}$, corresponds to that of a transmission line cavity mode, $\nu_N = \frac{v}{2l}N$ in our previous model. If the current flowing in the inductance L is denoted by $i(t)$ and the voltage across the capacitance C is denoted by $v(t)$, the Kirchhoff laws are represented by

$$C \frac{dv(t)}{dt} = i(t) \quad , \quad (4.67)$$

$$L \frac{di(t)}{dt} = -v(t) \quad . \quad (4.68)$$

These equations can be expressed in terms of the normalized “voltage,” $q(t) \equiv Cv(t)$, and the normalized “current,” $p(t) \equiv Li(t)$:

$$\frac{dq(t)}{dt} = \frac{1}{L}p(t) \quad , \quad (4.69)$$

$$\frac{dp(t)}{dt} = -\frac{1}{C}q(t) \quad . \quad (4.70)$$

The total energy stored in the lossless LC circuit is given by

$$H = \frac{1}{2}Li^2 + \frac{1}{2}Cv^2 = \frac{p^2}{2L} + \frac{q^2}{2C} \quad . \quad (4.71)$$

Suppose one interprets the above expression in terms of an analogy with a mechanical harmonic oscillator with a mass m and spring constant k . One can then identify the inductance L corresponding to the mass m , the capacitance C corresponding to the inverse of the spring constant $\frac{1}{k}$, and p and q corresponding to the momentum and position, respectively. If one considers (4.71) as a Hamiltonian function of the system, the classical Hamilton equations for the position q and momentum p are written as[9]

$$\frac{dq}{dt} \equiv \frac{\partial H}{\partial p} = \frac{1}{L}p \quad , \quad (4.72)$$

$$\frac{dp}{dt} \equiv -\frac{\partial H}{\partial q} = -\frac{1}{C}q \quad , \quad (4.73)$$

which are identical in form to the Kirchhoff equations (4.67) and (4.68) for the “voltage” q and “current” p . Hence, one can conclude that the “voltage” and “current” in a lossless LC circuit are a pair of conjugate observables, just like the position and momentum of a mechanical harmonic oscillator. Quantum mechanically, conjugate observables q and p must satisfy the following commutation relation,

$$[q, p] \equiv qp - pq = i\hbar \quad , \quad (4.74)$$

where q and p are no longer complex numbers (c -numbers); rather, they are quantum mechanical operators (q -numbers). Using the Schwartz inequality, one can derive the following Heisenberg uncertainty principle for the product of the variances of q and p [see Chapter 2 for the detailed discussion]:

$$\overline{\Delta q^2} \overline{\Delta p^2} \geq \frac{\hbar^2}{4} . \quad (4.75)$$

Equation (4.75) indicates that the current and voltage of an LC circuit have intrinsic quantum noise which does not disappear even at absolute zero temperature.

We now introduce the (non-Hermitian) creation (a^\dagger) and annihilation (a) operators for a circuit photon defined by:

$$a = \frac{1}{\sqrt{2\hbar\omega_0 L}}(\omega_0 L q + ip) , \quad (4.76)$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar\omega_0 L}}(\omega_0 L q - ip) . \quad (4.77)$$

The Hamiltonian function (4.71) and the commutation relation (4.74) are rewritten as

$$H = \hbar\omega_0 \left(a^\dagger a + \frac{1}{2} \right) , \quad (4.78)$$

$$[a, a^\dagger] = 1 . \quad (4.79)$$

If the LC circuit is not excited at all, the quantum mechanical state of such an unexcited LC circuit is referred to as a ground state, or a “vacuum state.” The vacuum state is mathematically defined as an eigenstate of the annihilation operator a with an eigenvalue of zero:

$$a|0\rangle = 0 . \quad (4.80)$$

The operation of the circuit creation operator on the vacuum state creates a single-quantum:

$$a^\dagger|0\rangle = |1\rangle . \quad (4.81)$$

Similarly, the operation of the circuit annihilation operator on the one quantum state creates a vacuum state:

$$a|1\rangle = |0\rangle . \quad (4.82)$$

The mean and mean-square of the “voltage” and “current” of the vacuum state are calculated by taking the ensemble average of the respective operators:

$$\bar{q} \equiv \langle 0|q|0\rangle = \langle 0|\sqrt{\frac{\hbar\omega_0 C}{2}}(a^\dagger + a)|0\rangle = 0 , \quad (4.83)$$

$$\overline{q^2} \equiv \langle 0|q^2|0\rangle = \langle 0|\frac{\hbar\omega_0 C}{2}(a^{\dagger 2} + a^2 + a^\dagger a + a a^\dagger)|0\rangle = \frac{\hbar\omega_0}{2} C , \quad (4.84)$$

$$\bar{p} \equiv \langle 0|p|0\rangle = \langle 0|i\sqrt{\frac{\hbar\omega_0 L}{2}}(a^\dagger - a)|0\rangle = 0 , \quad (4.85)$$

$$\overline{p^2} \equiv \langle 0|p^2|0\rangle = \langle 0|\frac{\hbar\omega_0 L}{2}(-a^{\dagger 2} - a^2 + a^\dagger a + a a^\dagger)|0\rangle = \frac{\hbar\omega_0}{2} L . \quad (4.86)$$

Here, the orthogonality relations between energy eigenstates such as $\langle 0|1\rangle = \langle 0|2\rangle = 0$ are used. The variances of the “voltage” and “current” satisfy the minimum uncertainty product,

$$\frac{\overline{\Delta q^2} \equiv \overline{q^2} - \bar{q}^2 = \frac{\hbar\omega_0}{2} C}{\overline{\Delta p^2} \equiv \overline{p^2} - \bar{p}^2 = \frac{\hbar\omega_0}{2} L} \searrow \nearrow \overline{\Delta q^2} \overline{\Delta p^2} = \frac{\hbar^2}{4} . \quad (4.87)$$

The expectation value of the zero-point energy is given by

$$\langle 0|H|0\rangle = \frac{\hbar\omega_0}{2} . \quad (4.88)$$

If the system of Fig. 4.7 is at an absolute zero temperature, each transmission line cavity mode should have this zero-point fluctuation energy. This intrinsic noise must be provided by the two terminal resistors, because the lossless transmission line cavity does not have any dissipation and thus there should not be internal noise source.

If one adds this zero-point fluctuation contribution to (4.66), one obtains the full quantum mechanical expression for an open-circuit voltage fluctuation spectral density:

$$S_v(\omega) = 2\hbar\omega R \coth\left(\frac{\hbar\omega}{2k_B\theta}\right) . \quad (4.89)$$

As shown in Fig. 4.8, $S_v(\omega)$ is reduced to the thermal noise value ($4k_B\theta R$) in the high-temperature limit ($k_B\theta \gg \hbar\omega$) and is reduced to the quantum noise value ($2\hbar\omega R$) in the low-temperature limit ($k_B\theta \ll \hbar\omega$). Equation (4.89), including the quantum mechanical zero-point fluctuation, is referred to as the generalized Nyquist noise[5].

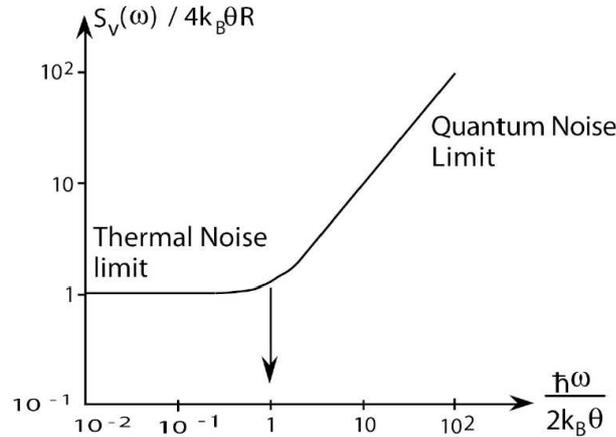


Figure 4.8: An open-circuit voltage noise spectrum including the zero-point fluctuation.

The generalized Nyquist formula (4.89) deserves some remarks. The thermal noise, $4k_B\theta R$, is a real fluctuation energy that can be measured directly. It is an intrinsic noise source in a sense that we cannot avoid it if a system is at a finite temperature θ . On the other hand, the quantum noise, $2\hbar\omega R$, is a virtual fluctuation energy that cannot be measured directly. It manifests itself only indirectly in a form of photon shot noise

in a laser[10] or mechanical (Casimir) force in metallic parallel plates[11]. Moreover, as (4.87) suggests, we can artificially squeeze the current fluctuation to zero by allowing the voltage fluctuation to increase indefinitely by some nonlinear processes[12]. Nevertheless, the generalized Nyquist formula (4.89) provides an adequate basis for an open dissipative device operating at a regime of $\hbar\omega \gg k_B\theta$. A laser and photon detector are such classic examples. We will see in Chapter 10 that a cavity loss and stimulated emission gain in a laser can be described by a positive and negative resistance, respectively, and that noise associated with such dissipative elements are given by $2\hbar\omega|R|$ under standard operation.

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