Chapter 1

Mathematical Methods

In this chapter we will study basic mathematical methods for characterizing noise processes. The two important analytical methods, probability distribution functions and Fourier analysis, are introduced here. These two methods will be used frequently throughout this text not only for classical systems but also for quantum systems. We try to present the two mathematical methods in a compact and succinct way as much as possible. The readers may find more detailed discussions in excellent texts [1]-[6]. In particular, most of the discussions in this chapter follow the texts by M.J. Buckingham [1] and by A.W. Drake [2].

1.1 Time Average vs. Ensemble Average

Noise is a stochastic process consisting of a randomly varying function of time and space, and thus is only statistically characterized. One cannot argue a single event at a certain time or position; one can only discuss the averaged quantity of a single system over a certain time (or space) interval or the averaged quantity of many identical systems at a certain time instance (or spatial position). The former is called time (or space) average and the latter ensemble average. Let us consider N systems which produce noisy waveforms $x^{(i)}(t)$, as shown in Fig. 1.1.
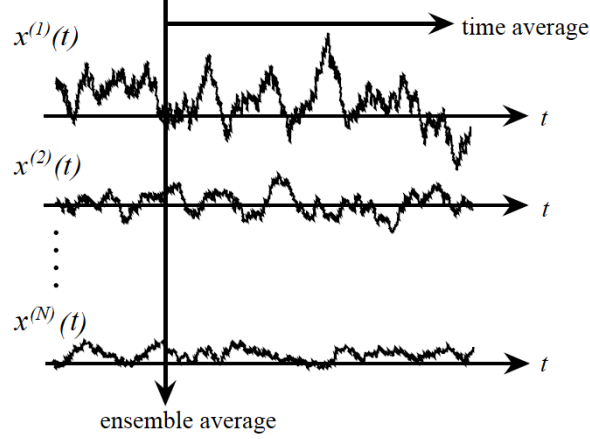
Figure 1.1: Ensemble average vs. time average.

One can define the following time-averaged quantities for the i-th member of the ensemble:

$$\overline{x^{(i)}(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^{(i)}(t)dt \quad,$$

$$\text{(mean = first-order time average)} \tag{1.1}$$

$$\overline{x^{(i)}(t)^2} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[x^{(i)}(t)\right]^2 dt \quad,$$

$$\text{(mean square = second-order time average)} \tag{1.2}$$

$$\phi_x^{(i)}(\tau) \equiv \overline{x^{(i)}(t)x^{(i)}(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^{(i)}(t)x^{(i)}(t+\tau)dt \quad.$$

$$\text{(autocorrelation function)} \tag{1.3}$$

One can also define the following ensemble-averaged quantities for all members of the ensemble at a certain time:

$$\langle x(t_1)\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x^{(i)}(t_1) = \int_{-\infty}^{\infty} x_1 p_1(x_1,t_1)dx_1 \quad,$$

$$\text{(mean = first-order ensemble average)} \tag{1.4}$$

$$\langle x(t_1)^2\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left[x^{(i)}(t_1)\right]^2 = \int_{-\infty}^{\infty} x_1^2 p_1(x_1,t_1)dx_1 \quad,$$

$$\text{(mean square = second-order ensemble average)} \tag{1.5}$$

$$\langle x(t_1)x(t_2)\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x^{(i)}(t_1)x^{(i)}(t_2) \tag{1.6}$$

$$= \int_{-\infty}^{\infty} x_1 x_2 p_2(x_1, x_2; t_1, t_2)dx_1 dx_2 \quad .$$

$$\text{(covariance)}$$

Here, $x_1 = x(t_1)$, $x_2 = x(t_2)$, $p_1(x_1, t_1)$ is the first-order probability density function (PDF), and $p_2(x_1, x_2; t_1, t_2)$ is the second-order joint probability density function. $p_1(x_1, t_1)dx_1$ is the probability that x is found in the range between x_1 and $x_1 + dx_1$ at a time t_1 and $p_2(x_1, x_2; t_1, t_2)dx_1 dx_2$ is the probability that x is found in the range between x_1 and $x_1 + dx_1$ at a time t_1 and also in the range between x_2 and $x_2 + dx_2$ at a different time t_2.

An ensemble average is a convenient theoretical concept since it is directly related to the probability density functions, which can be generally obtained by the theoretical analysis of a given physical system. On the other hand, a time average is more directly related to real experiments. One cannot prepare an infinite number of identical systems in a real situation. Theoretical predictions based on ensemble averaging are equivalent to experimental measurement results corresponding to time averaging when, and only when, the system is a so-called "ergodic ensemble." It is often said that ensemble averaging and time averaging are identical for a statistically-stationary system, but are different for a statistically-nonstationary system. We will see those concepts next and show there is a subtle difference between ergodicity and statistical stationarity.

1.2 Statistically Stationary vs. Nonstationary Processes

If $\langle x(t_1)\rangle$ and $\langle x(t_1)^2\rangle$ are independent of the time t_1 and if $\langle x(t_1)x(t_2)\rangle$ is independent of absolute times t_1 and t_2 but dependent only on the time difference $\tau = t_2 - t_1$, such a noise process is called a "statistically-stationary" process. For a "statistically-nonstationary" process, the above is not true. In such a case, the concept of ensemble averaging is still valid, but the concept of time averaging fails.

The statistics of a stationary process do not change in time. To be more precise, we make the following definitions. A stochastic process is stationary of order k if the k-th order joint probability density function satisfies,

$$P(\alpha_1, \ldots, \alpha_k; t_1, \ldots, t_k) = P(\alpha_1, \ldots, \alpha_k; t_1 + \varepsilon, \ldots, t_k + \varepsilon) \text{ for all } \varepsilon \quad . \tag{1.7}$$

Thus, if $P_1(x; t_1) = P_1(x; t_1 + \varepsilon)$, the process is stationary of order 1. If $P_2(x_1, x_2; t_1, t_2) = P_2(x_1, x_2; t_1 + \varepsilon, t_2 + \varepsilon)$, the process is stationary of order 2.

Since there are several types of stationarity, some special terminology has arisen. A process is strictly stationary if it is stationary for any order, $k = 1, 2, \ldots$. A process is called *wide-sense* (or weakly) stationary if its mean value is constant and its autocorrelation function depends only on $\tau = t_2 - t_1$. Wide-sense stationary processes can be analyzed by the Wiener-Khinchine theorem of Fourier transform, which we will discuss shortly. If a process is wide-sense stationary, the autocorrelation function and the power spectral density function form a Fourier transform pair. Therefore, if we know—or can measure—the autocorrelation function, we can find the power spectral density function, i.e. which frequencies contain how much power in the signal.

The idea of ergodicity arises if we have only one sample function of a stochastic process, instead of the entire ensemble. A single sample function will often provide little information about the statistics of the process. However, if the process is ergodic, that is, time averages equal ensemble averages, then all statistical information can be derived from just one sample function.

When a process is ergodic, any one sample function represents the entire process. A little thought should convince you that the process must necessarily be stationary for this to occur. Thus ergodicity implies stationarity. There are levels of ergodicity, just as there are levels (degrees) of stationarity. We will discuss two levels of ergodicity; ergodicity in the mean and correlation.

Level 1. A process is ergodic in the mean if

$$\overline{x(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \langle x(t) \rangle \tag{1.8}$$

We can compute the left-hand side of Eq. (1.8) by first selecting a particular member function $x(t)$ and then averaging in time. To compute the right-hand side, we must know the first-order PDF $P_1(x;t)$. The left-hand side of Eq. (1.8) is independent of t. Hence the mean must be a constant value. Therefore, ergodicity of the mean implies stationarity of the mean. However, stationarity of the mean does not imply ergodicity of the mean, as our example below indicates.

Level 2. A process is ergodic in the autocorrelation if

$$\phi_x(\tau) = \overline{x(t)x(t+\tau)} \quad = \quad \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt$$
$$= \quad \langle x(t)x(t+\tau)\rangle \tag{1.9}$$

We can compute the left-hand side of Eq. (1.9) by using a particular function $x(t)$. To compute the right-hand side, we must know the second-order PDF $P_2(x_1, x_2; t_1, t_2)$.

EXAMPLE 1. Consider a basket full of batteries. There are some flashlight batteries, some car batteries, and several other kinds of batteries. Suppose that a battery is selected at random and its voltage is measured. This battery voltage $v(t)$ is a member function selected from a certain sub-group of constant battery voltages. This process is stationary but not ergodic in the mean or correlation. The time average is equal to the particular battery voltage selected (say, 1.5V). The statistical average is some other number, depending on what is in the basket. Thus Eq. (1.8) does not hold.

EXAMPLE 2. Let $x(t) = \sin(\omega t + \theta)$ be a member function from a stochastic process specified by a transformation of variables. Let θ be a random variable with uniform distribution over the interval $0 < \theta \leq 2\pi$.

$$P(\theta) = \frac{1}{2\pi}, \quad 0 < \theta \leq 2\pi \quad . \tag{1.10}$$

Then each θ determines a time function $x(t)$, which means that the stochastic process is specified by a transformation of variables. This stochastic process is ergodic in both the mean and autocorrelation. You can see that the time average of $x(t)$ is 0. The ensemble average at any one time is over an infinite variety of sinusoids of all phases, and so must also be 0. Since the time average equals the ensemble average, the process is ergodic in the mean. It is also true that Eq. (1.9) holds, so the process is ergodic in the correlation. For any other distribution of θ, the process is not stationary and hence not ergodic.

1.3 Basic Stochastic Processes

A noisy waveform $x(t)$ often consists of a very large number of random and discrete pulses, and are represented by

$$x(t) = \sum_{k=1}^{K} a_k f(t - t_k) \quad . \tag{1.11}$$

One assumes the pulse amplitude a_k and the time of pulse emission event t_k are random variables, but the pulse-shape function $f(t)$ is a fixed function, as shown in Fig. 1.2. In a real physical situation, $f(t)$ is often determined by an inherent property of a system, for example, by the relaxation time of a circuit or the transit time of a charged carrier. Therefore we assume here that $f(t)$ is a fixed function.

Next let us consider the characteristics of several stochastic processes, which such random variables a_k and t_k obey. It is convenient to use the probability density functions and moment generating functions for this purpose.
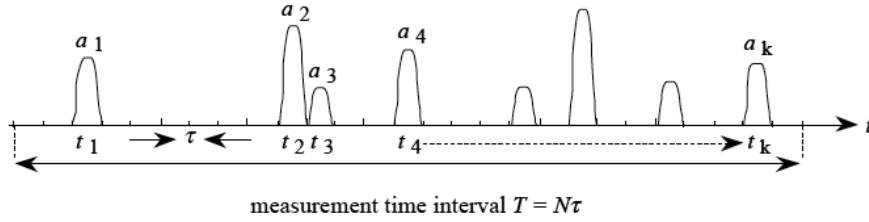
Figure 1.2: A random pulse train.

1.3.1 Probability density (or mass) functions and moment generating functions

The probability mass function $P(x)$ is defined for a discrete random variable x. Its discrete transform (or Z-transform) is defined by $P_x^T(Z) \equiv \sum_x Z^x P(x)$. It is straightforward to calculate the expectation values of any order moment by

$$\langle x \rangle = \frac{d}{dZ} P_x^T(Z)|_{Z=1} \quad , \tag{1.12}$$

$$\langle x^2 \rangle = \frac{d}{dZ^2} P_x^T(Z) + \frac{d}{dZ} P_x^T(Z)|_{Z=1} \quad , \tag{1.13}$$

and so on. Because of the above properties, the Z-transform is often referred to as a moment generating function.

If a random variable x takes a continuous value, we can define the probability density function $f(x)$. Its continuous transform (or s-transform) is defined by $f_x^T(s) \equiv \int_{-\infty}^{\infty} dx e^{-sx} f(x)$. We can obtain the expectation values of any order moment by

$$\langle x \rangle = -\frac{d}{ds} f_x^T(s)|_{s=0} \quad , \tag{1.14}$$

$$\langle x^2 \rangle = -\frac{d^2}{ds^2} f_x^T(s)|_{s=0} \quad , \tag{1.15}$$

and so on.

1.3.2 The Bernoulli Process

A. Bernoulli distribution

A single Bernoulli trial generates a discrete binary random nonnegative integer x, described by the probability mass function (PMF),

$$P_x(x_0) = \begin{cases} 1 - P & x_0 = 0 \\ P & x_0 = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.16}$$

Random variable x, described above, is known as a Bernoulli random variable. We define the z transform (or discrete transform) of the PMF as,

$$P_x{}^T(z) \equiv \sum_{x_0=0}^{\infty} z^{x_0} P_x(x_0) = z^0(1-P) + zP = 1 - P + zP \quad . \tag{1.17}$$

It is easily understood by the definition of the PMF and Z transform that the mean and mean-square are given by

$$\langle x \rangle = \left[\frac{d}{dz} P_x^T(z) \right]_{z=1} = \sum_{x_0} x_0 P_x(x_0) \quad , \tag{1.18}$$

$$\langle x^2 \rangle = \left[\frac{d^2}{dz^2} P_x^T(z) + \frac{d}{dz} P_x^T(z) \right]_{z=1} = \sum_{x_0} x_0^2 P_x(x_0) \quad . \tag{1.19}$$

By use of these relations and (1.17), we find the mean, mean-square and variance of the Bernoulli process:

$$\langle x \rangle = p, \qquad \langle x^2 \rangle = p, \qquad \sigma_x{}^2 \equiv \langle x^2 \rangle - \langle x \rangle^2 = p(1-p) \quad . \tag{1.20}$$

We refer to the outcome of a Bernoulli trial as a pulse emission when the experimental value of x is unity and as no emission when the experimental value of x is zero.

B. Binomial distribution

A Binomial distribution is obtained by a series of independent Bernoulli trials, each with the same probability of success. Suppose that n independent Bernoulli trials are to be performed, and define discrete random veriable k to be the number of successes in the n trials. Note that random variable k is the sum of n independent Bernoulli random variables, i.e. $k = x_1 + x_2 \cdots + x_n$, so the z transform of the PMF for the Bernoulli process is

$$P_k^T(z) \sum_{x_1 \cdots x_n} Z^{x_1 \cdots x_n} P_x(x_1) \cdots P_x(x_n) = [P_x^T(z)]^n = (1 - p + zp)^n \tag{1.21}$$

$$= P_{x_1}^T(Z) \cdots P_{x_1}^T(Z)$$

There are several ways to determine $P_k(k_0)$, the probability of exactly k_0 successes out of n independent Bernoulli trials. One way would be to apply the binomial theorem,

$$(a + b)^n = \sum_{l=0}^{n} \binom{n}{l} a^l b^{n-l} \quad , \tag{1.22}$$

to expand $P_k^T(z)$ in a power series,

$$P_k^T(z) = \sum_{l=0}^{n} \binom{n}{l} (zP)^l (1-P)^{n-l} \quad ,$$

and then compare the coefficients of z^{k_0} in the expansion for the definition of the z transform (1.17),

$$P_k^T(z) = \sum_{k_0=0}^{n} z^{k_0} P_k(k_0) = P_k(0) + z P_k(1) + z^2 P_k(2) + \cdots.$$

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This leads to the result known as the binomial PMF,

$$P_k(k_0) = \binom{n}{k_0} p^{k_0}(1-p)^{n-k_0} \quad , \quad k_0 = 0, 1, 2, \ldots, n \quad , \tag{1.23}$$

where

$$\binom{n}{k_0} = \frac{n!}{(n-k_0)!k_0!} \quad ,$$

as commonly used.

We can determine the expected value and variance of the binomial random variable k by any of the following three techniques. To evaluate $\langle k \rangle$ and $\sigma_k{}^2$ we may

1. perform the expected value summations directly,

2. use the moment-generating properties (1.18) and (1.19) of the z transform, or

3. recall that the expected value of a sum of random variables is always equal to the sum of their expected values and that the variance of a sum of linearly independent random variables is equal to the sum of their individual variances.

Since we know that a binomial random variable k is the sum of n independent Bernoulli random variables, the last of the above methods is the easiest and we obtain

$$\langle k \rangle = n\langle x \rangle = np \quad , \quad \sigma_k{}^2 = n\sigma_x{}^2 = np(1-p) \quad . \tag{1.24}$$

C. Geometric distribution

It is often convenient to refer to the successes in a Bernoulli process as pulse emission. Let a discrete random variable l_1 be the number of Bernoulli trials after any pulse emission and before the next pulse emission, including this pulse emission. The random variable l_1 is known as the first-order interarrival time of pulses, and it can take on the experimental values 1, 2, ... We begin by determining the PMF $P_{l_1}(l)$.

We shall determine $P_{l_1}(l)$ from a sequential sample space for the experiment of performing independent Bernoulli trials until we obtain our first success. Using the notation of the last section, we have

$$P_{l_1}(l) = p(1-p)^{l-1} \qquad l = 1, 2, \ldots \quad , \tag{1.25}$$

and since its successive terms decrease in a geometric progression, this PMF for the first-order interarrival times is known as the geometric PMF. The z transform for the geometric PMF is

$$P_{l_1}{}^T(z) = \sum_{l=1}^{\infty} P_{l_1}(l)z^l = \sum_{l=1}^{\infty} p(1-p)^{l-1}z^l = \frac{zp}{1-z(1-p)} \quad . \tag{1.26}$$

Since direct calculation of $\langle l_1 \rangle$ and $\sigma_{l_1}{}^2$ in an l_1 event space involves difficult summations, we shall use the moment-generating property of the z transform to evaluate these quantities.

$$\langle l_1 \rangle = \left[\frac{d}{dz} P_{l_1}{}^T(z) \right]_{z=1} = \frac{1}{p} \quad , \tag{1.27}$$

$$\sigma_{l_1}{}^2 = \left\{ \frac{d^2}{dz^2} P_{l_1}{}^T(z) + \frac{d}{dz} P_{l_1}{}^T(z) - \left[\frac{d}{dz} P_{l_1}{}^T(z) \right]^2 \right\}_{z=1} = \frac{1-p}{p^2} \quad . \tag{1.28}$$

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1.3.3   The Poisson Process

A. Poisson distribution

We defined the Bernoulli process by a particular probabilistic description of the "arrivals" of successes in a series of independent identical discrete trials. A Poisson process will be defined by a probabilistic description of the behavior of arrivals of successes at points on a continuous line.

For convenience, we shall generally refer to this line as a time (t) axis. By definition of the process, we shall see that a Poisson process may be considered to be in the limit of $\Delta t \to 0$ of a series of identical independent Bernoulli trials at intervals of Δt, with the probability of a success, $p = \lambda \Delta t$.

For our study of the Poisson process we shall adopt the probability that there are exactly k arrivals during any interval of duration t, $\wp(k,t)$. This notation is compact and particularly convenient for the types of equations to follow. We observe that $\wp(k,t)$ is a PMF for a random variable k for any fixed value of parameter t. In any interval of length t, with $t \geq 0$, we must have exactly zero, or exactly one, or exactly two, etc., arrivals of successes. Thus we have

$$\sum_{k=0}^{\infty} \wp(k,t) = 1 \quad . \tag{1.29}$$

We also note that $\wp(k,t)$ is not a probability density function (PDF) for t. Since $\wp(k,t_1)$ and $\wp(k,t_2)$ are not mutually exclusive events, we can state only that

$$0 \leq \int_0^{\infty} \wp(k,t)\, dt < \infty \quad . \tag{1.30}$$

The use of a random variable k to count arrivals is consistent with our notation for counting successes in a Bernoulli process.

There are several equivalent ways to define a Poisson process. We shall define it directly in terms of those properties which are most useful for the analysis of problems based on physical situations.

1. Any events defined on nonoverlapping time intervals are mutually independent.

2. The following statements are correct in the limit of $\Delta t \to 0$:

$$\wp(k, \Delta t) = \begin{cases} 1 - \lambda\Delta t & k = 0 \\ \lambda\Delta t & k = 1 \\ 0 & k > 1 \end{cases} . \qquad (1.31)$$

The first of the above two defining properties establishes the no-memory attribute of the Poisson process. The second defining property of the Poisson process states that, for small intervals, the probability of having exactly one arrival within one such interval is proportional to the duration of the interval and that, to the first order, the probability of more than one arrival within one such interval is zero. This simply means that $\wp(k, \Delta t)$ can be expanded in a Taylor series about $\Delta t = 0$, and when we neglect terms of order $(\Delta t)^2$ or higher, we obtain the given expressions for $\wp(k, \Delta t)$.

We wish to determine the expression for $\wp(k, t)$ for $t \geq 0$ and for k=0, 1, 2, …. Before doing mathematical derivation, let us reason out how we would expect the result to behave. By definition of the Poisson process and our interpretation of it as a series of Bernoulli trials in incremental intervals, we expect that

1. $\wp(0, t)$ as a function of t will be unity at t=0 and decrease monotonically toward zero as t increases. (The event of exactly zero arrivals in an interval of length t requires more and more successive failures in incremental intervals as t increases.)

2. $\wp(k, t)$ as a function of t, for $k > 0$, should start out at zero for t=0, increase for a while, and then decrease toward zero as t gets very large. [The probability of having exactly k arrivals (with $k > 0$) should be very small for intervals which are too long or too short.]

For a Poisson process, if Δt is small enough, we need to consider only the possibility of zero or one arrivals between t and $t + \Delta t$. Taking advantage also of the independence of events in nonoverlapping time intervals, we may write

$$\wp(k, t + \Delta t) = \wp(k, t)\wp(0, \Delta t) + \wp(k - 1, t)\wp(1, \Delta t) . \qquad (1.32)$$

The two terms summed on the right-hand side are the probabilities of the only two (mutually exclusive) histories of the process which may lead to having exactly k arrivals in an interval of duration $t + \Delta t$. Our definition of (1.31) for the process specified $\wp(0, \Delta t)$ and $\wp(1, \Delta t)$ for a small Δt. We substitute for these quantities to obtain,

$$\wp(k, t + \Delta t) = \wp(k, t)(1 - \lambda\Delta t) + \wp(k - 1, t)\lambda\Delta t . \qquad (1.33)$$

Collecting terms, dividing through by Δt, and taking the limit of $\Delta t \to 0$, we find

$$\frac{d}{dt}\wp(k, t) + \lambda\wp(k, t) = \lambda\wp(k - 1, t) . \qquad (1.34)$$

This may be solved iteratively for $k = 0$ and then for $k = 1$, and so on, with the initial conditions,

$$\wp(k, 0) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} . \qquad (1.35)$$

The solution for $\wp(k,t)$, which may be verified by direct substitution, is

$$\wp(k,t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad t \geq 0, \; k = 1, \, 2, \ldots \quad . \tag{1.36}$$

We find that $\wp(k,t)$ does have the properties we anticipated earlier, as shown in Fig. 1.3.

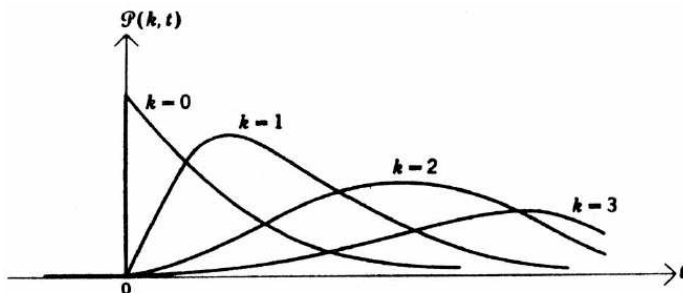
Figure 1.3: $\mathcal{P}(k,t)$ in a Poisson process.

Letting $\mu = \lambda t$, we may write this result in the more proper notation for a PMF as

$$P_k(k_0) = \frac{\mu^{k_0} e^{-\mu}}{k_0!} \qquad \mu = \lambda t, \; k_0 = 0, \, 1, \, 2, \ldots \quad . \tag{1.37}$$

This is known as the Poisson PMF. Although we derived the Poisson PMF by considering the number of arrivals in an interval of length t for a certain process, this PMF arises frequently in many other situations.

To obtain the mean value and variance of the Poisson PMF, we will use the z transform,

$$P_k{}^T(z) = \sum_{k_0=0}^{\infty} P_k(k_0) z^{k_0} = e^{-\mu} \sum_{k_0=0}^{\infty} \frac{(\mu z)^{k_0}}{k_0!} = e^{\mu(z-1)} \quad , \tag{1.38}$$

$$\langle k \rangle = \left[\frac{d}{dz} P_k{}^T(z) \right]_{z=1} = \mu \quad , \tag{1.39}$$

$$\sigma_k{}^2 = \left\{ \frac{d^2}{dz^2} P_k{}^T(z) + \frac{d}{dz} P_k{}^T(z) - \left[\frac{d}{dz} P_k{}^T(z) \right]^2 \right\}_{z=1} = \mu \quad . \tag{1.40}$$

Thus the mean value and variance of Poisson random variable k are both equal to μ.

We may also note that, since $\langle k \rangle = \lambda t$, we have an interpretation of the constant λ used in

$$\wp(k, \Delta t) = \begin{cases} 1 - \lambda \Delta t & k = 0 \\ \lambda \Delta t & k = 1 \\ 0 & k = 2, \, 3, \, \ldots \end{cases} \tag{1.41}$$

as part of the definition of the Poisson process. The relation $\langle k \rangle = \lambda t$ indicates that λ is the expected number of arrivals per unit time in a Poisson process. The constant λ is referred to as the average arrival rate for the process.

B. Erlang distribution

Let l_r be a continuous random variable defined to be an interval of time between any arrivals in a Poisson process and the r-th arrival after that. The continuous random variable l_r, the r-th order interarrival time, has the same interpretation here as the discrete random variable l_r had for the Bernoulli process.

We wish to determine the PDF's

$$f_{l_r}(l) \qquad l \geq 0; \quad r = 1, 2, 3, \ldots$$

For a small Δl we may write

$$\text{Prob}(l < l_r \leq l + \Delta l) = f_{l_r}(l)\Delta l \qquad , \tag{1.42}$$

$$f_{l_r}(l)\Delta l = \underbrace{\wp(r-1, l)}_{A} \underbrace{\lambda\Delta l}_{B} = \frac{(\lambda l)^{r-1}e^{-\lambda l}}{(r-1)!}\lambda\Delta l \qquad l \geq 0; \quad r = 1, 2, \ldots \quad , \tag{1.43}$$

where

A = probability that there are exactly $r - 1$ arrivals in an interval of duration l

B = conditional probability that rth arrival occurs in next Δl, given exactly $r-1$ arrivals in previous interval of duration l

Thus we have obtained the PDF for the rth-order interarrival time

$$f_{l_r}(l) = \frac{\lambda^r l^{r-1}e^{-\lambda l}}{(r-1)!} \qquad l \geq 0; \quad r = 1, 2, \ldots \quad , \tag{1.44}$$

which is known as the Erlang family of PDF's. Random variable l_r is said to be an Erlang random variable of order r.

The first-order interarrival time, described by random variable l_1, has the PDF

$$f_{l_1}(l) = \lambda e^{-\lambda l} \tag{1.45}$$

which is the exponential PDF. We may obtain its mean and variance by use of the s transform:

$$f_{l_1}{}^T(s) = \int_{-\infty}^{\infty} e^{-sl} f_{l_1}(l)dl = \frac{\lambda}{s+\lambda} \quad , \tag{1.46}$$

$$\langle l_1 \rangle = -\left[\frac{d}{ds} f_{l_1}{}^T(s)\right]_{s=0} = \frac{1}{\lambda} \quad , \tag{1.47}$$

$$\sigma_{l_1}{}^2 = \left\{\frac{d^2}{ds^2} f_{l_1}{}^T(s) - \left[\frac{d}{ds} f_{l_1}{}^T(s)\right]^2\right\}_{s=0} = \frac{1}{\lambda^2} \quad . \tag{1.48}$$

The random variable l_r is the sum of r independent experimental values of random variable l_1. Therefore we have

$$f_{l_r}{}^T(s) = \int dl_1 \cdots \int dl_r e^{-s(l_1+\cdots+l_r)} f_{l_1}(l) \cdots f_{l_r}(l)$$

$$= f_{l_1}^T(s) \cdots f_{l_r}^T(s)$$

$$= \left[f_{l_1}{}^T(s)\right]^r = \left(\frac{\lambda}{s+\lambda}\right)^r \quad ,$$

$$\langle l_r \rangle = r\langle l_1 \rangle = \frac{r}{\lambda} , \qquad \sigma_{l_r}{}^2 = r\sigma_{l_1}{}^2 = \frac{r}{\lambda^2} \quad .$$

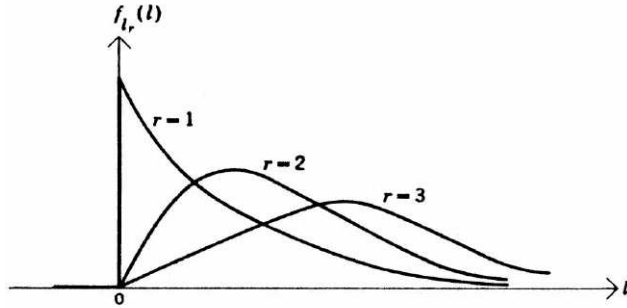Figure 1.4 shows a sketch of some members of Erlang family of PDF's.

Figure 1.4: $f_{l_r}(l)$ in a Poisson process.

C. Addition and random deletion of Poisson processes

Consider discrete random variable ω, the sum of two independent Poisson random variables x and y, with expected values $\langle x \rangle$ and $\langle y \rangle$. There are at least three ways to establish that $P_\omega(\omega_0)$ is also a Poisson PMF. One method involves direct summation in the x_0, y_0 event space. Or we may use z transforms as follows,

$$
\begin{aligned}
P_x{}^T(z) &= e^{\langle x \rangle (z-1)} \qquad P_y{}^T(z) = e^{\langle y \rangle (z-1)} \\
\omega &= x + y \qquad x, y \text{ independent} \\
P_\omega{}^T(z) &= P_x{}^T(z) P_y{}^T(z) = e^{(\langle x \rangle + \langle y \rangle)(z-1)} \quad ,
\end{aligned}
\tag{1.49}
$$

which we recognize to be the z transform of the Poisson PMF

$$
P_\omega(\omega_0) = \frac{(\langle x \rangle + \langle y \rangle)^{\omega_0} e^{-(\langle x \rangle + \langle y \rangle)}}{\omega_0!} \qquad \omega_0 = 0, 1, \ldots \quad ,
\tag{1.50}
$$

A third way would be to note that $\omega = x + y$ could represent the total number of arrivals for two independent Poisson processes within a certain interval. A new process which contains the arrivals due to both of the original processes would still satisfy our definition of the Poisson process with $\lambda = \lambda_1 + \lambda_2$ and would generate experimental values of random variable ω for the total number of arrivals within the given interval.

We have learned that the arrival process representing all the arrivals in several independent Poisson processes is also Poisson.

Furthermore, suppose that a new arrival process is formed by performing an independent Bernoulli trial for each arrival in a Poisson process. With probability p, any arrival in the Poisson process is also considered an arrival at the same time in the new process. With probability $1-p$, any particular arrival in the original process does not appear in the new process. The new process formed in this manner (by "independent random erasures") still satisfies the definition of a Poisson process and has an average arrival rate equal to λp and the expected value of the first-order interarrival time is equal to $(\lambda p)^{-1}$.

If the erasures are not independent, then the derived process has memory. For instance, if we erase alternate arrivals in a Poisson process, the remaining arrivals do not form a Poisson process. It is clear that the resulting process violates the definition of the Poisson process, since, given that an arrival in the new process just occurred, the probability of another arrival in the new process in the next Δt is zero (this would require two arrivals in Δt in the underlying Poisson process). This particular derived process is called an Erlang process since the first-order interarrival times are independent and have (second-order) Erland PDF's. This derived process is one example of how we can use the memoryless Poisson process to model more complicated situations with memory.

1.3.4 The Gaussian Process

A. Gaussian PDF

When the total number of trials n is very large and both the success and failure probabilities p and $1-p$ are not very close to zero, the binomial distribution (1.23) tends to exhibit a pronounced maximum at some value $k_0 = \tilde{k}_0$, and to decrease rapidly as one goes away from $\tilde{k}_0$. If n is large and we consider regions near the maximum of $P_k(k_0)$ where k_0 is also large, the fractional change in $P_k(k_0)$ when k_0 changes by unity is relatively small, i.e.

$$|P_k(k_0 + 1) - P_k(k_0)| \ll P_k(k_0) \quad . \tag{1.51}$$

Thus $P_k(k_0)$ can, to good approximation, be considered as a continuous function of the variable k_0, although only integral values of k_0 are of physical relevance. The location $k_0 = \tilde{k}_0$ of the maximum of P_k is then approximately determined by the condition

$$\frac{dP_k}{dk_0} = 0 \quad \text{or} \quad \frac{d \ln P_k}{dk_0} = 0 \quad , \tag{1.52}$$

where the derivatives are evaluated at $k_0 = \tilde{k}_0$. To evaluate the behavior of $P_k(k_0)$ near its maximum, we shall put

$$k_0 = \tilde{k}_0 + \eta \quad , \tag{1.53}$$

and expand $\ln P_k(k_0)$ in a Taylor's series about $\tilde{k}_0$. The reason for expanding $\ln P_k$, rather than P_k itself, is that $\ln P_k$ is a much more slowly varying function of k_0 than P_k. Thus the power series expansion for $\ln P_k$ should converge much more rapidly than the one for P_k.

Expanding $\ln P_k$ in Taylor's series, we obtain

$$\ln P_k(k_0) = \ln P_k(\tilde{k}_0) + B_1 \eta + \frac{1}{2} B_2 \eta^2 + \frac{1}{6} B_3 \eta^3 + \cdots \quad , \tag{1.54}$$

where

$$B_l = \frac{d^l \ln P_k}{dk_0{}^l} \quad , \tag{1.55}$$

is the l-th derivative of $\ln P_k$ evaluated at $\tilde{k}_0$. Since we are expanding about a maximum, $B_1 = 0$ by (1.52). Since P_k is a maximum, the term $\frac{1}{2} B_2 \eta^2$ must be negative. Let us write $B_2 = -|B_2|$, and we obtain

$$P_k(k_0) = \tilde{P}_k e^{-\frac{1}{2}|B_2|\eta^2} e^{\frac{1}{6} B_3 \eta^3} \cdots \quad . \tag{1.56}$$

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In the region where η is sufficiently small, higher-order terms in the expansion can be neglected, *i.e.* $B_3 = \cdots\cdots = 0$. By the binomial distribution (1.23), we obtain

$$\ln P_k(k_0) = \ln n! - \ln k_0! - \ln(n - k_0)! + k_0 \ln p + (n - k_0)\ln(1 - p) \quad . \tag{1.57}$$

If n is any large integer so that $n \gg 1$, then $\ln n!$ can be considered an almost continuous function of n, since $\ln n!$ changes only by a small fraction of itself if n is changed by a small integer. Here

$$\frac{d}{dn}\ln n! \simeq \frac{\ln(n+1)! - \ln n!}{(n+1) - n} \quad = \quad \ln(n+1)$$
$$\simeq \quad \ln n \quad . \tag{1.58}$$

Thus Eq. (1.57) yields

$$\frac{d}{dk_0}\ln P_k = -\ln k_0 + \ln(n - k_0) + \ln p - \ln(1 - p) \quad . \tag{1.59}$$

By equating this first derivative to zero, we find an expected result

$$\tilde{k}_0 = np \quad . \tag{1.60}$$

Further differentiation of (1.59) yields

$$\frac{d^2}{dk_0{}^2}\ln P_k = -\frac{1}{k_0} - \frac{1}{n - k_0} \quad . \tag{1.61}$$

Evaluating this for the value $k_0 = \tilde{k}_0$ given in (1.61), we get

$$B_2 = -\frac{1}{np(1 - p)} \quad . \tag{1.62}$$

The value of the constant $\tilde{P}_k$ in (1.56) can be determined from the normalization condition $\sum_{k_0=1}^{\infty} P_k(k_0) = 1$. Since P_k and k_0 can be considered as continuous variables, the sum over all integral values of k_0 can be approximately replaced by an integral. Thus the normalization condition can be written

$$\int_{-\infty}^{\infty} P_k(k_0)dk_0 = \tilde{P}_k \int_{-\infty}^{\infty} e^{-\frac{1}{2}|B_2|\eta^2} d\eta = \tilde{P}_k \sqrt{\frac{2\pi}{|B_2|}} = 1 \quad . \tag{1.63}$$

The final expression for $P_k(k_0)$ is thus given by

$$P_k(k_0) = \frac{1}{\sqrt{2\pi\sigma_{k_0}^2}} \exp\left[-\frac{(k_0 - \tilde{k}_0)^2}{2\sigma_{k_0}^2}\right] \quad , \tag{1.64}$$

where

$$\sigma_{k_0}^2 = np(1 - p) \quad . \tag{1.65}$$

This is the so-called Gaussian distribution. The Gaussian distribution is very general in nature and occur very frequently in statistical mechanics whenever we are dealing with large numbers of particles.

B. Gaussian s-transform

The s-transform of the Gaussian PDF (1.64) is written as

$$
\begin{aligned}
f_{k_0}^T(s) &= \int_{-\infty}^{\infty} dk_0 e^{-sk_0} p_k(k_0) \\
&= e^{-s\tilde{k}_0 + \frac{1}{2}s^2 \sigma_{k_0}^2}.
\end{aligned}
\tag{1.66}
$$

Using the moment generating properties (1.18) and (1.19), we find the mean and variance are identical to $\tilde{k}_0$ and $\sigma_{k_0}^2$ as expected.

1.4 Burgess Variance Theorem

In the discussion of the Bernoulli process, we start with a fixed (constant) number of trials n and introduce random deletion with the probability $1 - p$. The variance of the output event (1.24) is a rather general result for such a stochastic process and the fluctuation associated with such random deletion is referred to as "partition noise." In some cases, the total number of trials n itself fluctuates. In such a case, the probability of obtaining k_0 successes is given by

$$
P(k_0) = \sum_{n=k_0}^{\infty} W(n) P_k(k_0) \quad , \tag{1.67}
$$

where $W(n)$ is the distribution (PMF) of the total number of trials and $P_k(k_0)$ is the binomial distribution of obtaining k_0 successes out of n trials. The mean and mean-square of k_0 can be evaluated by using (1.24) and (1.67),

$$
\begin{aligned}
\langle k_0 \rangle \equiv \sum_{k_0=0}^{\infty} k_0 P(k_0) &= \sum_{n=0}^{\infty} \sum_{k_0=0}^{n} k_0 P_k(k_0) W(n) \\
&= \sum_{n=0}^{\infty} p n W(n) \\
&= p \langle n \rangle \quad ,
\end{aligned}
\tag{1.68}
$$

$$
\begin{aligned}
\langle k_0{}^2 \rangle \equiv \sum_{k_0=0}^{\infty} k_0{}^2 P(k_0) &= \sum_{n=0}^{\infty} \sum_{k_0=0}^{n} k_0{}^2 P_k(k_0) W(n) \\
&= \sum_{n=0}^{\infty} \left[(pn)^2 + p(1-p)n \right] W(n) \\
&= p^2 \langle n^2 \rangle + p(1-p) \langle n \rangle \quad .
\end{aligned}
\tag{1.69}
$$

The variance of the number of successful events is given by

$$
\sigma_{k_0}^2 = \langle k_0^2 \rangle - \langle k_0 \rangle^2 = \sigma_n^2 p^2 + \langle n \rangle p(1-p) \quad , \tag{1.70}
$$

where $\sigma_n^2 = \langle n^2 \rangle - \langle n \rangle^2$. The above relation is known as the "Burgess variance theorem". The first term on the right-hand side of (1.70) indicates that the fluctuation of the initial number of trials is suppressed by a "loss" probability p^2. The second term, on the other

hand, indicates the new noise term introduced by a random deletion process, which is called a partition noise.

Figure 1.5 shows the change of the variance due to random deletion process. If the initial distribution $W(n)$ is a Poisson distribution, the distribution $P(k_0)$ stays as a Poisson distribution with arbitrary deletion, as stated already. The variance is always equal to the mean value, $i.e.$ $\sigma_{k_0}^2 = \langle k_0 \rangle$. However, if the initial distribution $W(n)$ has a larger or smaller variance than a Poisson distribution, the Poisson limit ($\sigma_{k_0}^2 = \langle k_0 \rangle$) is obtained only at a very large deletion limit, as shown in Fig. 1.5.
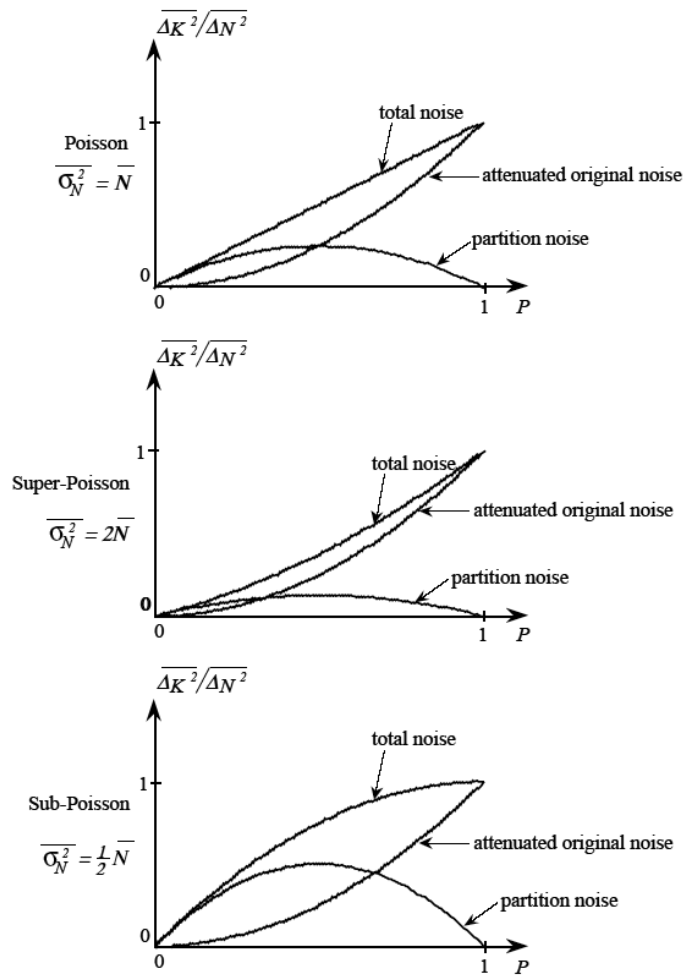
Figure 1.5: The change of the variance for a Poisson ($\sigma_N^2 = \overline{N}$), super-Poisson ($\sigma_N^2 = 2\overline{N}$) and sub-Poisson ($\sigma_N^2 = \frac{1}{2}\overline{N}$) distributions due to a partition process.

1.5 Fourier Analysis

When $x(t)$ is absolutely integrable, *i.e.*,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \quad , \tag{1.71}$$

the Fourier transform of $x(t)$ exists and is defined by

$$X(i\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \quad . \tag{1.72}$$

The inverse transform is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(i\omega) e^{i\omega t} d\omega \quad . \tag{1.73}$$

This inverse relation is proven by substituting for $X(i\omega)$ from (1.72), and interchanging the order of integration to obtain

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} X(i\omega) e^{i\omega t} d\omega &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \int_{-\infty}^{\infty} x(t') e^{-i\omega t'} dt' \\
&= \int_{-\infty}^{\infty} x(t') \delta(t' - t) dt' \\
&= x(t) \quad ,
\end{aligned} \tag{1.74}$$

where we use

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} = \delta(t' - t) \quad .$$

When $x(t)$ is a real function of time, as it always is the case for an "observable" waveform, the real part of $X(i\omega)$ is an even function of ω and the imaginary part is an odd function of ω [*i.e.*, $X(i\omega) = X^*(-i\omega)$].

When $x(t)$ is a statistically-stationary process, condition (1.71) is not satisfied and thus the Fourier transform cannot be defined. The total energy of the noisy waveform $x(t)$ is infinite, but in any practical noise measurement, a measurement time interval T is finite and the energy of such a gated function $x_T(t)$, defined by

$$x_T(t) = \begin{cases} x(t) & |t| \le \frac{T}{2} \\ 0 & |t| > \frac{T}{2} \end{cases} \quad , \tag{1.75}$$

is also finite. The Fourier transform of such a gated function "does" exist.

1.5.1 Parseval theorem

If $x_1(t)$ and $x_2(t)$ have Fourier transforms $X_1(i\omega)$ and $X_2(i\omega)$, one obtains

$$\begin{aligned}
\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt &= \int_{-\infty}^{\infty} dt x_1(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega X_2(i\omega)^* e^{-i\omega t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega X_2(i\omega)^* \int_{-\infty}^{\infty} dt x_1(t) e^{-i\omega t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(i\omega) X_2^*(i\omega) d\omega \quad .
\end{aligned} \tag{1.76}$$

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This relation is known as the Parseval theorem. If one uses $x_1(t) = x_T(t + \tau)$ and $x_2(t) = x_T(t)$ in (1.76), one obtains

$$\int_{-\infty}^{\infty} x_T(t + \tau)x_T(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(i\omega)|^2 e^{i\omega\tau} d\omega \quad , \tag{1.77}$$

where $\int_{-\infty}^{\infty} x_T(t + \tau)e^{-i\omega t}dt = X_T(i\omega)e^{i\omega\tau}$ is used. When $\tau = 0$, (1.77) is reduced to

$$\int_{-\infty}^{\infty} [x_T(t)]^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(i\omega)|^2 d\omega \quad . \tag{1.78}$$

The physical interpretation of $X_T(i\omega)$ and $|X_T(i\omega)|^2$ is now clear from the above relations. $X_T(i\omega)$ is the (complex) amplitude of the harmonic ($e^{i\omega t}$) component in a gated function $x_T(t)$ and $|X_T(i\omega)|^2$ is the energy density of this harmonic component with units of energy per Hz. Equation (1.78) is the total energy of $x_T(t)$ and increases linearly with T for a statistically-stationary process.

1.5.2 Power spectral density and Wiener-Khintchine theorem

The average power of $x_T(t)$, defined by

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} [x_T(t)]^2 dt = \lim_{T\to\infty} \frac{1}{2\pi} \int_{0}^{\infty} \frac{2|X_T(i\omega)|^2}{T} d\omega \quad , \tag{1.79}$$

is independent of T and a constant universal quantity, if $x(t)$ is a statistically stationary process. However, if $x(t)$ is a statistically nonstationary process, the average power is dependent of T and we are not allowed to take the limit of $T \to \infty$. If ensemble averaging is taken first for many identical gated functions $x_T(t)$ in (1.79), the order of $\lim_{T\to\infty}$ and $\int_0^\infty d\omega$ can be interchanged. In this way, the power spectral density is defined as

$$S_x(\omega) = \lim_{T\to\infty} \frac{2\langle |X_T(i\omega)|^2 \rangle}{T} \quad .$$
$$\text{(unilateral power spectral density)} \tag{1.80}$$

Note that the power spectral density is an ensemble averaged quantity and has the different form for a stationary and nonstationary process.

When $\tau \neq 0$ in (1.77), one can also divide both sides of (1.77) by T, take an ensemble average, and take a limit of $T \to \infty$ to obtain,

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \langle x_T(t+\tau)x_T(t)\rangle dt = \lim_{T\to\infty} \frac{1}{2\pi} \int_{0}^{\infty} \frac{2\langle |X(i\omega)|^2 \rangle}{T} \cos \omega\tau \, d\omega \quad . \tag{1.81}$$

The left-hand side of this expression is the ensemble averaged autocorrelation function $\phi_x(\tau)$. Using (1.80) in the right-hand side of this expression, one obtains

$$\phi_x(\tau) = \frac{1}{2\pi} \int_{0}^{\infty} S_x(\omega) \cos \omega\tau \, d\omega \quad . \tag{1.82}$$

The inverse relation of this expression is

$$\begin{aligned} 4 \int_{0}^{\infty} \phi_x(\tau) \cos \omega\tau \, d\tau &= \frac{2}{\pi} \int_{0}^{\infty} d\omega' S_x(\omega') \int_{0}^{\infty} d\tau \cos(\omega\tau)\cos(\omega'\tau) \\ &= \int_{0}^{\infty} d\omega' S_x(\omega')[\delta(\omega + \omega') + \delta(\omega - \omega')] \\ &= S_x(\omega) \quad . \end{aligned} \tag{1.83}$$

Here we use the relation,

$$\int_0^\infty d\tau \cos(\omega\tau)\cos(\omega'\tau) = \frac{\pi}{2}[\delta(\omega + \omega') + \delta(\omega - \omega')] \quad . \tag{1.84}$$

Equations (1.82) and (1.83) constitute the Wiener-Khintchine theorem and indicate that $2\phi_x(\tau)$ and $S_x(\omega)$ are the Fourier transform pairs.

If a noisy waveform $x(t)$ is a nonstationary process, we cannot take a limit as $T \to \infty$ in (1.81). The Wiener-Khintchine theorem for such a case is given by

$$\phi_x(\tau, T) = \frac{1}{2\pi}\int_0^\infty S_x(\omega, T)\cos(\omega\tau)d\omega \quad , \tag{1.85}$$

$$S_x(\omega, T) = 4\int_0^T \phi_x(\tau, T)\cos(\omega\tau)d\tau \quad . \tag{1.86}$$

1.5.3 Examples

Let us consider a few examples for demonstrating how to use the Wiener-Khintchine theorem.

EXAMPLE 1. Suppose a noisy waveform $x(t)$ is a statistically-stationary process, as shown in Fig. 1.6, with an exponentially decaying autocorrelation function

$$\phi_x(\tau) = \phi_x(0)\exp\left(-\frac{|\tau|}{\tau_1}\right) \quad , \tag{1.87}$$

where $\phi_x(0) = \langle x^2 \rangle$ by definition and τ_1 is a relaxation time constant which is a system's memory time. Substituting (1.87) into (1.83), one obtains the unilateral power spectral density

$$S_x(\omega) = \frac{4\phi_x(0)\tau_1}{1 + \omega^2\tau_1^2} \quad . \tag{1.88}$$

The spectrum is Lorentzian with a cut-off frequency of $\omega_c = 1/\tau_1$ and the low-frequency spectral density is $S_x(\omega = 0) = 4\phi_x(0)\tau_1$. The autocorrelation function and the unilateral power spectrum are shown in Fig. 1.7.

EXAMPLE 2. A time-integrated function $y(t) = \int_0^t x(t')\, dt'$ of a statistically-stationary process $x(t')$ goes through a random walk diffusion, as shown in Fig. 1.6. If $x(t)$ has an infinitesimally short correlation time, $\tau_1 \to 0$, its time-integrated waveform $y(t)$ is called a Wiener-Levy process and is a classic example of a statistically-nonstationary process. Let us define a gated function by

$$y(t) = \begin{cases} \int_0^t x(t')\, dt' & (0 \le t \le T) \\ 0 & (\text{otherwise}) \end{cases} \quad . \tag{1.89}$$

If $x(t)$ is a stationary noisy waveform with a finite memory time, we first have to evaluate the covariance function,

$$\langle y(t)y(t + \tau)\rangle = \int_0^t \int_0^{t+\tau} \langle x(t')x(t'')\rangle dt'dt'' \quad , \tag{1.90}$$
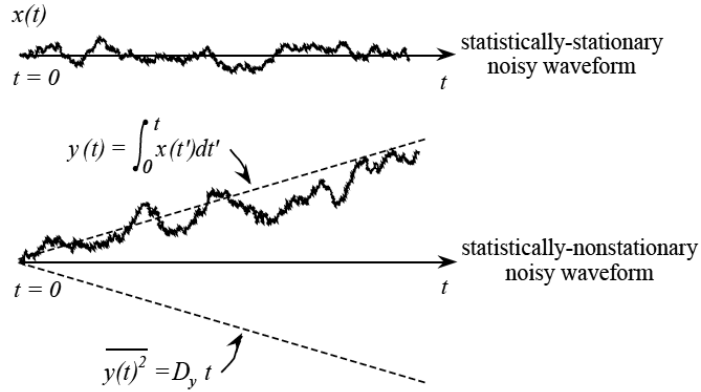
21

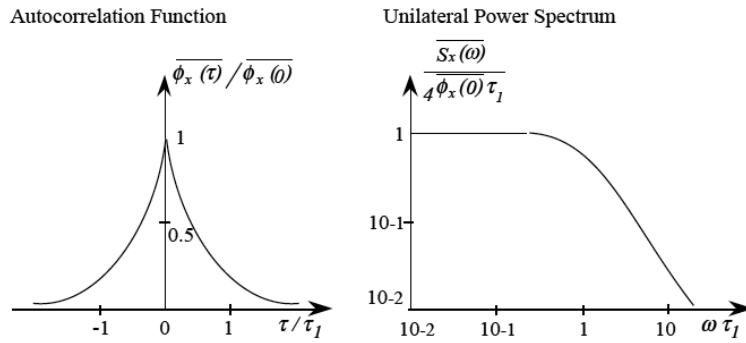Figure 1.6: Stationary vs. nonstationary noisy waveforms.

Figure 1.7: The autocorrelation function and unilateral power spectrum of a stationary noisy waveform.

to obtain the power spectrum of $y(t) = \int_0^t x(t')dt'$. If $x(t)$ is ergodic, the covariance $\langle x(t')x(t'')\rangle$ can be replaced by the autocorrelation $\phi_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t+\tau)x(t)dt$. Using the Wiener-Khintchine theorem (1.85) in (1.90), we have

$$
\begin{aligned}
\langle y(t)y(t+\tau)\rangle &= \int_0^t \int_0^{t+\tau} dt'dt'' \frac{1}{2\pi}\int_0^\infty S_x(\omega)\cos(\omega\tau)d\omega \\
&= \frac{1}{2\pi}\int_0^\infty S_x(\omega)\int_0^t \int_0^{t+\tau} \cos\omega(t'-t'')dt'dt'' \\
&= \frac{1}{2\pi}\int_0^\infty S_x(\omega)\frac{1}{\omega^2}\left[1+\cos(\omega\tau)-\cos(\omega t)-\cos\left(\omega(t+\tau)\right)\right]d\omega \quad .
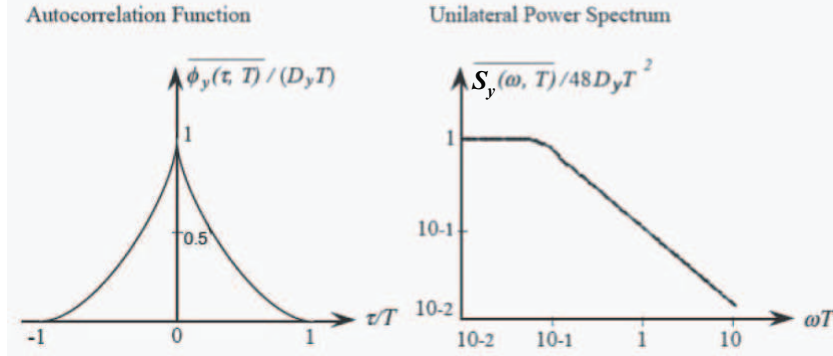\end{aligned}
$$

(1.91)

Figure 1.8: The autocorrelation function and unilateral power spectrum of a nonstationary noisy waveform $y(t)$.

The mean-square of $y(t)$ is now evaluated as

$$\langle y(t)^2 \rangle = \frac{1}{\pi} \int_0^\infty S_x(\omega) \frac{1}{\omega^2} [1 - \cos(\omega t)] \, d\omega \quad . \tag{1.92}$$

The relations given by (1.91) and (1.92) are called MacDonald's functions.

When the memory time of $x(t)$ becomes infinitesimally short, $S_x(\omega)$ is independent of ω, *i.e.* white noise. Then the mean-square of $y(t)$ is reduced to

$$
\begin{aligned}
\langle y(t)^2 \rangle &= \frac{S_x(\omega = 0)}{\pi} \int_0^\infty \frac{1}{\omega^2} [1 - \cos(\omega t)] \, d\omega \\
&= \frac{S_x(\omega = 0)}{2} t \quad .
\end{aligned}
\tag{1.93}
$$

Here the mathematical identity, $\lim_{a \to 0} \int_0^\infty \frac{1 - \cos(\omega t)}{\omega^2 + a^2} d\omega = \frac{\pi}{2} t$ is used. The diffusion constant D_y of the Wiener-Levy process appeared in (1.95) is thus related to the power spectral density of $x(t)$ at $\omega = 0$,

$$D_y = \frac{S_x(\omega = 0)}{4} \quad . \tag{1.94}$$

The corresponding autocorrelation function for y is calculated as

$$
\begin{aligned}
\phi_y(\tau, T) &= \frac{1}{T} \int_0^{T - |\tau|} \langle y(t + \tau) y(t) \rangle \, dt \\
&= T \left(1 - \frac{|\tau|}{T} \right)^2 D_y \quad .
\end{aligned}
\tag{1.95}
$$

In order to derive the second line of (1.95), the fact was used that $y(t)$ is a cumulative process of a memoryless noisy waveform and thus $\langle y(t + \tau) y(t) \rangle = \langle [y(t) + \Delta y(\tau)] y(t) \rangle = \langle y(t)^2 \rangle = 2D_y t$, where D_y is a diffusion constant. It is assumed that $\langle y(t) \Delta y(\tau) \rangle = 0$

because of the assumed zero correlation time for $x(t)$. The unilateral power spectral density is given by

$$
\begin{aligned}
S_y(\omega, T) &= 4 \int_0^T \phi_y(\tau, T) \cos(\omega\tau) \, d\tau \\
&= \frac{8D_y}{\omega^2} \left[1 - \frac{\sin(\omega T)}{\omega T} \right] \quad .
\end{aligned}
\tag{1.96}
$$

The correlation time is now proportional to the measurement time interval T as shown in Fig. 1.8. A finite measurement time T prevents the divergence of the power spectral density at $\omega = 0$, as shown in Fig. 1.8.

1.5.4 Cross-correlation

If we are interested in the correlation between two noisy waveforms $x(t)$ and $y(t)$, we can evaluate such cross-correlation by the following formula. The cross-correlation function and cross-spectral density are defined by

$$
\phi_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \langle x(t+\tau)y(t) \rangle dt \quad ,
\tag{1.97}
$$

$$
S_{xy}(\omega) = \lim_{T \to \infty} \frac{2 \langle X(i\omega)Y(i\omega)^* \rangle}{T} = S_{yx}(\omega)^* \quad .
\tag{1.98}
$$

Substituting $x_1(t) = x(t+\tau)$, $x_2(t) = y(t)$, $X_1(i\omega) = X(i\omega)e^{i\omega\tau}$ and $X_2^*(i\omega) = Y^*(i\omega)$ in the Parseval theorem (1.76), we obtain

$$
\phi_{xy}(\tau) = \frac{1}{4\pi} \int_{-\infty}^{\infty} S_{xy}(\omega)e^{i\omega\tau} d\omega \quad ,
\tag{1.99}
$$

$$
S_{xy}(\omega) = 2 \int_{-\infty}^{\infty} \phi_{xy}(\tau)e^{-i\omega\tau} dt \quad .
\tag{1.100}
$$

Equations (1.99) and (1.100) are often called generalized Wiener-Khintchine theorem. The degree of cross-correlation between $x(t)$ and $y(t)$ is often evaluated by the coherence function defined by

$$
\Gamma_{xy}(\omega) = \frac{S_{xy}(\omega)}{\left[S_{xx}(\omega) \ S_{yy}(\omega) \right]^{\frac{1}{2}}} \quad ,
\tag{1.101}
$$

where $S_{xx}(\omega)$ and $S_{yy}(\omega)$ are the power spectral density of $x(t)$ and $y(t)$, respectively. Note that $\Gamma_{xy}(\omega)$ is a c-number, so that it includes both correlation amplitude and relative phase.

1.6 Random pulse train

1.6.1 Carson's Theorem

The Fourier transform of a random pulse train (1.11) is now calculated as

$$
X(i\omega) = F(i\omega) \sum_{k=1}^{K} a_k \ e^{-i\omega t_k} \quad .
\tag{1.102}
$$

The unilateral power spectral density for such a random pulse train is given by

$$S_x(\omega) = \lim_{T \to \infty} \frac{2\langle |X(i\omega)|^2 \rangle}{T} = \lim_{T \to \infty} \frac{2|F(i\omega)|^2}{T} \sum_{k,m=1}^{K} \langle a_k a_m \exp\left[-i\omega(t_k - t_m)\right]\rangle \quad . \quad (1.103)$$

The summation in (1.103) over k and m can be split into the summation for $k = m$ and for $k \neq m$,

$$S_x(\omega) = \lim_{T \to \infty} \frac{2|F(i\omega)|^2}{T} \left\{ \sum_{k=1}^{K} \left\langle a_k^2 \right\rangle + \sum_{k \neq m} \langle a_k a_m \exp\left[-i\omega(t_k - t_m)\right]\rangle \right\} \quad . \quad (1.104)$$

Suppose $\nu = \lim_{T \to \infty} \frac{K}{T}$ is the average rate of pulse emission and $\langle a^2 \rangle = \lim_{T \to \infty} \frac{1}{K}\sum_{k=1}^{K}\langle a_k^2 \rangle$ is the mean-square of the pulse amplitude. Then the first term of the right-hand side of (1.104) is expressed by $2\nu\langle a^2 \rangle |F(i\omega)|^2$. If we assume that different pulse emission events are completely independent, the second term of the right-hand side of (1.104) can be evaluated

$$\lim_{T \to \infty} \frac{2|F(i\omega)|^2}{T} \sum_{k \neq m} \langle a_k \rangle \langle a_m \rangle \langle e^{-i\omega t_k} \rangle \langle e^{i\omega t_m} \rangle = \lim_{T \to \infty} \frac{2|F(i\omega)|^2}{T} \sum_{k \neq m} \langle a \rangle^2 \frac{4\sin^2\left(\frac{\omega T}{2}\right)}{\omega^2 T^2}$$

$$= 4\pi \, \overline{x(t)}^2 \delta(\omega) \quad . \quad (1.105)$$

Here the mean of the noisy waveform $x(t)$ is

$$\overline{x(t)} = \nu\langle a \rangle \int_{-\infty}^{\infty} f(t)dt \quad , \quad (1.106)$$

and $\langle a \rangle = \lim_{T \to \infty} \frac{1}{K}\sum_{k=1}^{K} a_k$ is the mean of the pulse amplitude. The second equality in (1.105) is obtained by identifying $F(\omega = 0)$ with $\int_{-\infty}^{\infty} f(t)dt$, and replacing $\lim_{T \to \infty} \frac{2\sin^2(\omega T/2)}{\omega^2 T}$ with $\pi\delta(\omega)$. For a symmetric distribution of a_k about zero, (1.105) is zero because $\langle a \rangle$ is zero. However, when a_k are not symmetrically distributed about zero, the dc term appears in the power spectral density. Final result is

$$S_x(\omega) = 2\nu\langle a^2 \rangle |F(i\omega)|^2 + 4\pi\overline{x(t)}^2 \delta(\omega)$$
$$\text{(Carson's theorem)} \quad . \quad (1.107)$$

This is the Carson theorem.

1.6.2 Campbell's theorem

From the Wiener-Khintchine theorem, the autocorrelation function

$$
\begin{aligned}
\Phi_x(\tau) &= \frac{1}{2\pi} \int_0^{\infty} S_x(\omega)\cos(\omega\tau)dw \\
&= \frac{\nu\langle a^2 \rangle}{\pi} \int_0^{\infty} |F(i\omega)|^2 \cos\omega\tau \, d\omega + 2\overline{x(t)}^2 \int_0^{\infty} \delta(\omega)\cos\omega\tau \, d\omega \\
&= \nu\langle a^2 \rangle \int_{-\infty}^{\infty} f(t)f(t+\tau)dt + \overline{x(t)}^2 \quad ,
\end{aligned}
\quad (1.108)
$$

where the Parseval theorem (1.76) and $\int_0^\infty \delta(\omega)\cos(\omega\tau)d\omega = \frac{1}{2}$ are used to derive the third line. Since $\phi_x(\tau = 0) = \overline{x(t)^2}$, one obtains

$$
\begin{aligned}
\overline{x(t)^2} - \overline{x(t)}^2 &= \nu\langle a^2\rangle \int_{-\infty}^{\infty} [f(t)]^2 dt \\
&= \frac{\nu\langle a^2\rangle}{\pi} \int_0^\infty |F(i\omega)|^2 d\omega \quad ,
\end{aligned}
\tag{1.109}
$$

where the energy theorem (1.78) is used to derive the second line. This is the Campbell's theorem of mean square. On the other hand, the mean value is calculated by

$$
\overline{x(t)} = \nu\langle a\rangle \int_{-\infty}^{\infty} f(t)dt = \nu\langle a\rangle F(\omega = 0) \quad .
\tag{1.110}
$$

This is the Campbell's theorem of mean.

1.7   Shot noise in a vacuum diode

As an application of the Carson theorem, the current noise of a vacuum diode is calculated in this section.

1.7.1   Ramo theorem

Suppose an electron is emitted from the cathode and is in transit to the anode in a vacuum diode shown in Fig. 1.9 Assume the source resistance, Rs, is zero. We show that the external short-circuit current due to this moving charge carrier is given by

$$
i(t) = \frac{qv}{d},
\tag{1.111}
$$

where q is the electron charge, v is the electron drift velocity, and d is the distance between the two electrodes.
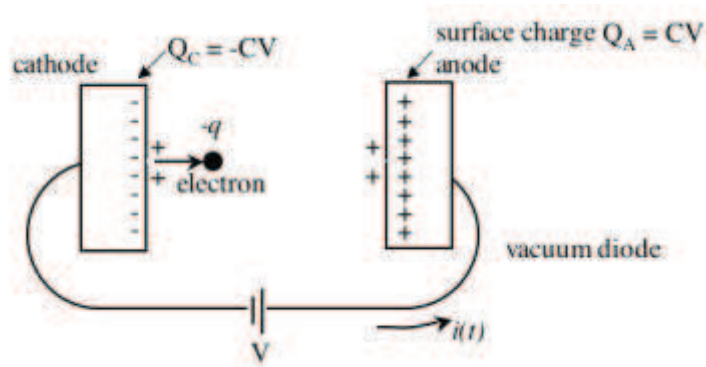

Figure 1.9: A vacuum diode.

In the short-circuit limit $(R_S = 0)$, as the electron moves from time $t' = 0$ to t, the energy the electron gains is given by:

$$
U' = \int_0^t dt' \vec{F} \cdot \vec{v} = -q \int_0^t dt'(-Ev) = q \int_0^t dt' Ev,
\tag{1.112}
$$

where $F = qE$ is an external force acting on the electron and the electric field $\vec{E}$ is anti-parallel to $\vec{v}$. If the current in the external circuit is $i(t)$, the total energy supplied by the external voltage source is

$$U'' = \int_0^t dt' V(t') i(t') = \int_0^t dt' V i(t'),$$ (1.113)

where $V(t) = V$ is constant. Since $E = V/d$ in the tube, from $U' = U''$ we obtain:

$$\int_0^t dt' qEv = \int_0^t dt' E di(t').$$ (1.114)

Therefore we have Eq.(1.111). This is called the Ramo theorem.

1.7.2 External Circuit Current

We consider the case that the external circuit has a finite source resistance,$R_s \neq 0$, and the circuit relaxation time, $\tau_c = R_s C$, is much longer than the electron transit time, τ_t, where C is the capacitance of the vacuum diode.

For $\tau_t = d/v \ll \tau_{CR} = R_s C$, the voltage developing due to the electron transit event occurs "instantly," whereas the relaxation through the external circuit is very slow. Immediately following the electron transit, the voltage across the vacuum diode is $V - q/C$, i.e., the voltage at anode is $V_A(t) = V - q/C$ at $t = 0$. Using Kirchoff's law, and noting that a current from battery to anode must be equal to a change in the surface charge, we have

$$\frac{V - V_A(t)}{R_s} = \frac{d}{dt}(C V_A(t)).$$ (1.115)

We rewrite (6) as

$$\frac{d}{dt} V_A(t) = -\frac{V_A(t)}{R_s C} + \frac{V}{R_s C},$$ (1.116)

and obtain the solution with the initial condition at $t = 0$ as

$$V_A(t) = V - \frac{q}{C} e^{-t/R_s C}.$$ (1.117)

The current in the external circuit is then

$$i(t) = \frac{V - V_A}{R_s} = \frac{q}{R_s C} e^{-t/R_s C}.$$ (1.118)

1.7.3 Surface Charge

We now calculate the surface charges of the cathode and the anode as a function of time for a single-electron traversal process in the following three cases:

(I) The electron drift velocity is assumed to be constant over the electron's transit from the cathode to the anode, and $\tau_c \ll \tau_t$.

(II) The electron drift velocity is initially zero at the cathode and is accelerated by the constant applied electric field, and $\tau_c \ll \tau_t$.

(III) $\tau_c \gg \tau_t$. In this case, we assume the electron transit to be an impulsive event.

(I) $\tau_c \ll \tau_t (R_s = 0)$ limit, constant v

Since there is a voltage of V across the vacuum diode, there is a surface charge of CV on the anode and $-CV$ on the cathode. When an electron with charge $-q$ is emitted from the cathode, it induces a net charge of $+q$ on the cathode. Over the time, d/V, this charge is compensated by the current supplied from the external circuit. The surface charge on the cathode is:

$$Q_c(t) = -CV + q - \int_0^t dt' i(t'). \tag{1.119}$$

We perform the integration and obtain,

$$Q_c(t) = \begin{cases} -CV + q(1 - \frac{v}{d}t) & 0 < t < \frac{d}{v}. \\ -CV & \text{otherwise} \end{cases} \tag{1.120}$$

The surface charge on the anode starts increasing by $+q$ over the time d/v, due to the external current, from its $t = 0$ value of CV. Then, it is compensated for by the electron from the cathode. The surface charge on the anode is,

$$Q_A(t) = CV + \int_0^t dt' i(t') = \begin{cases} CV + q\frac{v}{d}t & 0 < t < \frac{d}{v}. \\ CV & \text{otherwise} \end{cases} \tag{1.121}$$

Since the external voltage source supplies an external current (without delay) to keep up with the change inside the diode, the voltage across the diode is kept constant.

(II) $\tau_c \ll \tau_t (R = 0)$ limit, accelerated ν

Now we allow the electron to be accelerated by the electric field. The electron acquires a velocity,

$$\nu(t) = \frac{1}{m}p(t)\frac{1}{m}\int_0^t dt' F(t') = \frac{qE}{m}t. \tag{1.122}$$

The transit time across the vacuum diode is,

$$\frac{dr}{dt} = v(t). \tag{1.123}$$

This leads to,

$$\int_0^d dr = \int_0^{T_{tr}} dt' v(t'), \quad T_{tr} = \sqrt{\frac{2md^2}{qV}}. \tag{1.124}$$

The current can then be calculated from the current density.

$$J(t) = \frac{q}{Ad}v(t), \quad i(t) = J(t) \cdot A = \frac{q}{d}v(t) = \frac{q^2 V}{md^2}t. \tag{1.125}$$

The surface charge on the cathode is,

$$Q_c(t) = -CV + q - \int_0^t dt' i(t') \tag{1.126}$$

$$= \begin{cases} -CV + q\left(1 - \frac{qV}{2md^2}t^2\right) = -CV + q\left(1 - \frac{v(t)}{2d}t\right) & 0 < t < T_{tr} \\ -CV & \text{otherwise} \end{cases} \tag{1.127}$$

The surface charge on the anode starts increasing by $+q$ over the time T_{tr}, due to the external current, from its $t = 0$ value of CV. Then, it is compensated by the electron from the cathode. The surface charge on the cathode is:

$$Q_A(t) = CV + \int_0^t dt' i(t') \tag{1.128}$$

$$= \begin{cases} CV + \frac{q^2 V}{2md^2} t^2 = CV + \frac{qv(t)}{2d} t & 0 < t < T_{tr}. \\ CV & \text{otherwise} \end{cases} \tag{1.129}$$

Since the external voltage source supplies an external current (without delay) to keep up with the change inside the diode, the voltage across the diode is still kept constant.

(III) $\tau_t \ll \tau_C$ limit, impulsive electron transit
The charge on the anode is given by

$$Q_A(t) = CV_A(t) = \begin{cases} CV - qe^{-t/R_s C} & t > 0 \\ CV & t < 0. \end{cases} \tag{1.130}$$

and that on the cathode is,

$$Q_C(t) = -CV_A(t) = -Q_A(t) = \begin{cases} -CV + qe^{-t/R_s C} & t > 0 \\ -CV & t < 0. \end{cases} \tag{1.131}$$

Here, the voltage across the diode has an $R_s C$ relaxation form.

1.7.4 Independent Emission of Electrons: a Poisson Point Process

For the case where the electron emission event and the transport process are mutually independent, *i.e.* the electron emission obeys a Poisson point process. We calculate the external current noise spectra for the above three cases.

(I) $\tau_C \ll \tau_t (R_s = 0)$ limit, constant v
The Carson theorem states that for a random pulse train $i(t) = \sum_{k=1}^{K} a_k f(t - t_k)$ with an identical pulse shape $f(t)$, the unilateral power spectrum is given by,

$$S(\omega) = 2\nu \langle a_k^2 \rangle |F(i\omega)|^2 + 4\pi \left[\nu a_k \int_{-\infty}^{\infty} dt f(t) \right]^2 \delta(\omega), \tag{1.132}$$

where ν is the average rate of arrival and $F(i\omega)$ is the Fourier transform of $f(t)$. In this case, each current pulse is given by

$$f(t) = \begin{cases} q\frac{v}{d} & 0 < t < \frac{d}{v}, \\ 0 & \text{otherwise} \end{cases} \tag{1.133}$$

and the Fourier transform is

$$F(i\omega) = \int_{-\infty}^{\infty} dt f(t) e^{-i\omega t} = \int_0^{d/v} dt \frac{qv}{d} e^{-i\omega t} \tag{1.134}$$

$$= \frac{qv(1 - e^{-i\omega d/v})}{i\omega d} = qe^{-i\omega d/2v} \frac{\sin(\omega d/2v)}{(\omega d/2v)}. \tag{1.135}$$

Using (1.132), we obtain

$$S_i(\omega) = 2\nu q^2 \frac{\sin^2(\omega d/2v)}{(\omega d/2v)^2} + 4\pi v^2 q^2 \delta(\omega). \tag{1.136}$$

Since the average rate is ν, the current is given by $I = q\nu$. Therefore (1.136) can be written as

$$S_i(\omega) = 2qI \left[\operatorname{sinc}(\omega\mathrm{d}/2\mathrm{v})\right]^2 + 4\pi I^2 \delta(\omega). \tag{1.137}$$

In the low-frequency limit, $0 < \omega \ll v/d$, since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we have

$$S_i(\omega \ll 2v/d) = 2qI, \tag{1.138}$$

which is a full shot noise.

(II) $\tau_C \ll \tau_t (R_s = 0)$ limit, accelerated v
In this case, each current pulse is given by

$$a = \frac{q^2 V}{d^2 m} \text{ and } f(t) = \begin{cases} t & 0 < t < T_{tr} \\ 0 & \text{otherwise.} \end{cases} \tag{1.139}$$

It follows that:

$$I = \overline{i(t)} = a\frac{1}{T_{tr}} \int_0^{T_{tr}} dt' f(t') = \frac{a\nu T_{tr}^2}{2} = q\nu, \tag{1.140}$$

$$\langle a^2 \rangle = a^2, \tag{1.141}$$

$$F(i\omega) = \int_{-\infty}^{\infty} dt f(t) e^{-i\omega t} = \int_0^{T_{tr}} dt\, t\, e^{-i\omega t} = iT_{tr}\frac{e^{-i\omega T_{tr}}}{\omega} - \frac{1 - e^{-i\omega T_{tr}}}{\omega^2}, \tag{1.142}$$

where an integration by parts is used in the last line. The magnitude squared is,

$$|F(i\omega)|^2 = \frac{2 + \omega^2 T_{tr}^2 - 2\omega T_{tr} \sin(\omega T_{tr}) - 2\cos(\omega T_{tr})}{\omega^4}. \tag{1.143}$$

Plugging into the unilateral power spectral density as per the Carson theorem, we have

$$S_i(\omega) = 2\nu \left(\frac{q^2 V}{d^2 m}\right)^2 \left[\frac{2 + \omega^2 T_{tr}^2 - 2\omega T_{tr} \sin(\omega T_{tr}) - 2\cos(\omega T_{tr})}{\omega^4}\right] + 4\pi \nu^2 q^2 \delta(\omega). \tag{1.144}$$

We use

$$\sin(\omega T_{tr}) = \omega T_{tr} - \frac{1}{3!}(\omega T_{tr})^3 + O(\omega^5), \tag{1.145}$$

$$\cos(\omega T_{tr}) = 1 + \frac{1}{2!}(\omega T_{tr})^2 + \frac{1}{4!}(\omega T_{tr})^4 + O(\omega^6), \tag{1.146}$$

in the small frequency limit, to write the power spectral density as

$$S_i(\omega) = 2\nu \left(\frac{q^2 V}{d^2 m}\right)^2 \left(\frac{2}{3!}T_{tr}^4 - \frac{2}{4!}T_{tr}^4 + O(\omega^5)\right) + 4\pi \nu^2 q^2 \delta(\omega). \tag{1.147}$$

In the low-frequency limit, we ignore $O(\omega^5)$, and we have

$$S_i(\omega) = 2qI + 4\pi I^2\delta(\omega). \tag{1.148}$$

In the low-frequency limit, $\quad 0 < \omega \ll 1/T_{tr}$, the power spectral density is,

$$S_i(\omega \ll \frac{1}{T_{tr}}) = 2qI, \tag{1.149}$$

which is again a full shot noise.

(III) $\tau_t \ll \tau_C$ limit, impulsive electron transit
In this case, each current pulse is given by

$$f(t) = \begin{cases} \frac{q}{CR_s}e^{-t/R_sC} & t > 0 \\ 0 & t < 0, \end{cases} \tag{1.150}$$

and the Fourier transform is

$$F(i\omega) = \int_{-\infty}^{\infty} dt f_{iii}(t)e^{-i\omega t} = \frac{q}{1 + i\omega R_s C}. \tag{1.151}$$

The power spectral density is then,

$$S_i(\omega) = 2\nu\frac{q^2}{1 + \omega^2 R_s^2 C^2} + 4\pi\nu^2 q^2\delta(\omega) = 2qI\frac{1}{1 + \omega^2 R_s^2 C^2} + 4\pi I^2\delta(\omega). \tag{1.152}$$

In the low-frequency limit, $0 < \omega \ll 1/R_sC$,

$$S_i(\omega \ll 1/R_sC) = 2qI, \tag{1.153}$$

which is again a full shot noise.

1.7.5 Noise Suppression in Vacuum Diodes

For statistically independent emission of an electron to occur, the condition for electron emission has to be identical for each emission event. For the case in which $\tau_t \gg \tau_C$ with constant electron velocity, the relevant time scale is the electron transit time $\tau_t = d/v$. To ensure statistical independence, we would require that no electron be emitted while one is currently in transit through the vacuum. Incidentally, this is over the time scale for which the voltage across the vacuum diode is recovered to its initial value. Therefore, in this case, we would require the electron emission rate to be

$$\nu \ll \frac{1}{\tau_t} \tag{1.154}$$

for statistically independent emission of electrons. In the other limit,$\tau_C \gg \tau_t$, we realize that the above condition is not satisfied. This is because the voltage across the vacuum diode is not recovered within time τ_t after the electron emission event. The rate of electron emission would be a function of the voltage across the diode, and is only fully recovered

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after a time τ_C has elapsed. Only then is the emission condition identical to ensure statistically independent emission. Thus, we require the emission rate to be,

$$\nu \ll \frac{1}{\tau_C} \tag{1.155}$$

If these conditions are *not* met, then there is a statistical dependence between the electron emission events. In this system, this dependence manifests itself as a negative feedback process in which subsequent electron emissions are suppressed following an electron emission. This is due to:

1. a space-charge effect in the $\tau_t \gg \tau_C$ limit, in which the existence of an electron in the vacuum creates a repulsive potential such that the rate of the subsequent electron emissions is suppressed.

2. a memory effect in the external circuit in the $\tau_C \gg \tau_t$ limit, in which the slow recovery of the voltage across the diode suppresses the rate of the subsequent electron emissions.

In both cases, the tendency is to regulate the emission events. This regulation leads to a quieter stream of electrons, and the noise is suppressed below the full shot noise value.

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