Chapter 3

Classical and Quantum Circuit Theory

A noisy electrical network can be represented by a noise-free network with external noise generators. The magnitude of the external noise generator is expressed either by an equivalent noise resistance or an equivalent noise temperature. When a network is dominated by the granular property of changed carriers, the noise generator is more conveniently described by a shot noise suppression factor. If such a network has four terminals or two (input and output) ports, the noise figure is often used as a figure of merit expressing the inherent noisiness of the circuit. The technique is particularly useful to analyze a cascaded amplifier system. The linear or linearized network technique is also useful to express the quantum noise properties of complicated quantum systems. Most of the arguments in this chapter follow the excellent text on the circuit model of noise by H.A. Haus [1].

3.1 Two-Terminal Networks –Thevenin Equivalent Circuit–

A noisy two-terminal network with impedance $Z(\omega) = R(\omega) + iX(\omega)$ generates the open circuit voltage fluctuation $v(t)$ as shown in Fig. 3.2 (a). The two equivalent circuits based on Thevenin's theorem[2] are shown in Fig. 3.2 (b) and (c). One is the noise-free network with impedance $Z(\omega)$ in series with a voltage generator $v(t)$. The other is the noise-free network with admittance $Y(\omega) = G(\omega) + i\ B(\omega)$ in parallel with a current generator $i(t)$.

Consider the parallel RC circuit shown in Fig. 3.1 (a) as an example of such a two terminal network. The noise of the register is represented by the parallel current source $i(t)$ with the spectral density of $S_i(\omega)$. The series voltage source $v(t)$ in the Thevenin equivalent circuit shown in Fig. 3.1 (b) has then the spectral density of

$$S_v(\omega) = \frac{R^2}{1+\omega^2(CR)^2} S_i(\omega) \quad . \tag{3.1}$$

The frequency dependent (Lorentzian) power spectral density Eq. (3.1) is due to the impedance of the capacitor. We assume the spectrum $S_i(\omega)$ is constant and flat in a frequency range of interest. Using the Wiener-Khintchine theorem, or more specifically Parseval theorem of Chapter 1, we obtain the mean-square value of the voltage noise

$$\langle v^2 \rangle = \frac{1}{2\pi} \int_0^\infty S_v(\omega) d\omega = \frac{R}{4C} S_i(\omega) \quad . \tag{3.2}$$

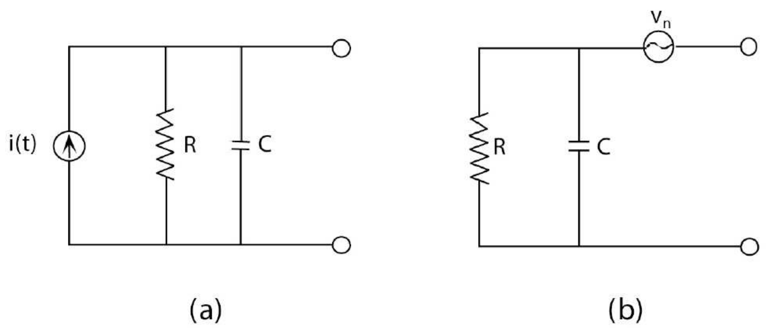

Figure 3.1: (a) A parallel RC circuit with thermal noise current source. (b) The Thevenin equivalent circuit with thermal noise voltage source.

The energy stored in the capacitor is thus equal to

$$\frac{1}{2}C\langle v^2 \rangle = \frac{1}{8}RS_i(\omega) \quad . \tag{3.3}$$

According to the equipartition theorem[3], if the system energy is of the form of quadratic dependence on generalized coordinate (the voltage in this case), the average energy of the system under thermal equilibrium condition is equal to $\frac{1}{2}k_B\theta$ per degree of freedom. This means current spectral density must be given by

$$S_i(\omega) = \frac{4k_B\theta}{R} \quad . \tag{3.4}$$

This is Johnson-Nyguist thermal noise of a simple register which we will discuss in the next section. Notice that the noise energy $\frac{1}{2}k_B\theta$ is independent of the resistance R while the magnitude and the bandwidth of the noise spectrum are dependent on R.

In a more general content, the single-sided power spectral density of $v(t)$ in Fig. 3.2(b) is often expressed by

$$S_{\mathrm{v}}(\omega) = 4k_{\mathrm{B}}\theta R_{\mathrm{n}} \quad , \tag{3.5}$$

where θ is the absolute temperature and R_n is the equivalent noise resistance.

The spectral density of $i(t)$ is in Fig. 3.2(c) expressed by

$$S_{\mathrm{i}}(\omega) = 4k_{\mathrm{B}}\theta G_{\mathrm{n}} \quad , \tag{3.6}$$

where G_{n} is the equivalent noise conductance.

If the network is linear and passive, and there is no net energy flow, *i.e.* the circuit is at thermal equilibrium condition, then $R_{\mathrm{n}} = R(\omega)$ and $G_{\mathrm{n}} = G(\omega) = R(\omega)/[R(\omega)^2 + X(\omega)^2]$. The noise in this case is reduced to Johnson-Nyquist thermal noise, as mentioned above. However, in nonlinear active circuits, or in non-equilibrium condition with a net energy flow, these equalities generally do not hold. This is due to the fact that equipartition theorem of statistical mechanics, on which Johnson-Nyquist thermal noise is based, does not hold for a non-equilibrium system.

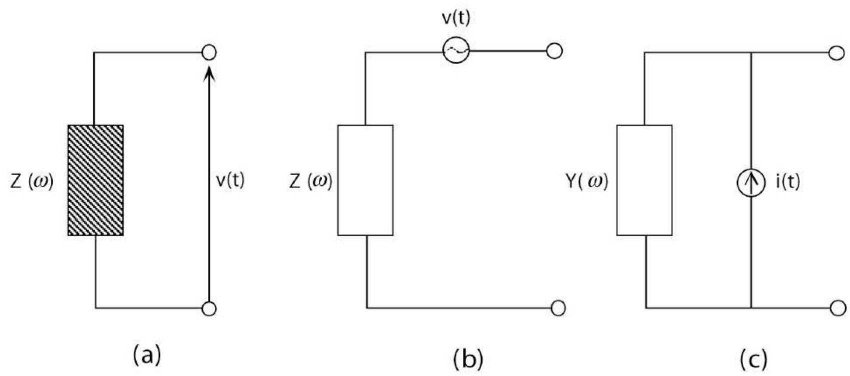

Figure 3.2: (a) A noisy two-terminal network. (b) Thevenin equivalent circuit with an external voltage source. (c) Thevenin equivalent circuit with an external current source.

When the electron temperature is different from the lattice temperature, which is the case for hot electron devices, it is convenient to express Eqs. (3.5) and (3.6) in the alternative forms:

$$S_v(\omega) = 4k_B\theta_n R \quad , \tag{3.7}$$

$$S_i(\omega) = 4k_B\theta_n G \quad , \tag{3.8}$$

where θ_n is the equivalent noise temperature. In circuits containing shot noise sources as primary noise sources, it is convenient to use the expression

$$S_i(\omega) = 2q\xi I \quad , \tag{3.9}$$

where I is the terminal current and ξ is the shot noise suppression factor. If some smoothing mechanisms are dominant, for instance due to space charge effect, ξ becomes smaller than unity. If there is no smoothing mechanism in the system, the power spectral density is full-shot noise ($\xi = 1$).

Next let us consider a parallel LCR circuit shown in Fig. 3.3. The open circuit voltage $v(t)$ has the spectral density of

$$S_v(\omega) = \left[\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2\right]^{-1} S_i(\omega) \quad . \tag{3.10}$$

$S_v(\omega)$ now concentrates on the resonant frequency $\omega_0 = \frac{1}{\sqrt{LC}}$ of a LC circuit and decays toward $\omega = 0$ and $\omega = \infty$. Using the Parseval theorem, we obtain the mean-square value of the voltage generator

$$\langle v^2 \rangle = \frac{1}{2\pi}\int_0^\infty S_v(\omega)d\omega = \frac{k_B\theta}{C} \quad , \tag{3.11}$$

which is identical to Eq. (3.2), even though the spectral shape Eq. (3.10) is very different from Eq. (3.1). The energy stored in the system is now given by

$$\begin{aligned}\frac{1}{2}C\langle v(t)^2\rangle + \frac{1}{2}L\langle i(t)^2\rangle &= \frac{1}{2}k_B\theta + \frac{1}{2}L\left[\frac{\langle v(t)^2\rangle}{\omega_0^2 L^2}\right] \\ &= k_B\theta \quad . \end{aligned} \tag{3.12}$$

We have two degrees of freedom in this system (the voltage across the capacitor and the current through the inductor), so the average thermal energy is doubled.

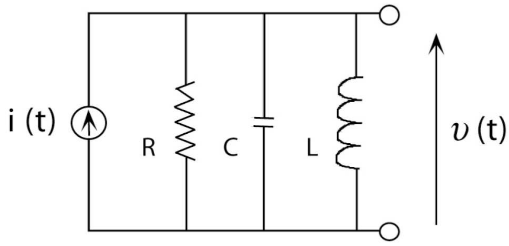

Figure 3.3: A parallel LCR circuit with thermal noise current source.

3.2 Four-Terminal Networks (Two Ports)

A network with two pairs of terminals, input and output ports, is known as a four-terminal or two-port network. For a noiseless four-terminal network, the currents and voltages at the terminals are related to each other in terms of the impedance matrix Z or the admittance matrix Y as follows:

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \quad , \tag{3.13}$$

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad . \tag{3.14}$$

The subscripts 1 and 2 refer to the input and output ports, respectively, and the sign convention is that currents flowing into the network are positive, as shown in Fig. 3.4. The upper case letters I and V indicate the Fourier transforms of the time dependent current and voltage, which are in general dependent on frequency.

A noisy four-terminal network is represented by an extension of Thevenin's theorem[2]. In Fig. 3.5 (a), a series noise voltage generator appears at each port. Some degree of correlation may exist between these two generators since the same internal noise mechanism may be responsible, at least in part, for the open circuit voltage fluctuations at the two terminals. The dual of Fig. 3.5 (a) is shown in Fig. 3.5 (b), in which the internal noise is represented by external parallel current generators.

The current-voltage relation of a noisy four-terminal network becomes

$$\begin{pmatrix} V_1 + V_{n1} \\ V_2 + V_{n2} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \quad , \tag{3.15}$$

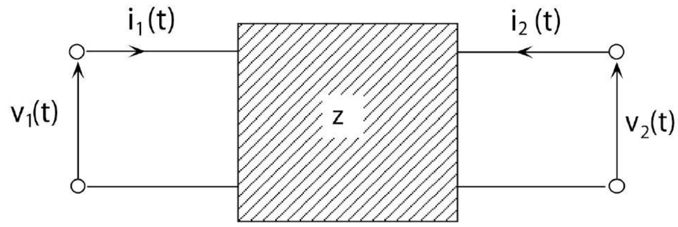

Figure 3.4: A noiseless four-terminal network.

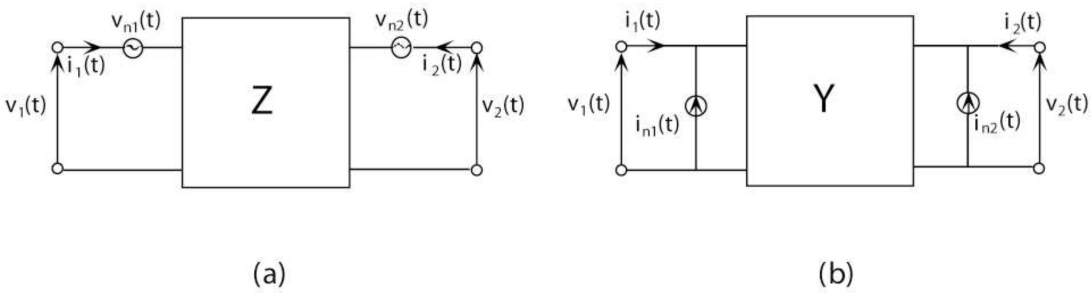

Figure 3.5: (a) Thevenin equivalent circuit with two external series voltage generators. (b) Thevenin equivalent circuit with two external parallel current generators.

$$\begin{pmatrix} I_1 + I_{n1} \\ I_2 + I_{n2} \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad . \tag{3.16}$$

It is often more convenient to refer both external generators to the input port. The equivalent circuit shown in Fig. 3.6 has a series voltage generator and a parallel current generator at the input, for which the current-voltage characteristic is expressed by the relation

$$\begin{pmatrix} I_1 + I_{na} \\ I_2 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} V_1 + V_{na} \\ V_2 \end{pmatrix} \quad . \tag{3.17}$$

Comparing Eqs. (3.16) and (3.17), the Fourier transform of the new voltage generator $v_{na}(t)$ should be related to that of the current generator $i_{n2}(t)$ is similarly given by:

$$V_{na} = -\frac{I_{n2}}{Y_{21}} \quad . \tag{3.18}$$

The Fourier transform of the new current generator $i_{na}(t)$ is similarly given by

$$I_{na} = I_{n1} - \frac{Y_{11}}{Y_{21}} I_{n2} \quad . \tag{3.19}$$

The arrangement of Fig. 3.6 is particularly convenient for calculating the noise figure of the two-port network. However, this equivalent circuit is valid only for calculating the

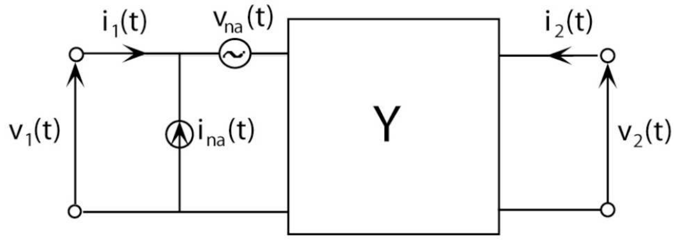

Figure 3.6: The equivalent circuit of a noisy two-port with external current and voltage generators in the input port.

noise in the output port. It does not give the correct description of the noise in the input port. This can be easily seen by the fact that I_{na} is not equal to I_{n1}. That is, if the input and output ports are shorted in the circuits shown in Fig. 3.5 (b) and Fig. 3.6, the short-circuit current is i_{n1} in the former case, while it is i_{na} in the latter case.

3.3 Noise Figure of a Linear Two-Port

When a weak signal is amplified, the noise associated with the signal is also amplified. If the amplifier is free from internal noise, the signal-to-noise (S/N) ratio is preserved. In many cases, an amplifier has internal noise that is added to the output signal, so the S/N ratio is usually degraded. The noisiness of a linear amplifier is evaluated by a noise figure:

$$F = (S/N)_{\text{in}}/(S/N)_{\text{out}}$$

where $(S/N)_{\text{in}}$ and $(S/N)_{\text{out}}$ are the input and output S/N ratios.

The noise figure of a linear two-port is generally defined by

$$F = \frac{\text{total output noise power per unit bandwidth}}{\text{output noise power per unit bandwidth due to input noise}}$$

at a specific frequency and temperature. A signal input $i_s(t)$ is transferred to the output through an input admittance Y_s and noisy two-port network, as shown in Fig. 3.7. The noise $i_{ns}(t)$ generated in the input admittance Y_s and the noise $i_{na}(t)$ and $v_{na}(t)$ in the two-port are independent, and so the noise figure of the whole system can be expressed as

$$\begin{aligned} F &= \frac{\overline{|I_{ns} + I_{na} + Y_s V_{na}|^2}}{\overline{|I_{ns}|^2}} \\ &= 1 + \frac{S_{ia}(\omega)}{S_{is}(\omega)} + |Y_s|^2 \frac{S_{va}(\omega)}{S_{is}(\omega)} + 2Re(\Gamma_{iv} Y_s^*) \frac{[S_{ia}(\omega) \cdot S_{va}(\omega)]^{1/2}}{S_{is}(\omega)} \quad , \end{aligned} \tag{3.20}$$

where $S_{ia}(\omega), S_{va}(\omega)$ and $S_{is}(\omega)$ are the power spectra of $i_{na}(t), v_{na}(t)$ and $i_{ns}(t)$, respectively, and Γ_{iv} is the normalized cross-correlation spectral density (coherence function)

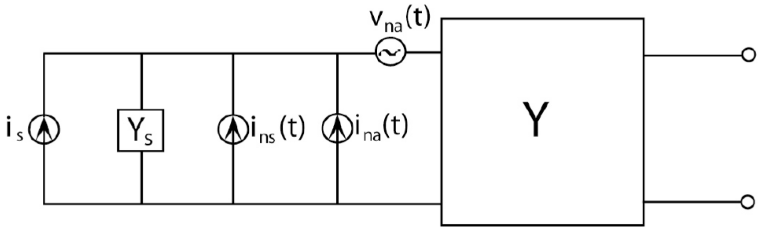

Figure 3.7: A circuit for calculating the noise figure of a two-port network.

between $i_{na}(t)$ and $v_{na}(t)$,

$$\begin{aligned}\Gamma^*_{iv}(\omega) &= \frac{\overline{I^*_{na}V_{na}}}{\left[\overline{|I_{na}|^2}\cdot\overline{|V_{na}|^2}\right]^{\frac{1}{2}}} \\ &= \frac{S_{iva}(\omega)}{\left[S_{ia}(\omega)S_{va}(\omega)\right]^{\frac{1}{2}}} \quad . \end{aligned} \tag{3.21}$$

The power spectral densities can be expressed as

$$S_{ia}(\omega) = 4k_B\theta G_{ni} \quad , \tag{3.22}$$

$$S_{va}(\omega) = 4k_B\theta / G_{nv} \quad , \tag{3.23}$$

$$S_{is}(\omega) = 4k_B\theta G_s \quad . \tag{3.24}$$

Here G_{ni} and G_{nv} are the equivalent noise conductances, which are not necessarily actual conductances of the two-port network. On the other hand, $G_s = Re(Y_s)$ is the actual source conductance. In the spirit of Eqs. (3.18) and (3.19), the current generator $i_{na}(t)$ is split into two parts, one part of which is uncorrelated with $v_{na}(t)$ and the other part is fully correlated with $v_{na}(t)$. Therefore we obtain

$$I_{na} = I_{nb} + Y_c V_{na} \quad , \tag{3.25}$$

where Y_c is called the correlation admittance of $i_{na}(t)$ and $v_{na}(t)$. Since $\overline{I_{nb}V^*_{na}} = 0$, we have

$$\Gamma_{iv} \equiv \frac{\overline{I_{na}V^*_{na}}}{\left[\overline{|I_{na}|^2}\ \overline{|V_{na}|^2}\right]^{1/2}} = Y_c\left[\frac{\overline{|V_{na}|^2}}{\overline{|I_{na}|^2}}\right]^{1/2} = \frac{Y_c}{\sqrt{G_{ni}G_{nv}}} \quad . \tag{3.26}$$

The noise figure of Eq. (3.20) is now rewritten as

$$F = 1 + \frac{G_{ni}}{G_s} + \frac{(G_s+G_c)^2 + (B_s+B_c)^2 - (G_c^2+B_c^2)}{G_{nv}G_s} \quad , \tag{3.27}$$

where G_c and B_c are the real and imaginary parts of the correlation admittance Y_c. The optimum source admittance to minimize the noise figure and the minimum noise figure are obtained by the conditions:

$$\frac{\partial F}{\partial B_s} = 0 \quad \text{and} \quad \frac{\partial F}{\partial G_s} = 0 \quad .$$

Using this technique, Eq. (3.27) is easily transformed into the form

$$F = F_0 + \frac{(G_s - G_{so})^2 + (B_s - B_{so})^2}{G_{nv} G_s} \quad . \tag{3.28}$$

Here, $F_0 = 1 + \frac{2}{G_{nv}}(G_{so} + G_c)$ is the minimum noise figure achieved when the source admittance satisfies the following matching condition:

$$G_s = G_{so} = (G_{nv} G_{ni} - B_c^2)^{1/2} \quad , \tag{3.29}$$

$$B_s = B_{so} = -B_c \quad . \tag{3.30}$$

The conditions Eqs. (3.29) and (3.30) for the source conductance and susceptance are referred to as noise tuning or noise matching. The noise figure increases quadratically when G_s and B_s are deviated from the optimum values. The four parameters F_o, G_{so}, B_{so} and G_{nv} completely characterize the noise of the two-port network.

The measurement of these four parameters and full characterization for a given linear two-port runs as follows:

1. Adjust the source conductance G_s and susceptance B_s to achieve the minimum noise figure F_0. The three parameters F_0, G_{s_0} and B_{s_0} can be directly obtained in this noise matching process.

2. Make a measurement of the noise figure at some non-optimum source admittance ($G_s \neq G_{s_0}$ and/or $B_s \neq B_{s_0}$). The remaining parameter G_{nv} can be obtained from $F - F_0$.

3. Three other parameters B_c, G_{ni} and G_c to fully characterize Eq. (3.27) are obtained from Eqs. (3.29), (3.30) and $G_c = \frac{G_{nv}}{2}(F_0 - 1) - G_{s_0}$.

4. Calculate the power spectral densities $S_{ia}(\omega)$ and $S_{va}(\omega)$ from Eqs. (3.22) and (3.23) and Γ_{iv} from Eq. (3.26).

If an amplifier is noise-free, the noise figure takes a lower bound, $F_0 = 1$.

3.4 Noise Figure of Amplifiers in Cascade

One of the important applications of the equivalent circuit discussed in the previous section is the overall noise figure of the system when several amplifiers are connected in cascade as shown in Fig. 3.8. Suppose each amplifier in the cascade is connected to a matched load, *i.e.* the output and input admittances of adjoining amplifiers are equal: $Y_{1,out} = Y_{2,in} = Y_1$, $Y_{2,out} = Y_{3,in} = Y_2$, and so on. The noise figure of the whole system in such a case can be written as

$$\begin{aligned} F \;=\; & \frac{\overline{|I_{sn} + I_{na1} + Y_s V_{na1}|^2}}{\overline{|I_{ns}|^2}} + \frac{\overline{|I_{na2} + Y_1 V_{na2}|^2}}{\eta_1 \overline{|I_{ns}|^2}} + \frac{\overline{|I_{na3} + Y_2 V_{na3}|^2}}{\eta_1 \eta_2 \overline{|I_{ns}|^2}} \\ & + \cdots \quad . \end{aligned} \tag{3.31}$$

Here Y_i and η_i are the output admittance and power gain of the i-th amplifier. The power gain η is defined by P_{out}/P_{in}, where P_{out} is the output power delivered to a matched load

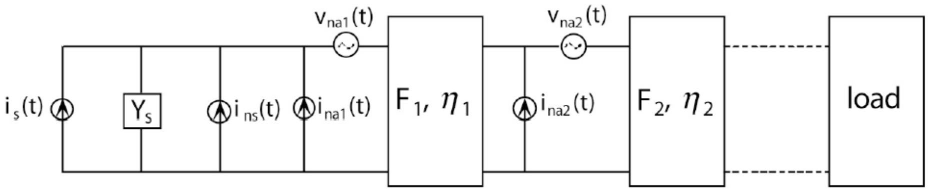

Figure 3.8: The equivalent circuit of a cascade amplifier system.

to the output admittance, $Y_L = Y_{out}$, and P_{in} is the input power delivered from a matched source to the input admittance, $Y_s = Y_{in}$. If F_i is the noise figure of the i-th amplifier defined by Eq. (3.20), the overall noise figure is

$$F = F_1 + (F_2 - 1)/\eta_1 + (F_3 - 1)/\eta_1\eta_2 + \cdots \quad . \tag{3.32}$$

Equation (3.32) is known as Friiss's formula[4]. The expression indicates that the noise figure of the cascade amplifier system is essentially determined by the noise figure of the first stage if the power gain of the first stage is sufficiently high. It is important to use a low-noise amplifier in the first stage in order to realize a small overall noise figure.

3.5 Thermal Noise of a Linear n-port Network

Classical noise at thermal equilibrium assigns $\frac{1}{2}k_B\theta$ to each degree of freedom. As shown in the previous argument, every resistor R at the thermal equilibrium with the circuit equation (see Fig. 3.9).

$$v = Ri + e \tag{3.33}$$

must be assigned mean square voltage fluctuations $S_e(\omega) = 4k_B\theta R$. From now on, v and i are the terminal voltage and current, and e is a noise source. Twiss[5] generalized this formula to an n-port with impedance matrix Z, showing that the correlation spectral density matrix of the voltage sources is (see Fig. 3.10)

$$S_{ee^\dagger}(\omega) = 2(Z + Z^\dagger)k_B\theta \tag{3.34}$$

where the superscript † indicates complex conjugate transposition of a matrix. We do not distinguish matrices from scalars by a change of typeface. Whether a quantity is a matrix or scalar will be apparent from the context.

The expression (3.34) can be adapted into, often more convenient, scattering matrix formalism, using the linear (matrix) equation (see Fig. 3.11)

$$b = Sa + \beta \tag{3.35}$$

and the correlation spectral density matrix for the noise source is

$$S_{\beta\beta^\dagger}(\omega) = (1 - SS^\dagger)k_B\theta \quad . \tag{3.36}$$

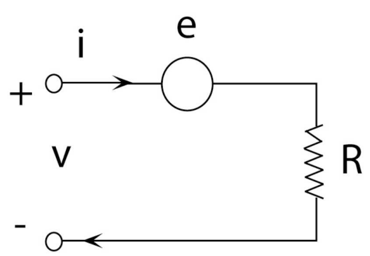

Figure 3.9: Equivalent circuit of linear noisy resistor.

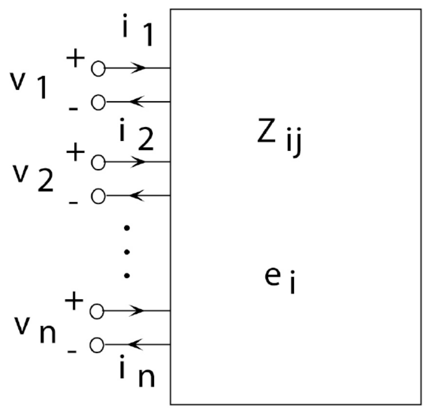

Figure 3.10: The impedance representation of a linear n-port network.

The terminal voltage and current column vectors v and i are related to input and output waves, a and b, by

$$\sqrt{\frac{1}{4Z_0}}(v + Z_0 i) = a \tag{3.37}$$

$$\sqrt{\frac{1}{4Z_0}}(v - Z_0 i) = b \tag{3.38}$$

where Z_0 is the normalization impedance. The inverse relations are

$$v = \sqrt{Z_0}(a + b) \tag{3.39}$$

$$i = \frac{1}{\sqrt{Z_0}}(a - b) \quad . \tag{3.40}$$

The scattering matrix formalism is cast into the impedance matrix formalism by manipulation of Eq. (3.35).

$$\frac{1}{2}(1 - S)(b + a) = \frac{1}{2}(1 + S)(a - b) + \beta \quad . \tag{3.41}$$

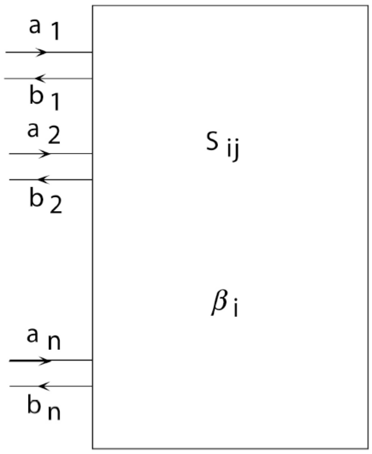

Figure 3.11: Scattering matrix representation of a n-port network.

Comparison of Eq. (3.41) with the impedance formulation

$$v = Zi + e \tag{3.42}$$

gives

$$\begin{aligned} Z &= (1-S)^{-1}(1+S)Z_0 & (3.43)\\ e &= 2\sqrt{Z_0}(1-S)^{-1}\beta \quad . & (3.44)\end{aligned}$$

The correlation spectral density matrix of e is

$$\begin{aligned} S_{ee^\dagger}(\omega) &= 4Z_0(1-S)^{-1}S_{\beta\beta^\dagger}(\omega)(1-S^\dagger)^{-1} \\ &= 4Z_0(1-S)^{-1}(1-SS^\dagger)(1-S^\dagger)^{-1}k_B\theta \quad . \end{aligned} \tag{3.45}$$

Let us check the expression for

$$Z + Z^\dagger = Z_0\left[(1-S)^{-1}(1-S) + (1+S^\dagger)(1-S^\dagger)^{-1}\right] \quad . \tag{3.46}$$

Multiplying by $(1-S)$ from the left and by $(1-S^\dagger)$ from the right we obtain

$$\begin{aligned} (1-S)(Z+Z^\dagger)(1-S^\dagger) &= Z_0\left[(1+S)(1-S^\dagger) + (1-S)(1+S^\dagger)\right] \\ &= 2Z_0[1-SS^\dagger] \quad . \end{aligned} \tag{3.47}$$

Thus, we have proven that

$$2Z_0(1-S)^{-1}(1-SS^\dagger)(1-S^\dagger)^{-1} = Z + Z^\dagger \quad . \tag{3.48}$$

Using this fact, we have for Eq. (3.45)

$$S_{ee^\dagger}(\omega) = 2(Z+Z^\dagger)k_B\theta \quad . \tag{3.49}$$

3.6 Quantum Circuit Theory

If an optical wave propagates in a complicated noisy system, a noise equivalent circuit model introduced in sec.(3.5) also provides a convenient tool. However, since $\hbar\omega \gg k_B\theta$ at optical frequencies, quantum mechanical zero-point fluctuation dominates over thermal noise. We need to establish a new rule to handle such a quantum system.

Linear quantum systems with loss must contain noise sources in order to provide for conservation of commutator brackets, which would decay to zero in the absence of such sources[6]. This property is analogous to that of lossy systems at thermal equilibrium which would lose their thermal excitation were it not for the thermal noise sources in the lossy element, which was first introduced by Langevin[7]. The noise sources associated with loss are simply a manifestation of the fluctuation-dissipation theorem[8]. Linear phase-insensitive systems with gain contain noise sources as well[1] because, if noise were not present, classical measurements could be performed on the amplified output, performing simultaneous measurements on two noncommuting observables (say in-phase and quadrature field components) with no increase in uncertainty, in violation of the doubling of uncertainty associated with a simultaneous measurement of two noncommuting observables[9]. Phase-sensitive systems with gain do not necessarily permit a simultaneous measurement, and thus do not necessarily contain noise sources[10].

The Langevin noise sources are particularly well adapted to a circuit theoretical treatment of linear systems. Much work has been done in classical systems, such as oscillators and amplifiers, using the terminology of electrical circuit theory[11]. Therefore, it is advantageous to express the terminology of quantum electrodynamics in circuit "language."

3.6.1 Analogy of Thermal Noise with Commutator Bracket Conservation

A quantum-mechanical linear circuit has to obey commutator bracket conservation. The input and output wave amplitudes, a and b in Fig. 3.11 are now considered as the photon annihilation operators. The standard commutator of a single-mode field operator a is

$$[a,\, a^+] = 1 \tag{3.50}$$

where the normalization is such that $< a^+a >$ gives the expectation of photon number. We can renormalize the photon annihilation and creation operators in such a way,

$$[a,\, a^+] = \hbar\omega \tag{3.51}$$

is satisfied, where the normalization of $< a^+a >$ is to energy. Here, the superscript $+$ indicates the Hermitian adjoint of the operator. If a quantum system is characterized by the scattering coefficient Γ for a single (scalar) incident wave a and reflected wave b, a noise source β has to be assigned to conserve commutators:

$$b = \Gamma a + \beta \quad . \tag{3.52}$$

because the output wave must satisfy the same commutator bracket as (3.51):

$$[b,\, b^+] = |\Gamma|^2[a,\, a^+] + [\beta,\, \beta^+] = \hbar\omega \quad . \tag{3.53}$$

One then finds for the commutator of the noise source

$$[\beta,\, \beta^+] = (1 - |\Gamma|^2)\hbar\omega \quad . \tag{3.54}$$

Note that $[\beta,\, \beta^+]$ is negative, when the system has gain ($|\Gamma|^2 > 1$). In this case β must be interpreted as a creation operator, β^+ as an annihilation operator. The noise components can be separated into in-phase and quadrature components

$$\beta_1 = \frac{1}{2}(\beta + \beta^+) \quad \text{and} \quad \beta_2 = \frac{i}{2}(\beta^+ - \beta) \quad . \tag{3.55}$$

The mean square fluctuations of β_1 are

$$\begin{aligned} \langle \Delta\beta_1^2 \rangle &= \frac{1}{4}\langle (\beta + \beta^+)^2 \rangle - \frac{1}{4}\langle (\beta + \beta^+) \rangle^2 \\ &= \frac{1}{4}\langle (\beta + \beta^+)^2 \rangle \end{aligned} \tag{3.56}$$

because the expectation value of the noise $\beta + \beta^+$ is zero. Thus

$$\begin{aligned} \langle \Delta\beta_1^2 \rangle &= \frac{1}{4}\langle \beta^+\beta^+ + \beta\beta + \beta\beta^+ + \beta^+\beta \rangle \\ &= \frac{1}{4}\langle \beta^+\beta^+ + \beta\beta + 2\beta\beta^+ \rangle - \frac{1}{4}[\beta,\, \beta^+] \\ &= \frac{1}{4}\langle \beta^+\beta^+ + \beta\beta + 2\beta^+\beta \rangle + \frac{1}{4}[\beta,\, \beta^+] \quad . \end{aligned} \tag{3.57}$$

Suppose that the reservoir responsible for the noise source is in the ground state. This assumption corresponds to an ideal attenuator and amplifier which impose a minimum allowable noise on the signal. The expectation values of $\beta^+\beta^+$ and $\beta\beta$ are then zero. Further, $\langle \beta^+\beta \rangle$ is zero when β is an annihilation operator ($|\Gamma|^2 < 1$). Then

$$\langle \Delta\beta_1^2 \rangle = (1 - |\Gamma|^2)\frac{1}{4}\hbar\omega \quad . \tag{3.58}$$

When $|\Gamma|^2 > 1$ and β is interpreted as a creation operator, then $\langle \beta\beta^+ \rangle$ is zero, and

$$\langle \Delta\beta_1^2 \rangle = (|\Gamma|^2 - 1)\frac{1}{4}\hbar\omega \quad . \tag{3.59}$$

An analogous derivation for $\langle \Delta\beta_2^2 \rangle$ shows that

$$\langle \Delta\beta_2^2 \rangle = \langle \Delta\beta_1^2 \rangle \tag{3.60}$$

for the present case of phase-insensitive quantum systems.

The generalization of the scattering relation (3.54) to passive n-ports is easy. Classically, Eq. (3.36) expresses the self- and cross-correlation spectra by the formula

$$S_{\beta_i\beta_j^*}(\omega) = (1 - SS^\dagger)_{ij} k_B \theta \tag{3.61}$$

Quantum mechanically we ask for the commutator

$$[\beta_i\, ,\, \beta_j^+] = \beta_i\beta_j^+ - \beta_j^+\beta_i \tag{3.62}$$

which has no classical analog, because classically the expression is zero. From Eq. (3.35), one has

$$b_i b_j^+ = S_{ik} a_k a_l^+ S_{jl}^* + \beta_i \beta_i^+ \quad . \tag{3.63}$$

Similarly,

$$b_j^+ b_i = S_{ik} a_l^+ a_k S_{jl}^* + \beta_i^+ \beta_i \quad . \tag{3.64}$$

Note that only the operators have been reversed in order, not the matrix multipliers. Therefore,

$$[b_i\, ,\, b_j^+] = \hbar\omega S_{ik} S_{jk}^* + [\beta_i\, ,\, \beta_j^+] = \hbar\omega \quad . \tag{3.65}$$

We denote by the dagger the Hermitian adjoint of an operator as well as the conjugate transpose of the matrix, whose elements are the operators. Then, one may write Eq. (3.65) in matrix form

$$[\beta\, ,\, \beta^+] = [1 - SS^\dagger]\hbar\omega \quad . \tag{3.66}$$

This is the generalization of (3.54) for a single port circuit.

3.6.2 The Characteristic Noise Matrix

To see more clearly what is involved in the transition from a system with loss to a system with gain, consider the case of uncoupled resistors, or reflectors. Then $(1-SS^\dagger)$ is diagonal and $S_{\beta\beta^\dagger}(\omega)$ is also diagonal. The characteristic noise matrix[12] defined by

$$N \equiv S_{\beta\beta^\dagger}(\omega)(SS^\dagger - 1)^{-1} \tag{3.67}$$

is diagonal and has all diagonal elements (eigenvalues) equal to $-k_B\theta$ for a classical case. If one or more reflections have gain, an equivalent noise temperature can be assigned to the noise source associated with each of the reflections. The noise temperature should be considered negative[13]. Then,

$$S_{\beta_i}(\omega) \equiv (1 - |S_{ii}|^2) k_B \theta_i \tag{3.68}$$

is a positive quantity, as it must be. Negative temperatures have been assigned to inverted media[14].

A lossless (noise-free) imbedding of the network (Fig. 3.12) is represented by a unitary transformation and therefore, leaves the eigenvalues of the characteristic noise matrix invariant.

In the quantum-mechanical case one may define two different characteristic "noise matrices." One is more properly called characteristic commutator matrix N_C and can be defined by

$$N_C \equiv [\beta\, ,\, \beta^\dagger](SS^\dagger - 1)^{-1} = -I\hbar\omega \quad . \tag{3.69}$$

Here I is an identity matrix. This matrix has all identical eigenvalues of $-\hbar\omega$ and thus is proportional to the identity matrix. The transformation of N_C by a lossless imbedding network follows the same laws as those of N, and thus is unitary. Therefore the commutator matrix remains diagonal after such an imbedding.

In analogy with Eq. (3.67) one may define a characteristic noise matrix for the quantum-mechanical system. The commutators fix the minimum amount of noise that must be associated with a particular reflection, or resistor. Indeed, since for the uncoupled network

$$[\beta_i\, ,\, \beta_i^\dagger] = [1 - |S_{ii}|^2]\hbar\omega \tag{3.70}$$

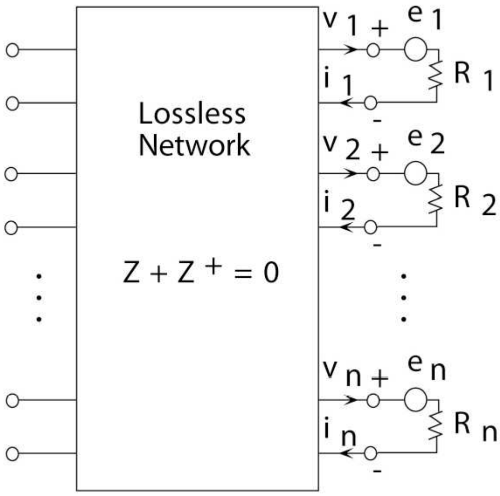

Figure 3.12: A lossless "imbedding."

then, with the states of the operators β_i's in the ground states, one finds

$$\begin{aligned} \langle \beta_i^2 \rangle &\equiv \frac{1}{2}\langle \beta_i \beta_i^+ + \beta_i^+ \beta_i \rangle \\ &= (1 - |S_{ii}|^2)\frac{1}{2}\hbar\omega \end{aligned} \tag{3.71}$$

when $|S_{ii}| < 1$ and

$$\begin{aligned} \langle \beta_i^2 \rangle &= \frac{1}{2}\langle \beta_i \beta_i^+ + \beta_i^+ \beta_i \rangle \\ &= (|S_{ii}|^2 - 1)\frac{1}{2}\hbar\omega \end{aligned} \tag{3.72}$$

for $|S_{ii}| > 1$. The characteristic noise matrix of the uncoupled network is diagonal and has eigenvalues $\pm\frac{1}{2}\hbar\omega$; the plus sign corresponds to the terminations that exhibit gain, the minus sign to the terminations exhibiting loss.

Again, a lossless noise-free imbedding may cast N into nondiagonal form, but leaves the eigenvalues unchanged. The importance of the eigenvalues rests on the fact that they determine the optimum noise performance as expressed by the "noise measure."

3.6.3 Noise Measure

Let us review briefly the concept of noise measure M of a two-port defined by

$$M = \frac{F-1}{1-\frac{1}{\eta}} = \frac{\eta(F-1)}{\eta - 1} \quad . \tag{3.73}$$

Here, F is the conventional noise figure and η is the exchangeable gain, both of which are defined in sec. (3.5). The exchangeable gain, in turn is the ratio of exchangeable output power to input power. Finally, exchangeable power is defined for a one-port as

$$\text{exchangeable power} \equiv \frac{\overline{|e|^2}}{4R} \tag{3.74}$$

where e is the amplitude of the internal source (Fig. 3.9) and R is the resistance. The exchangeable power reduces to the well-known "available" power, when $R > 0$, and becomes negative for $R < 0$. When R is negative, the exchangeable power is the maximum amount of power "exchangeable" with another negative resistor. In scattering matrix notation, the exchangeable power of a one-port is

$$\text{exchangeable power} \equiv \frac{\overline{|\beta|^2}}{1-|\Gamma|^2} \quad . \tag{3.75}$$

The excess noise figure $F-1$ times Γ is the exchangeable noise power at the amplifier output due to the amplifier noise sources normalized to $k_B\theta_0$, the thermal power at standard temperature. The gain must be defined as exchangeable gain when either the source resistance is negative and/or the output resistance (looking back into the amplifier) is negative. In applications to quantum phenomena it is best to drop the normalization to standard temperature, *i.e.*, the division by $k_B\theta_0$.

These results have profound implications for linear, phase-insensitive, quantum amplifiers. Fig. 3.13 shows schematically a passive environment connected to an active linear system, with one output port. The characteristic noise matrix of the active "network" has eigenvalues $\frac{1}{2}(\hbar\omega)$ (analogous to $k_B\theta$). The exchangeable output power at the output port with gain η due to the internal noise is $\eta(F-1) \geq \frac{1}{2}\hbar\omega(\eta-1)$. The passive environment causes an additional noise of $\geq \frac{1}{2}\hbar\omega\eta$.

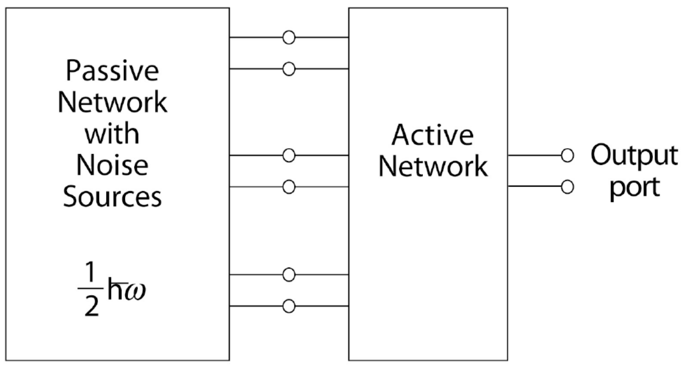

Figure 3.13: Active network with output excited by noise from passive network.

Consider first the case when the impedance as seen from the output port has a positive real part (the reflection coefficient is less that unity). Then the exchangeable power is equal to available power and the total noise output is

$$\text{available output power} \geq \hbar\omega(\eta - \frac{1}{2}) \quad . \tag{3.76}$$

When the gain is unity, the output noise power is equal to that of the zero-point fluctuations. When the gain is much larger than unity, the output noise power referred to the input by division by the gain η is $\hbar\omega$. This is in agreement with the results of Arthurs and Kelly[9]. A large gain allows a simulataneous measurement of the noncommuting observables a_1 and a_2 and, thus, must at least double the uncertainty product, *i.e.*, at least double the noise.

3.6.4 Phase sensitive system

If a quantum system is characterized by the phase sensitive scattering coefficients:

$$\begin{aligned} b_1 &= \Gamma_1 a_1 + \beta_1 \quad , \\ b_2 &= \Gamma_2 a_1 + \beta_2 \quad , \end{aligned} \tag{3.77}$$

the commutator bracket of the output wave is given by

$$[b_1, b_2] = \Gamma_1\Gamma_2 \left[a_1, a_2\right] + [\beta_1, \beta_2] \quad . \tag{3.78}$$

Since $[a_1, a_2] = [b_1, b_2] = \frac{i}{2}\hbar\omega$, we obtain the new commutator bracket for the noise:

$$[\beta_1, \beta_2] = (1 - \Gamma_1\Gamma_2)\,\frac{1}{2}\hbar\omega \quad . \tag{3.79}$$

One then immediately note that if $\Gamma_1 = 1/\Gamma_2$ is satisfied, the commutator bracket (3.79) disppears and such a system does not need to add the quantum noise on the amplified/deamplified signal. When $\Gamma_1 = 1/\Gamma_2 > 1$, the in-phase amplitude a_1 is amplified while the quadrature amplitude a_2 is deamplified. The noise figure F for such a phase sentive amplifier is 1(0dB), in contrast to the minimum noise figure of 2 (3dB) of a phase insensitive amplifier [15]. We will see in chapter 10 and chapter 11 that a laser amplifier and degenerate parametric amplifier as a respective example of such phase insensitive and phase sensitive amplifiers.

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