Chapter 9

Master Equation Approach to Matter-Wave Lasers

In the previous chapter we have presented the quantum theory of matter-wave lasers in the Heisenberg picture. We started with the classical equations of motion, Gross-Pitaevskii equation for the (c-number) condensate order parameter and the rate equation for the (real number) pump reservoir population. In order to conserve the proper commutator bracket for the corresponding operators (q-numbers) against dissipation processes, we introduced the noise operators. The resulting equations are the Heisenberg-Langevin equations. In spite of the transparency in their physical interpretation in terms of the corresponding classical equations, it is generally impossible to solve the Heisenberg-Langevin equations in a nonlinear regime. We introduced the linearization approximation to circumvent this difficulty and calculated the noise spectra. In this chapter we will present an alternative approach based on the master equations in the Schödinger picture. Even though we lose the classical-quantum correspondence in this second approach, we can solve the master equations exactly. Therefore, it is useful to study this alternative approach to gain more quantitative results.

9.1 Four fundamental assumptions in the quantum theory of an open-dissipative system

When we study a small quantum system which dissipatively couples to a large external world and in return receives a fluctuating force from the external world, we can usually introduce the following assumptions [1].

9.1.1 Born approximation

In spite of the mutual coupling between a small quantum system A and large reservoir (external world) B, the reservoir has a very fast internal relaxation process so that a quantum correlation between the two systems is quickly lost. Therefore, we can write the total density operator in a product state:

$$\hat{\rho}_{AB}(t) = \hat{\rho}_A(t) \otimes \hat{\rho}_B(t), \tag{9.1}$$

where $\hat{\rho}_A$ and $\hat{\rho}_B$ are the density operators for the system and reservoir, respectively.

9.1.2 Markov approximation

A reservoir consists of many (or infinite) degrees of freedom, so that it has a very short memory (correlation) time. During such a short memory time τ of the reservoir, the change in the system state can be neglected. Therefore, the system density operator in (9.1) satisfies

$$\hat{\rho}_A(t+\tau) \simeq \hat{\rho}_A(t). \tag{9.2}$$

9.1.3 Reservoir approximation

The reservoir is so large that the reservoir state is not affected by its coupling to the system. Therefore, the reservoir density operator is independent of time and often at thermal equilibrium condition:

$$\hat{\rho}_B(t) \simeq \hat{\rho}_B(0). \tag{9.3}$$

9.1.4 Rotating wave approximation

The coupling K_{AB} between the system and the reservoir is orders of magnitude smaller than the transition frequencies $\omega_n - \omega_m$ of both systems:

$$K_{AB} \ll \omega_n - \omega_m, \tag{9.4}$$

where ω_n and ω_m are the adjacent discrete energy levels of the system or reservoir. Therefore, only energy conserving (or nearly energy conserving) transitions are taken into account in the analysis.

9.2 Master equation

9.2.1 Basic modeling

We start with the Liouville-von Neumann equation for the total density operator [2]:

$$\frac{d}{dt}\hat{\rho}_{AB} = \frac{1}{i\hbar}\left[\hat{\mathcal{H}}_{\mathrm{int}}, \hat{\rho}_{AB}\right], \tag{9.5}$$

where $\hat{\mathcal{H}}_{\mathrm{int}}$ is the total Hamiltonian of the coupled system and reservoir in the interaction picture. The iterative solution for the reduced density operator for the system is written as

$$\begin{aligned}\hat{\rho}_A(t+\tau) \equiv Tr_B\left[\hat{\rho}_{AB}(t+\tau)\right] = Tr_B\Bigg\{&\hat{\rho}_A(t)\otimes\hat{\rho}_B(t) + \frac{1}{i\hbar}\int_t^{t+\tau} dt_1 \left[\hat{\mathcal{H}}_{\mathrm{int}}(t_1), \hat{\rho}_A(t)\otimes\hat{\rho}_B(t)\right] \\ &+ \left(\frac{1}{i\hbar}\right)^2 \int_t^{t+\tau} dt_1 \int_t^{t+\tau} dt_2 \left[\hat{\mathcal{H}}_{\mathrm{int}}(t_1), \left[\hat{\mathcal{H}}_{\mathrm{int}}(t_2), \hat{\rho}_A(t)\otimes\hat{\rho}_B(t)\right]\right]\Bigg\},\end{aligned} \tag{9.6}$$

where we truncate the interaction at the second order.

Let us study first the particular interaction Hamiltonian $\hat{\mathcal{H}}_{\text{int}}$ of the form

$$\hat{\mathcal{H}}_{\text{int}} = \hbar g \left(\hat{a}\hat{b}^+ + \hat{a}^+\hat{b}\right), \tag{9.7}$$

where $\hat{a}$ ($\hat{a}^+$) and $\hat{b}$ $\left(\hat{b}^+\right)$ are the annihilation (creation) operator of the condensate particle (quantum system) and the pump reservoir particle (reservoir), respectively. The pump reservoir density operator is given by the statistical mixture of the one particle state and the vacuum state:

$$\begin{aligned} \hat{\rho}_{\text{res}} &= \begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{00} \end{pmatrix} \begin{matrix} |1\rangle \\ |0\rangle \end{matrix} \\ &= \rho_{11}|1\rangle\langle 1| + \rho_{00}|0\rangle\langle 0|. \end{aligned} \tag{9.8}$$

The pump reservoir state does not have passes a quantum coherence (off-diagonal term) and is independent of time due to the reservoir approximation. The initial state for a combined system-reservoir density operator at a time $t = 0$ is given in the matrix form of the reservoir coordinate $\{|0\rangle, |1\rangle\}$ [3]:

$$\hat{\rho}_{AB}(0) = \begin{pmatrix} \rho_{11}\hat{\rho}_A(0) & 0 \\ 0 & \rho_{00}\hat{\rho}_A(0) \end{pmatrix}. \tag{9.9}$$

If we recall the reservoir creation and annihilation operators are identical to the projectors: $\hat{b}^+ = |1\rangle\langle 0|$ and $\hat{b} = |0\rangle\langle 1|$, the interaction Hamiltonian (9.7) is rewritten in the same matrix form as

$$\hat{\mathcal{H}}_{\text{int}} = \hbar g \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^+ & 0 \end{pmatrix}. \tag{9.10}$$

Because of the Born-Markov approximation, the commutator $\left[\hat{\mathcal{H}}_{\text{int}}, \hat{\rho}_A(t) \otimes \rho_B(t)\right]$ does not change during a reservoir relaxation time (or rather interaction time in this case) τ. Also because of the reservoir approximation, $\hat{\rho}_B(t)$ is time-independent. Based on these considerations, we can rewrite (9.6)

$$\begin{aligned} \hat{\rho}_A(t+\tau) &= Tr_B \left\{\hat{\rho}_A(t) \otimes \hat{\rho}_B(t) + \frac{\tau}{i\hbar}\left[\hat{\mathcal{H}}_{\text{int}}, \hat{\rho}_A(t) \otimes \hat{\rho}_B(t)\right]\right. \\ &\quad \left. + \frac{\tau^2}{2(i\hbar)^2}\left[\hat{\mathcal{H}}_{\text{int}}, \left[\hat{\mathcal{H}}_{\text{int}}, \hat{\rho}_A(t) \otimes \hat{\rho}_B(t)\right]\right]\right\}. \end{aligned} \tag{9.11}$$

The first-order interation term (second term of R.H.S) of (9.11) is evaluated as

$$\begin{aligned} \left[\hat{\mathcal{H}}_{\text{int}}, \hat{\rho}_A(t) \otimes \hat{\rho}_B(t)\right] &= \hbar g \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^+ & 0 \end{pmatrix} \begin{pmatrix} \rho_{11}\hat{\rho}_A & 0 \\ 0 & \rho_{00}\hat{\rho}_A \end{pmatrix} - \hbar g \begin{pmatrix} \rho_{11}\hat{\rho}_A & 0 \\ 0 & \rho_{00}\hat{\rho}_A \end{pmatrix} \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^+ & 0 \end{pmatrix} \\ &= \hbar g \begin{pmatrix} 0 & \hat{a}\rho_{00}\hat{\rho}_A - \rho_{11}\hat{\rho}_A\hat{a} \\ \hat{a}^+\rho_{11}\hat{\rho}_A - \rho_{00}\hat{\rho}_A\hat{a}^+ & 0 \end{pmatrix}, \end{aligned} \tag{9.12}$$

so that this term vanishes after taking the trace over the reservoir coordinate

$$Tr_B \left\{\left[\hat{H}_{\text{int}}, \hat{\rho}_A \otimes \hat{\rho}_B\right]\right\} = 0. \tag{9.13}$$

Similarly we can evaluate the second-order interation term (third term of R.H.S) of (9.11) as

$$\left[\hat{\mathcal{H}}_{\text{int}}, \left[\hat{\mathcal{H}}_{\text{int}}, \hat{\rho}_A(t) \otimes \hat{\rho}_B(t)\right]\right]$$
$$= (\hbar g)^2 \begin{pmatrix} \hat{a}\hat{a}^+\rho_{11}\hat{\rho}_A - 2\hat{a}\rho_{00}\hat{\rho}_A\hat{a}^+ + \rho_{11}\hat{\rho}_A\hat{a}\hat{a}^+ & 0 \\ 0 & \hat{a}^+\hat{a}\rho_{00}\hat{\rho}_A - 2\hat{a}^+\rho_{11}\hat{\rho}_A\hat{a} + \rho_{00}\hat{\rho}_A\hat{a}^+\hat{a} \end{pmatrix}, \tag{9.14}$$

so that this term has a non-zero contribution to (9.11):

$$Tr_B\left\{\left[\hat{\mathcal{H}}_{\text{int}}, \left[\hat{\mathcal{H}}_{\text{int}}, \hat{\rho}_A(t) \otimes \hat{\rho}_B(t)\right]\right]\right\}$$
$$= (\hbar g)^2 \left\{\rho_{11}\left[\hat{a}\hat{a}^+\hat{\rho}_A + \hat{\rho}_A\hat{a}\hat{a}^+ - 2\hat{a}^+\hat{\rho}_A\hat{a}\right] + \rho_{00}\left[\hat{a}^+\hat{a}\hat{\rho}_A + \hat{\rho}_A\hat{a}^+\hat{a} - 2\hat{a}\hat{\rho}_A\hat{a}^+\right]\right\} \tag{9.15}$$

Using (9.15) in (9.11), we obtain the system operator evolution to the second order

$$\begin{aligned}\hat{\rho}_A(t+\tau) = \hat{\rho}_A(t) - \frac{1}{2}(g\tau)^2 \left\{\rho_{11}\left[\hat{a}\hat{a}^+\hat{\rho}_A + \hat{\rho}_A\hat{a}\hat{a}^+ - 2\hat{a}^+\hat{\rho}_A\hat{a}\right]\right.\\ \left. + \rho_{00}\left[\hat{a}^+\hat{a}\hat{\rho}_A + \hat{\rho}_A\hat{a}^+\hat{a} - 2\hat{a}\hat{\rho}_A\hat{a}^+\right]\right\}.\end{aligned} \tag{9.16}$$

We can model the system-pump reservoir interaction in a matter wave laser by the following picture. We inject a particle into the pump reservoir state at a rate r per second with a finite probability ρ_{11}. With the probability of $\rho_{00} = 1 - \rho_{11}$, we do not inject a particle. The injected particles interact with the condensate for a time interval τ, so that a total number of $r \times \tau$ particles interact with the condensate for every second. The coarse-grained time rate of change of the system density operator is then given by

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_A &= \frac{r\tau\left[\hat{\rho}_A(t+\tau) - \hat{\rho}_A(t)\right]}{\tau} \\ &= -\frac{1}{2}R_1\left[\hat{a}\hat{a}^+\hat{\rho}_A + \hat{\rho}_A\hat{a}\hat{a}^+ - 2\hat{a}^+\hat{\rho}_A\hat{a}\right] \\ &\quad -\frac{1}{2}R_0\left[\hat{a}^+\hat{a}\hat{\rho}_A + \hat{\rho}_A\hat{a}^+\hat{a} - 2\hat{a}\hat{\rho}_A\hat{a}^+\right],\end{aligned} \tag{9.17}$$

where $R_1 = r\rho_{11}(g\tau)^2$ is the cooling rate of pump reservoir particles into the condensate per second and $R_0 = r\rho_{00}(g\tau)^2$ is the reverse process, i.e. the excitation rate into the pump reservoir from the condensate per second. This deterministic linear differential equation (9.17) is called a master equation and fully equivalent to the Lionville-von Neumann equation (9.5) for the total system.

9.2.2 Master equation in Linblad form

The dynamical condensate in matter wave lasers has a net gain $R_1 - R_0$, which must be compensated for a net loss rate due to the output coupling of the condensate particles. The mutual coupling between the condensate and external particle field reservoir, which is responsible for a net loss, can be incorporated into (9.17) by replacing R_0 with $\gamma_c(1 + N_R)$ and R_1 with $\gamma_c N_R$, where γ_c is the condensate particle decay rate and N_R is the average particle number of the external particle field reservoir. The master equation of matter-wave lasers has thus a general form of

$$\frac{d}{dt}\hat{\rho}_A = -\frac{1}{2}\left\{(R_1 + \gamma_c N_R)\left(\hat{a}\hat{a}^+\rho_A + \rho_A\hat{a}\hat{a}^+ - 2\hat{a}^+\hat{\rho}_A\hat{a}\right)\right.$$

$$+\left[R_0+\gamma_c(1+N_R)\right]\left(\hat{a}^+\hat{a}\hat{\rho}_A+\hat{\rho}_A\hat{a}^+\hat{a}-2\hat{a}\rho_A\hat{a}^+\right)\big\}. \tag{9.18}$$

This equation is called a master equation of Linblad form and independent of specific system-reservoir coupling models. All we need to know are the rate-in and rate-out of the condensate particles as shown in Fig. 9.1 in order to write the master equation. As discussed already in Chapter 8, this is also true to write the Heisenberg-Langevin equation.

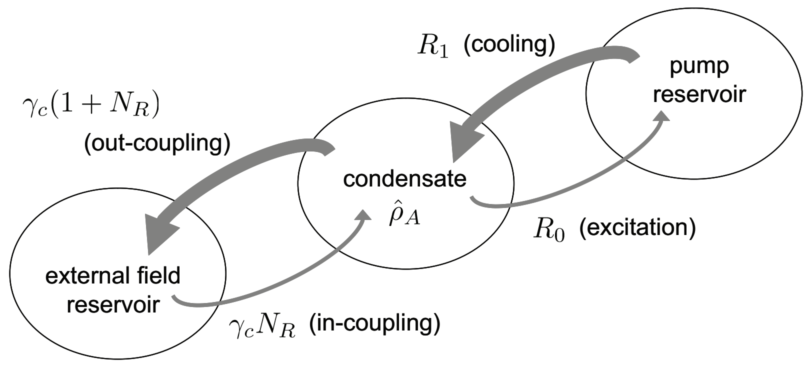

Figure 9.1: The coupling of the condensate field with the pump and external field reservoirs.

9.3 Particle statistics

The matrix element $\rho_{nm}=\langle n|\hat{\rho}_A|m\rangle$ of the condensate density operator in terms of particle number eigenstates satisfies the following equation [4]:

$$\frac{d}{dt}\rho_{nm}=-\frac{1}{2}\left\{(R_1+\gamma_c N_R)(n+m+2)+\left[R_0+\gamma_c(1+N_R)\right](n+m)\right\}\rho_{nm}$$

$$+(R_1+\gamma_c N_R)\sqrt{nm}\rho_{n-1,m-1}+\left[R_0+\gamma_c(1+N_R)\right]\sqrt{(n+1)(m+1)}\rho_{n+1,m+1}. \tag{9.19}$$

If the loss rate γ_c is set to zero (no output coupling), the diagonal element ρ_{nn} satisfies

$$\frac{d}{dt}\rho_{nn}=\left[-R_1(n+1)-R_0 n\right]\rho_{nn}+R_1 n\rho_{n-1,n-1}+R_0(n+1)\rho_{n+1,n+1}. \tag{9.20}$$

ρ_{nn} couples to only neighboring diagonal elements $\rho_{n-1,n-1}$ and $\rho_{n+1,n+1}$ as indicated in Fig. 9.2. The physical interpretation of each flow should be obvious from the figure. In order to have a steady state solution in (9.20), the cooling rate should not exceed the excitation rate, i.e. $R_1<R_0$. Under this constraint, the particle flow from the pump reservoir to the condensate (up-ward transition in Fig. 9.2) and that from the condensate to the pump-reservoir (down-ward transition in Fig. 9.2) should balance between any pair of neighboring states, so that we obtain the detailed balance:

$$R_0 n\rho_{nn}=R_1 n\rho_{n-1,n-1}. \tag{9.21}$$

Using (9.21) iteratively, we can relate ρ_{nn} to ρ_{00},

$$\rho_{nn}=\frac{R_1}{R_0}\rho_{n-1,n-1}=\cdots\cdots=\left(\frac{R_1}{R_0}\right)^n\rho_{00}, \tag{9.22}$$

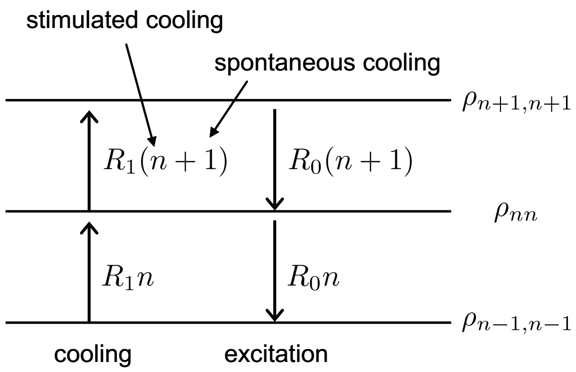

Figure 9.2: The coupling of the diagonal elements of the condensate density operator.

where the normalization condition, $\sum_n \rho_{nn} = 1$, must be satisfied and thus ρ_{00} is uniquely determined as

$$\rho_{00} = 1 - \left(\frac{R_1}{R_0}\right). \tag{9.23}$$

The average particle number $\bar{N}$ is thus calculated by

$$\bar{N} = \sum_{n=0}^{\infty} n\rho_{nn} = \rho_{nn} = \sum_{n=0}^{\infty} n\left[1 - \left(\frac{R_1}{R_0}\right)\right]\left(\frac{R_1}{R_0}\right)^n = \frac{R_1}{R_0 - R_1}. \tag{9.24}$$

Substituting (9.23) and (9.24) into (9.22), we obtain

$$\rho_{nn} = \frac{1}{1+\bar{N}}\left(\frac{\bar{N}}{1+\bar{N}}\right)^n. \tag{9.25}$$

This is the famous particle statistics for a single-mode thermal state. The experimental signatures for a single-mode thermal state are the so-called bunching in the particle correlation measurement and super-Poisson distribution in the particle number distribution measurement:

$$g^{(n)}(0) = n! \qquad \text{(bunching)}, \tag{9.26}$$

$$\left\langle \Delta\hat{N}^2 \right\rangle = \bar{N}\left(\bar{N}+1\right) \qquad \text{(super-Poisson)}, \tag{9.27}$$

where $g^{(n)}(0)$ is the n-th order coherence function and $\left\langle \Delta\hat{N}^2 \right\rangle = \left\langle \hat{N}^2 \right\rangle - \left\langle \hat{N} \right\rangle^2$ is the variance of the particle number.

If the condensate has a net gain, $R_1 > R_0$, there should be a finite output coupling loss $\gamma_c \neq 0$ in order to have a steady state solution in (9.20). In this case, the net gain $R_1 > R_0$ should balance the loss rate γ_c if the particle population in the external field reservoir is negligible. It is convenient to write down the net gain $R_1 - R_0$ in terms of a linear gain A, which is proportional to the pump rate, and nonlinear gain saturation B, which depletes the linear gain and make the net gain $R_1 - R_0$ pinned at the loss rate γ_c. Using these linear and nonlinear gain coefficients, the master equation (9.20) can be rewritten as

$$\begin{aligned}
\frac{d}{dt}\rho_{nn} &= -\frac{(n+1)A}{1+(n+1)\frac{A}{B}}\rho_{nn} + \frac{nA}{1+n\frac{B}{A}}\rho_{n-1,n-1} - \gamma_c n\rho_{nn} + \gamma_c(n+1)\rho_{n+1,n+1} \\
&\simeq -\left[A - B(n+1)\right](n+1)\rho_{nn} + (A - Bn)n\rho_{n-1,n-1} \\
&\quad -\gamma_c n\rho_{nn} + \gamma_c(n+1)\rho_{n+1,n+1} \qquad (9.28)
\end{aligned}$$

The physical interpretation for each term in R.H.S, of (9.28) should be clear from the probability flow diagram Fig. 9.3.

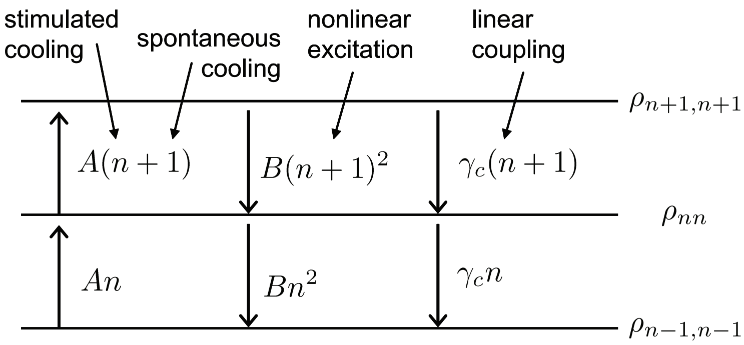

Figure 9.3: The flow of the diagonal elements due to linear gain, nonlinear gain saturation and output coupling.

The new detailed balance is given by

$$-\frac{(n+1)A}{1+(n+1)\frac{B}{A}}\rho_{nn} + \gamma_c(n+1)\rho_{n+1,n+1} = 0. \qquad (9.29)$$

From this relation, ρ_{nn} is related to ρ_{00} by

$$\rho_{nn} = \frac{\left(\frac{A^2}{\gamma_c B}\right)^n}{\left(n+\frac{A}{B}\right)!}\rho_{00}. \qquad (9.30)$$

At well above threshold, the average particle number in the condensate is much larger than one, $\left\langle \hat{N} \right\rangle \gg 1$ and thus $\rho_{nn} \simeq \rho_{n-1,n-1}$. The saturated gain should be equal to the loss, $\gamma_c = \frac{A}{1+n\frac{B}{A}}$. From this relation, the average particle number is approximately given by

$$\left\langle \hat{N} \right\rangle = \frac{A}{\gamma_c}\frac{A-\gamma_c}{B} \simeq \frac{A^2}{\gamma_c B}, \qquad (9.31)$$

in the limit of high pump rates ($A \gg \gamma_c$). Using (9.31), the diagonal element ρ_{nn} is reduced to

$$\rho_{nn} = \frac{\left\langle \hat{N} \right\rangle^n}{n!}\rho_{00} = e^{-\left\langle \hat{N} \right\rangle}\frac{\left\langle \hat{N} \right\rangle^n}{n!}. \qquad (9.32)$$

This is a Poisson distribution, for which the variance is equal to the average particle number, $\left\langle \Delta\hat{N}^2 \right\rangle = \left\langle \hat{N} \right\rangle$.

9.4 Quantum mechanical Fokker-Planck equation

The master equation (9.18) is a key working equation to calculate the quantum statistics of the matter-wave lasers. However, the particle number representation of the master equation such as (9.19) is poorly suited for analyzing the condensate statistics with a huge average number of particles $\bar{N} \gg 1$, since the matrix size is practically intractable. The Glauber-Sudarshan $p(\alpha)$ representation [5, 6] of the density operator does a much better job in such a case.

Let us assume the density operator of the condensate field is expanded by the diagonal elements in terms of the coherent state [5, 6]:

$$\hat{\rho}_A(t) = \int d^2\alpha p(\alpha,t)|\alpha\rangle\langle\alpha|, \tag{9.33}$$

where $p(\alpha, t)$ is a real number probability of finding a particular coherent state $|\alpha\rangle$ in the condensate field. If we substitute (9.33) into (9.18), we obtain

$$\begin{aligned}\int d^2\alpha \frac{d}{dt}p(\alpha,t)|\alpha\rangle\langle\alpha| &= -\frac{1}{2}\int d^2\alpha p(\alpha,t)\left\{(R_1+\gamma_c N_R)\left[\hat{a}\hat{a}^+|\alpha\rangle\langle\alpha| - \hat{a}^+|\alpha\rangle\langle\alpha|\hat{a}\right]\right. \\ &+ \left.[R_0+\gamma_c(N_R+1)]\left[\hat{a}^+\hat{a}|\alpha\rangle\langle\alpha| - \hat{a}|\alpha\rangle\langle\alpha|\hat{a}^+\right]\right\} + h.c. \end{aligned} \tag{9.34}$$

The coherent state is generated by projecting a displacement operator on a vacuum state [5, 6]

$$|\alpha\rangle = e^{\alpha\hat{a}^+ - \alpha^+\hat{a}}|0\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^+}|0\rangle, \tag{9.35}$$

where the Baker-Hausdorf relation for a non-commuting pair of observables $\hat{A}$ and $\hat{B}$ [7]

$$e^{\alpha\hat{A}+\beta\hat{B}} = e^{\frac{1}{2}\alpha\beta[\hat{A},\hat{B}]}e^{\alpha\hat{A}}e^{\beta\hat{B}}, \tag{9.36}$$

and the relation $e^{-\alpha^*\hat{a}}|0\rangle = |0\rangle$ are used in the second equality of (9.35). From (9.35) and its adjoint, $\langle\alpha| = e^{-\frac{|\alpha|^2}{2}}\langle 0|e^{\alpha^*\hat{a}}$, we can rewrite the projector $|\alpha\rangle\langle\alpha|$ as

$$|\alpha\rangle\langle\alpha| = e^{-|\alpha|^2}e^{\alpha\hat{a}^+}|0\rangle\langle 0|e^{\alpha^*\hat{a}}. \tag{9.37}$$

By differentiating (9.37) with respect to α and α^*, we obtain

$$\hat{a}^+|\alpha\rangle\langle\alpha| = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)|\alpha\rangle\langle\alpha|, \tag{9.38}$$

$$|\alpha\rangle\langle\alpha|\hat{a} = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|. \tag{9.39}$$

Using these relations, the two terms in R.H.S. of (9.34) are simplified to

$$\begin{aligned}\hat{a}\hat{a}^+|\alpha\rangle\langle\alpha| - \hat{a}^+|\alpha\rangle\langle\alpha|\hat{a} &= \left[\frac{\partial}{\partial\alpha}\cdot\alpha + \alpha^*\alpha - \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)\left(\frac{\partial}{\partial\alpha^+} + \alpha\right)\right]|\alpha\rangle\langle\alpha| \\ &= \left(-\alpha^*\frac{\partial}{\partial\alpha^*} - \frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)|\alpha\rangle\langle\alpha|, \end{aligned} \tag{9.40}$$

$$\begin{aligned}\hat{a}^{+}\hat{a}|\alpha\rangle\langle\alpha| - \hat{a}|\alpha\rangle\langle\alpha|\hat{a}^{+} &= \left[\alpha\frac{\partial}{\partial\alpha} + \alpha\alpha^{*} - \alpha\alpha^{*}\right]|\alpha\rangle\langle\alpha| \\ &= \alpha\frac{\partial}{\partial\alpha}|\alpha\rangle\langle\alpha|. \end{aligned} \tag{9.41}$$

Substituting (9.40) and (9.41) into (9.34), we finally obtain

$$\begin{aligned}\int d^2\alpha\frac{d}{dt}p(\alpha,t)|\alpha\rangle\langle\alpha| &= -\int d^2\alpha\left\{\frac{1}{2}\left(R_1 - R_0 - \gamma_c\right)\left[\frac{\partial}{\partial\alpha}\left(\alpha p(\alpha,t)\right) + C.C.\right]\right. \\ &\quad \left. - \left(R_1 + \gamma_c N_R\right)\frac{\partial^2}{\partial\alpha\partial\alpha^{*}}p(\alpha,t)\right\}|\alpha\rangle\langle\alpha|. \end{aligned} \tag{9.42}$$

By comparing the expansion coefficients for $|\alpha\rangle\langle\alpha|$ in both sides of (9.42), we can obtain the quantum mechanical Fokker-Planck equation:

$$\frac{d}{dt}p(\alpha,t) = -\frac{1}{2}\left(R_1 - R_0 - \gamma_c\right)\left[\frac{\partial}{\partial\alpha}(\alpha p(\alpha) + C.C.\right] + \left(R_1 + \gamma_c N_R\right)\frac{\partial^2}{\partial\alpha\partial\alpha^{*}}p(\alpha,t) \tag{9.43}$$

Instead of solving the time evolution for the matrix elements ρ_{nn} in the large-size matrix, we can solve the linear differential equation for $p\left(\alpha,t\right)$. This is a much easier task in both analytical and numerical approaches.

The first term $M_1 = -\frac{1}{2}\left(R_1 - R_0 - \gamma_c\right)\left[\frac{\partial}{\partial\alpha}(\alpha p(\alpha)) + C.C.\right]$ of R.H.S. of (9.43) is called a drift term. If $R_1 - R_0 - \gamma_c > 0$, the net gain $R_1 - R_0$ exceeds the loss rate γ_c. In such a case, $p(\alpha,t)$ distribution drifts to a higher excitation amplitude α as shown in Fig. 9.4. On the contrary, if $R_1 - R_0 - \gamma_c < 0, p(\alpha,t)$ distribution drifts forward a lower excitation amplitude α. The second term $M_2 = \left(R_1 + \gamma_c N_R\right)\frac{\partial^2}{\partial\alpha\partial\alpha^{*}}p(\alpha,t)$ of R.H.S. of (9.43) is called a diffusion term. Since $R_1 + \gamma_c N_R$ is always positive, the central part of $p(\alpha,t)$ distribution decreases its amplitude and the trailing part increases its amplitude as shown in Fig. 9.5. Because of this behavior, the $p(\alpha,t)$ distribution always broadens as a time elapses. An analogy of the above interpretation for the quantum mechanical Fokker-Planck equation to the classical problem of the random walk diffusion of a Brownian particle under a drift field is discussed in ref. [4].

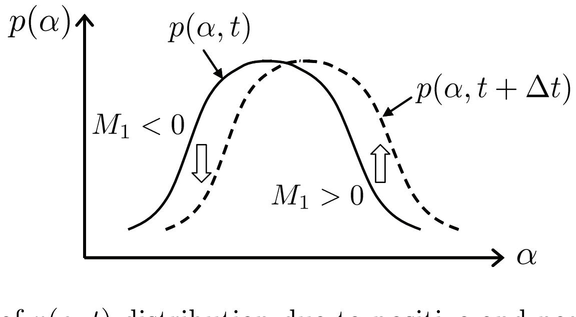

Figure 9.4: The drift of $p(\alpha,t)$ distribution due to positive and negative M_1 values when $R_1 - R_0 - \gamma_c > 0$.

9.5 Amplitude and phase noise

In order to calculate the amplitude and phase noise of the condensate, let us introduce the polar coordinate to express a c-number $\alpha = re^{i\theta}$. The quantum mechanical Fokker-Planck

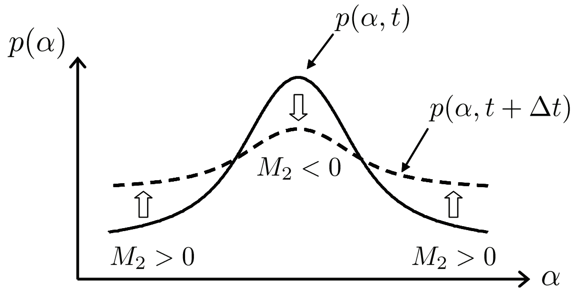

Figure 9.5: The diffusion of $p(\alpha,t)$ distribution due to positive and begative M_2 values.

equation (9.43) is rewritten in the polar coordinate:

$$\begin{aligned}\frac{d}{dt}p(r,\theta,t) &= -\frac{1}{2r}\left(R_1-R_0-\gamma_c\right)\frac{\partial}{\partial r}\left[r^2p(r,\theta,t)\right]+\frac{1}{4r^2}\left(R_1+\gamma_cN_{th}\right)\\ &\times\left[r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial\theta^2}\right]p(r,\theta,t). \end{aligned} \tag{9.44}$$

At well above threshold, the net gain is approximated by $R_1-R_0\simeq A-B|\alpha|^2=A-Br^2$ and the thermal population term γ_cN_R of the external reservoir can be neglected. Thus (9.44) is reduced to

$$\frac{d}{dt}p(r,\theta,t)=-\frac{1}{2r}\frac{\partial}{\partial r}\left[r^2\left(A-Br^2-\gamma_c\right)p(r,\theta,t)\right]+\frac{1}{4r^2}A\left(r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial\theta^2}\right)p(\alpha,\theta,t) \tag{9.45}$$

In a steady state condition, the drift term for the average amplitude $\bar{r}$ must vanish, so that $A-B\bar{r}^2-\gamma_c=0$ and thus the average particle number in the condensate is given by

$$\left\langle\hat{N}\right\rangle\equiv\bar{r}^2=\frac{A-\gamma_c}{B}. \tag{9.46}$$

If we substitute $r=\bar{r}+\Delta r$ into the first term of R.H.S. of (9.45), we obtain $r^2\left(A-Br^2-\gamma_c\right)\simeq -2rB\bar{r}^2\left(r-\bar{r}\right)$. We introduce the ansatz $p(r,\theta)=R(r)\Phi(\theta)$ as a steady state solution of (9.45). Then the amplitude wave function $R(r)$ satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rB\bar{r}^2\left(r-\bar{r}\right)R(r)\right]+\frac{A}{4r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}R(r)\right]=0, \tag{9.47}$$

or

$$\frac{\partial}{\partial r}R(r)=-\frac{4B\bar{r}^2}{A}\left(r-\bar{r}\right)R(r). \tag{9.48}$$

The steady state solution for the amplitude wave function can be immediately obtained from (9.48)

$$R(r)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{\left(r-\bar{r}\right)^2}{2\sigma^2}\right], \tag{9.49}$$

where the variance is given by

$$\sigma^2=\frac{A}{4B\bar{r}^2}=\frac{1}{4}\cdot\frac{A}{A-\gamma_c}. \tag{9.50}$$

At well above threshold, the linear gain exceeds the loss, $A \gg \gamma_c$, so that the variance approaches to $1/4$, which is the variance of the coherent state.

The steady state solution for the phase wave function satisfies $\frac{d}{d\theta}\Phi(\theta) = 0$, which means the phase is completely random. This is the universal characteristic of a system under random walk phase diffusion and seemingly in contradiction to our earlier discussion on the off-diagonal long range order and the superfluidity associated with Bose-Einstein condensation. In order to study the dynamics of this phase diffusion process, let us consider a transient solution of $\Phi(\theta)$ governed by

$$\frac{d}{dt}\Phi(\theta) = \frac{A}{4\bar{r}^2}\frac{\partial^2}{\partial\theta^2}\Phi(\theta). \tag{9.51}$$

Equation (9.51) indicates the phase exerts a Brownian motion, $\left\langle \Delta\left[\theta(t) - \theta(0)\right]^2 \right\rangle = 2D(\theta)t$, with a phase diffusion constant given by

$$2D(\theta) = \frac{A}{2\bar{r}^2} = \frac{\gamma_c n_{sp}}{2\left\langle \hat{N} \right\rangle}, \tag{9.52}$$

where $\frac{A}{\gamma_c} = \frac{A}{A - B\bar{r}^2} \equiv n_{sp}$ is called the population inversion parameter and (9.46) is used. If we introduce the output particle flux per second by $n_{\text{out}} = \gamma_c \left\langle \hat{N} \right\rangle$, the phase diffusion constant is rewritten as

$$2D(\theta) = \frac{\gamma_c^2 n_{sp}}{2n_{\text{out}}}. \tag{9.53}$$

This is the famous Schawlow-Toweres linewidth of a laser oscillator [4]. The phase diffusion constant (or FWHM spectral linewidth of the Lorentzian spectral profile) $2D(\theta)$ decreases abruptly at threshold by $\frac{Bn_{sp}}{\gamma_c}$ and continuously decreases in inversely proportioned to the pump rate, $\left(\frac{A}{\gamma_c} - 1\right)^{-1}$, at above threshold as shown in Fig. 9.6.

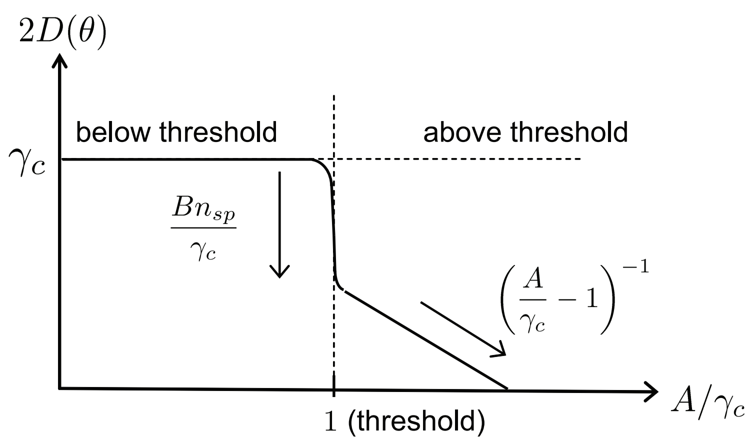

Figure 9.6: The spectral linewidth (FWHM) of the condensate $2D(\theta)$ vs. the normalized pump rate A/γ_c.

9.6 Superfluidity in an open dissipative condensate

In Chapter 5 we have described the quantized vortex and the bound pair of vortex-antivortex as an experimented evidence of superfluidity. Let us consider the following situation: a bound pair of quantized vortex and antivortex with opposite winding numbers (2π and -2π phase rotation) is produced at the center of a condensate, which is trapped in a circular potential. How does a vortex-pair move as a time elapses in such a system? This problem can be solved by the numerical simulation of the open dissipative Gross-Pitaevskii equation for the condensate and the rate equation for the pump reservoir. As shown in Fig. 9.7(a), the vortex-pair moves in parallel to the perpendicular direction due to the local velocity field (phase gradient induced by the vortex-pair). When the vortex-pair approach the trap potential, the vortex and antivortex move apart and make separate curved trajectories along the two side boundaries and form a bound pair again at the opposite trap boundary. This cyclic motion of the vortex-pair is an inherent property of an equilibrium condensate in a closed system.

If we introduce a finite gain and loss rate into the system, the completely different dynamics is expected as shown in Fig. 9.7(b). The vortex-pair continuously loses its potential energy and the vortex-antivortex separation decreases while it propagate along the perpendicular direction. Eventually the vortex-pair recombines and ends up as a simple density dip. Eventually, a simple condensate without any disturbances (steady state condensate) is recovered. As demonstrated in this example, an open dissipative condensate system has a function to eliminate any defect introduced into the system and restore the steady state condensate at a cost of stationary amplitude and phase noise.

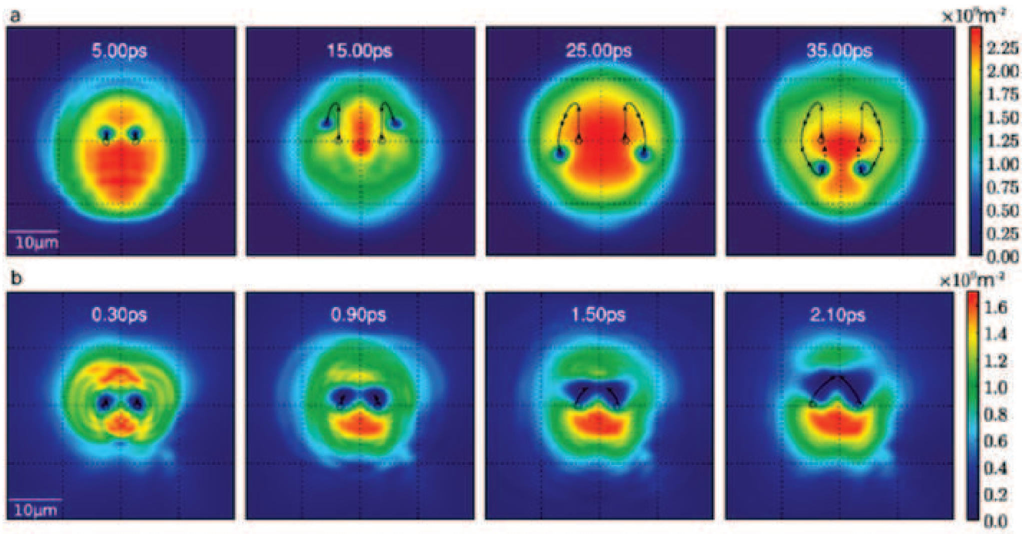

Figure 9.7: The time evolution of a bound pair of vortex and antivortex in a closed condensate (a) and open dissipative condensate (b). [8]

In Chapter 4, we derived the Bogoliubov excitation spectrum which is linear in a low momentum regime (sound mode). The critical velocity for superfluidity is given by the slope of the linear dispersion, sound velocity $c = \sqrt{\frac{gn}{m}}$, according to the Landau's criterion.

This conserved BEC behavior is also modified by the addition of loss and gain terms. The open dissipative Gross-Pitaevskii equation for the condensate order parameter is given by

$$i\frac{d}{dt}\psi_0(r,t) = \left\{-\frac{\hbar}{2m}\nabla^2 - \frac{i}{2}\left[\gamma_c - G\left(n_R\right)\right] + g|\psi_0(r,t)|^2 + g_R n_R\right\}\psi_0(r,t), \tag{9.54}$$

while the rate equation for the pump reservoir population is

$$\frac{d}{dt}n_R(r,t) = p - \gamma_R n_R(r,t) - G(n_R)|\psi_0(r,t)|^2. \tag{9.55}$$

We neglect the spacial diffusion of the pump reservoir particles, $D\nabla^2 n_R(r,t) = 0$, where D is a diffusion constant. The linearized solutions of (9.54) and (9.55) can be expressed as

$$\psi_0(r,t) = e^{-i\mu t}\psi_0\left[1 + \sum_k\left\{u_k e^{i(kr-\omega t)} + v_k^* e^{-i(kr-\omega t)}\right\}\right], \tag{9.56}$$

$$n_R(r,t) = n_R^0\left(1 + \sum_k\left\{w_k e^{i(kr-\omega t)} + w_k^* e^{-i(kr-\omega t)}\right\}\right). \tag{9.57}$$

Here we allow the amplitude and phase fluctuations for the condensate order parameter, corresponding to the two degrees of freedom u_k and v_k^*, but assume only amplitude fluctuation for the pump reservoir population.

Below the condensate threshold $(G\left(n_R\right) < \gamma_c)$, $\psi_0 = 0$ and $n_R^0 = p/\gamma_R$. This solution is dynamically stable. When the pump rate p is increased above the condensation threshold, the solution $\psi_0 = 0$ becomes dynamically unstable and a condensate order parameter appears with the steady state population $|\psi_0|^2 = \left(p - p_{th}\right)/\gamma_c$ where p_{th} is defined by $G\left(n_R^{th} = p_{th}/\gamma_R\right) = \gamma_c$. The oscillation frequency of the condensate is $\mu_T = \mu + g_R n_R^0$, where $\mu = g|\psi_0|^2$ and $g_R = 2g$ according to the standard Hartree-Fock argument.

Substituting (9.56) and (9.57) into (9.54) and (9.55) and using the above steady state solutions above the threshold, we can obtain the eigenvalue equation for the elementary excitations:

$$\begin{pmatrix} \mu + \frac{\hbar k^2}{2m} & \mu & \frac{i\beta\gamma_c}{2} + \frac{2\gamma_c}{\alpha\gamma_R}\mu \\ -\mu & -\mu - \frac{\hbar k^2}{2m} & \frac{i\beta\gamma_c}{2} - \frac{2\gamma_c}{\alpha\gamma_R}\mu \\ -i\alpha\gamma_R & -i\alpha\gamma_R & -i\eta\gamma_R \end{pmatrix}\begin{pmatrix} u_k \\ v_k \\ w_k \end{pmatrix} = \omega\begin{pmatrix} u_k \\ v_k \\ w_k \end{pmatrix} \tag{9.58}$$

Here $\alpha = p/p_{th} - 1$ is the relative pump rate, $\beta = n_R^0 \frac{\partial}{\partial n_R^0} G\left(n_R^0\right)/G\left(n_R^0\right)$ characterizes the dependence of the gain term on n_R^0 and $\eta = 1 + \alpha\beta$. For a phonon assisted gain process $\left(G\left(n_R^0\right) = A n_R^0\right), \beta = 1$, while for a two particle collision induced gain process $\left(G\left(n_R^0\right) = \beta n_R^{02}\right), \beta = 2$. The elementary excitation spectrum, ω vs. k, can be obtained by solving the above equation and the examples are shown in Fig. 9.8(a) and (b) [9]. When $\gamma_R \gg \gamma_c$, the pump reservoir is able to adiabatically follow the evolution of the condensate due to the slaving principle. The dispersion of elementary excitations shown in Fig. 9.8(a) is dramatically different from the Bogoliubov sound mode, indicated by the dashed line in Fig. 9.8(a), in equilibrium BEC. The Nambu-Goldstone modes (indicated by + and - in Fig. 9.8(a)) shows a diffusive and nonpropagating behavior at low k.

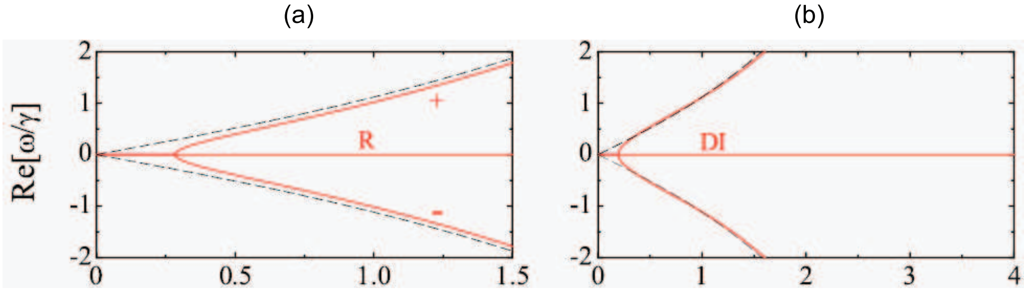

Figure 9.8: Real part of the excitation spectrum of an exciton-polariton condensate with $\gamma_R/\gamma_c = 5$ (a) and $\gamma_R/\gamma_c = 1$ (b) [9].

An analytical explanation of this behavior is obtained by eliminating the pump reservoir mode adiabatically, which leads to the following dispersion of the two branches of elementary excitations:

$$\omega_{\pm}(k) = -i\frac{\Gamma}{2} \pm \sqrt{\omega_B(k)^2 - \Gamma^2/4}, \tag{9.59}$$

where $\omega_B(k) = \sqrt{\omega_k\left(\omega_k + 2\mu\right)}, \omega_k = \hbar k^2/2m$ and $\Gamma = \alpha\beta\gamma_c/\left(1+\alpha\beta\right)$ whose value tends to 0 near the threshold ($\alpha \simeq 0$) and approaches to γ_c at well above the threshold ($\alpha \gg 1$). The + branch of (9.59) is the Nambu-Goldstone mode which corresponds to a slow rotation of the condensate phase for small k values. Indeed, the vector $(1, -1, 0)^T$ of the gobal phase rotation is an eigenvector of the matrix, LHS of (9.58), with a vanishing eigenvalue ω. The - branch of (9.59) corresponds to modulations of the condensate density. The width Δk of the k-space region where the Nambu-Goldstone mode is flat is given by $Re\left(\omega_{\pm}\right) = 0$ or $\omega_B(k) = \Gamma/2$. On the other hand, for $k \gg \Delta k$, the + modes recover the standard Bogoliubov dispersion.

If γ_R and γ_c are comparable, the pump reservoir mode takes full part in the condensate dynamics through the so-called relaxation oscillation and dynamical instability. The former (relaxation oscillation) is caused by the cross-saturation of the condensate and pump reservoir populations through the gain term $G\left(n_R\right)$. The local depletion of the pump reservoir density $n_R(r,t)$ results in the smaller gain and leads to the decreased condensate density $|\psi_0(r,t)|^2$. This results in the local increase of the pump reservoir density and the larger gain, which leads to the increased condensate density. In this way, the condensate and pump reservoir densities modulate with each other with $\pi/2$ phase delay. The latter (dynamical instability) is caused by the repulsive interaction g_R between the condensate and pump reservoir densities. A local depletion of the pump reservoir density creates a potential well which attracts the condensate particles. This in turn leads to a further drop of the pump reservoir density by the increased stimulated scattering rate. Fig. 9.8(b) shows the dispersion of the elementary excitations for the case of $\gamma_R = \gamma_c$ [9].

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