Chapter 8

Quantum theory of matter-wave lasers

If a cooling time is not negligible compared to a condensate particle escape time from a trap, such a system is necessarily dynamical and must be treated as a non-equilibrium open dissipative system rather than a thermal equilibrium closed system. The steady state population of a ground state in such a system has the pump (reservoir particle injection) rate dependence as shown in Fig. 8.1. At very low pump rates, injected particles are shared by many states so that the probability of finding them in the ground state, which is defined here as a quantum efficiency, is much lower than one. At a certain pump rate, the average population in the ground state reaches one, which is called a quantum degeneracy point. Once the pump rate exceeds this critical point a bosonic final state stimulation accelerates a cooling process of the reservoir particles into the ground state and the ground state population increases nonlinearly. Eventually the quantum efficiency approaches to one at well above threshold (quantum degeneracy point). In this case all injected particles are condensed into the ground state before they escape from a trap. This behavior is similar to the laser phase transition based on the stimulated emission of photons also shown in Fig. 8.1, so that such a dynamical condensate is often referred to as a matter-wave laser [1, 2]. A matter-wave laser is distinct from a photon laser, since the temperature and chemical potential can be defined and play important roles in the formation of an order, while such concepts do not exist in a photon laser. A matter-wave laser is also distinct from BEC, since the amplitude and phase fluctuate dynamically so that the finite spectral linewidth (first-order temporal coherence) and higher-order temporal coherence functions can be defined [3].

How is the quantum theory of matter-wave lasers distinct from the classical treatment? So far we have treated the condensate order parameter ψ_0 as a c-number since there is a macroscopic population in the ground state. The populations of the Bogoliubov quasi-particles at all energies are identically equal to zero at $T = 0$. Is this treatment compatible with quantum mechanics even if we consider a finite lifetime for the condensate particles? Suppose the condensate particles decay from a trap with a rate γ. Then the Heisenberg equation of motion for the field operator $\hat{\psi}_0$ must satisfy

$$\frac{d}{dt}\hat{\psi}_0 = -i\omega\hat{\psi}_0 - \frac{\gamma}{2}\hat{\psi}_0, \tag{8.1}$$

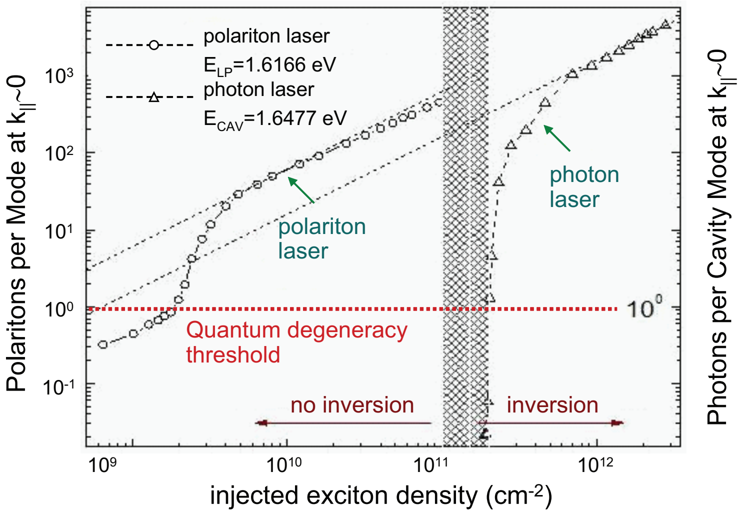

Figure 8.1: Ground state population N_0 vs. pump rate P of a matter-wave (polariton) laser and photon laser [2].

$$\frac{d}{dt}\hat{\psi}_0^+ = i\omega\hat{\psi}_0^+ - \frac{\gamma}{2}\hat{\psi}_0^+. \tag{8.2}$$

Here $\mu = \hbar\omega$ is the chemical potential of the condensate. From (8.1) and (8.2), we have the time dependence of the commutator bracket:

$$\left[\hat{\psi}_0(t), \hat{\psi}_0^+(t)\right] = \left[\hat{\psi}_0(0), \hat{\psi}_0^+(0)\right] e^{-\gamma t}. \tag{8.3}$$

Even though we assume a proper bosonic commutation relation at $t = 0$, i.e. $\left[\hat{\psi}_0(0), \hat{\psi}_0^+(0)\right] = 1$, the commutator bracket is not conserved as a time evolves ($t > 0$). What was wrong with our Heisenberg equations (8.1) and (8.2)? A mistake is that we have neglected the coupling between the condensate and external reservoir fields, which are responsible for a particle loss from a trap. The reservoirs inject "noise" whenever the system (the condensate in our case) dissipates into the reservoirs. This relation is the quantum mechanical analogue of the fluctuation-dissipation theorem [4, 5]. The classical counterpart of this problem is well established in satistical mechanics [6]. The quantum noise injected by the external reservoirs is an ultimate origin for the spectral linewidth and incomplete temporal coherence of the condensate formed in a matter wave laser.

8.1 Heisenberg-Langevin equation for an open dissipate trap

8.1.1 Derivation of the equation

The Hamiltonian of a total system consisting of the condensate in a trap and the external reservoir field outside of a trap should have the same form as that for a photon laser system with an open-dissipate cavity [7],

$$\hat{\mathcal{H}} = \hbar\omega_0\hat{\psi}_0^+\hat{\psi}_0 + \sum_p \hbar\omega_p\hat{\phi}_p^+\hat{\phi}_p + \hbar\sum_p \left(g_p\hat{\psi}_0^+\hat{\phi}_p + g_p^*\hat{\psi}_0\hat{\phi}_p^+\right). \tag{8.4}$$

The Heisenberg equations of motion for the condensate field operator $\hat{\psi}_0$ and the reservoir field operator $\hat{\phi}_p$ are

$$\frac{d}{dt}\hat{\psi}_0 = \frac{1}{i\hbar}\left[\hat{\psi}_0, \hat{\mathcal{H}}\right] = -i\omega_0\hat{\psi}_0 - i\sum_p g_p\hat{\phi}_p, \tag{8.5}$$

$$\frac{d}{dt}\hat{\phi}_p = \frac{1}{i\hbar}\left[\hat{\phi}_p, \hat{\mathcal{H}}\right] = -i\omega_p\hat{\phi}_p - ig_p^*\hat{\psi}_0. \tag{8.6}$$

We can formally integrate (8.6) and substitute the result, $\hat{\phi}_p(t) = \hat{\phi}_p(0)e^{-i\omega_p t} - ig_p^*\int_0^t dt'\hat{\psi}_0(t')e^{-i\omega_p}(t-t')$ into (8.5) to obtain the integro-differential equation:

$$\frac{d}{dt}\hat{\psi}_0(t) = -i\omega_0\hat{\psi}_0(t) - \int_0^t dt'\sum_p |g_p|^2\hat{\psi}_0(t')e^{-i\omega_p(t-t')} - i\sum_p g_p\hat{\phi}_p(0)e^{-i\omega_p t}. \tag{8.7}$$

The summation over the momentum p in (8.7) can be approximated by the integral over p, $\sum_p \to \frac{V}{(2\pi\hbar)^3}\int d^3p$, where V is a quantization volume. We can introduce the slowly varying field operator by $\hat{\psi}_0(t) = \hat{\Phi}_0(t)e^{-i\omega_0 t}$. Substituting these relations, (8.7) can be rewritten as

$$\frac{d}{dt}\hat{\Phi}_0(t) = -\frac{V}{(2\pi\hbar)^3}\int d^3p|g_p|^2\int_0^t dt'\hat{\Phi}_0(t')e^{-i(\omega_p-\omega_0)(t-t')}$$
$$-i\frac{V}{(2\pi\hbar)^3}\int d^3p g_p\hat{\phi}_p(0)e^{-i(\omega_p-\omega_0)t}. \tag{8.8}$$

The first term of R.H.S of (8.8) is simplified by introducing

$$\begin{aligned}\Gamma(\tau) &= \frac{V}{(2\pi\hbar)^3}\int d^3p|g_p|^2e^{-i(\omega_p-\omega_0)\tau} \\ &= \int d\omega_p D(\omega_p)|g(\omega_p)|^2e^{-i(\omega_p-\omega_0)\tau} \\ &= 2\pi D(\omega_0)|g(\omega_0)|^2\delta(\tau),\end{aligned} \tag{8.9}$$

where $\tau = t - t'$ and $D(\omega_p) = \frac{V}{(2\pi\hbar)^3}$ is the energy density of states for reservoir modes. Substituting (8.9) into the first term of R.H.S of (8.8) results in

$$-\int_0^\infty d\tau\Gamma(\tau)\hat{\Phi}_0(t-\tau) = -\frac{\gamma}{2}\hat{\Phi}_0(t), \tag{8.10}$$

where $\gamma = 2\pi D(\omega_0)|g(\omega_0)|^2$ is the Fermi's golden rule decay rate of the condensate particles [7].

The second term of R.H.S. of (8.8) constitutes the noise operator

$$\hat{F}(t) = -i\frac{V}{(2\pi\hbar)^3}\int d^3p g_p\hat{\phi}_p(0)e^{-i(\omega_p-\omega_0)t}. \tag{8.11}$$

The two-time correlation function of this operator can be calculated as

$$\begin{aligned}\left\langle \hat{F}^+(t)\hat{F}(t')\right\rangle &= \sum_p\sum_{p'} g_p^* g_{p'} \left\langle \hat{\phi}_p^+(0)\hat{\phi}_{p'}(0)\right\rangle e^{i(\omega_p-\omega_0)t-i(\omega_{p'}-\omega_0)t'} \\ &= \sum_p |g_p|^2 \left\langle \hat{N}_R\right\rangle e^{i(\omega_p-\omega_0)(t-t')} \\ &= \gamma\left\langle \hat{N}_R\right\rangle \delta(t-t'), \end{aligned} \tag{8.12}$$

where $\left\langle \hat{N}_R\right\rangle = \frac{1}{\exp[(\varepsilon_0-\mu_R)/k_BT_R]-1}$ is the reservoir particle population at the energy ε_0. Here μ_R and T_R are the chemical potential and temperature of the reservoir. Similarly we obtain

$$\left\langle \hat{F}(t)\hat{F}^+(t')\right\rangle = \gamma\left(1+\left\langle \hat{N}_R\right\rangle\right)\delta(t-t'). \tag{8.13}$$

Using (8.10) and (8.11), (8.8) is rewritten as

$$\frac{d}{dt}\hat{\Phi}_0(t) = -\frac{\gamma}{2}\hat{\Phi}_0(t) + \hat{F}(t), \tag{8.14}$$

where the noise operator $\hat{F}(t)$ satisfies the two-time correlation functions (8.12) and (8.13). This is the Heisenberg-Langevin equation, in which the dynamics of the external reservoirs is decoupled from the system (condensate). An important and implicit assumption behind the above derivation is that the average population $\left\langle \hat{N}_R\right\rangle$ of the reservoir can be evaluated independently from the system dynamics and it is determined by the thermal equilibrium properly of the reservoirs. This is called a reservoir approximation.

8.1.2 Commutator bracket conservation

We can write the time evolution of the condensate field operator as

$$\begin{aligned}\hat{\psi}_0(t) &= \hat{\psi}_0(t-\tau) + \int_{t-\tau}^t dt' \left[\frac{d}{dt}\hat{\psi}_0(t')\right] \\ &= \hat{\psi}_0(t-\tau) + \int_{t-\tau}^t dt' \left[\left(-i\omega_0 + \frac{\gamma}{2}\right)\hat{\psi}_0(t') + \hat{f}(t')\right], \end{aligned} \tag{8.15}$$

where $\hat{f}(t') = \hat{F}(t')e^{-i\omega_0 t'}$. Using the above expression, we can evaluate the expectation values of the following product operators at equal time:

$$\begin{aligned}\left\langle \hat{\psi}_0^+(t)\hat{f}(t)\right\rangle &= \left\langle \hat{\psi}_0^+(t-\tau)\hat{f}(t)\right\rangle + \int_{t-\tau}^t dt' \left\langle\left[\left(i\omega_0+\frac{\gamma}{2}\right)\hat{\psi}_0^+(t') + \hat{f}^+(t')\right]\hat{f}(t)\right\rangle \\ &= \int_{t-\tau}^t dt' \left\langle \hat{f}^+(t')\hat{f}(t)\right\rangle \\ &= \frac{1}{2}\gamma\left\langle \hat{N}_R\right\rangle, \end{aligned} \tag{8.16}$$

$$\left\langle \hat{f}^+(t)\hat{\psi}_0(t)\right\rangle = \frac{1}{2}\gamma\left\langle \hat{N}_R\right\rangle, \tag{8.17}$$

$$\left\langle \hat{\psi}_0(t)\hat{f}^+(t)\right\rangle = \left\langle \hat{f}(t)\hat{\psi}_0^+(t)\right\rangle = \frac{1}{2}\gamma\left(1 + \left\langle \hat{N}_R\right\rangle\right). \tag{8.18}$$

Here we used the facts that the past system operator $\hat{\psi}_0^+(t')$ does not depend on the future reservoir operator $\hat{f}(t)$ and that there is no order parameter in the reservoir, so that

$$\left\langle \hat{\psi}_0^+(t')\hat{f}(t)\right\rangle = \left\langle \hat{\psi}_0^+(t')\right\rangle \left\langle \hat{f}(t)\right\rangle = 0. \tag{8.19}$$

Using (8.16)-(8.18), we can now evaluate the time evolution of the commutator bracket:

$$\begin{aligned}
\frac{d}{dt}\left\langle \left[\hat{\psi}_0(t), \hat{\psi}_0^+(t)\right]\right\rangle &= \left\langle \dot{\hat{\psi}}_0(t)\hat{\psi}_0^+(t) + \hat{\psi}_0(t)\dot{\hat{\psi}}_0^+(t) - \dot{\hat{\psi}}_0^+(t)\hat{\psi}_0(t) - \hat{\psi}_0^+(t)\dot{\hat{\psi}}_0(t)\right\rangle \\
&= -\gamma\left\langle \left[\hat{\psi}_0(t), \hat{\psi}_0^+(t)\right]\right\rangle + \left\langle \hat{f}(t)\hat{\psi}_0^+(t) + \hat{\psi}_0(t)\hat{f}^+(t)\right\rangle - \left\langle \hat{f}^+(t)\hat{\psi}_0(t) + \hat{\psi}_0^+(t)\hat{f}(t)\right\rangle \\
&= \gamma\left\{1 - \left\langle \left[\hat{\psi}_0(t), \hat{\psi}_0^+(t)\right]\right\rangle\right\}.
\end{aligned} \tag{8.20}$$

Equation (8.20) guarantees that if the commutator bracket is conserved at a time t, it is conserved at all later time. The noise operator $\hat{f}(t)$ (or $\hat{F}(t)$) is needed to conserve the proper commutator bracket in such an open dissipative system.

8.1.3 Einstein relation between drift and diffusion coefficients

The Heisenberg-Langevin equation for a system operator $\hat{A}_\mu$ has a general form

$$\frac{d}{dt}\hat{A}_\mu(t) = \hat{D}_\mu(t) + \hat{F}_\mu(t), \tag{8.21}$$

where $\hat{D}_\mu(t)$ and $\hat{F}_\mu(t)$ are drift and diffusion (noise) operators, respectively. The coefficient for the drift operator can be obtained from the classical equation of motion for the same system, i.e.

$$\left\langle \hat{D}_\mu(t)\right\rangle = \frac{d}{dt}\left\langle \hat{A}_\mu(t)\right\rangle. \tag{8.22}$$

We can define the diffusion coefficient $D_{\mu\nu}$ by

$$\left\langle \hat{F}_\mu(t)\hat{F}_\nu(t')\right\rangle = 2D_{\mu\nu}\delta(t - t'). \tag{8.23}$$

Let us evaluate the following product of two system operators that satisfy the Heisenberg-Langevin equation.

$$\begin{aligned}
\frac{d}{dt}\left\langle \hat{A}_\mu(t)\hat{A}_\nu(t)\right\rangle &= \left\langle \dot{\hat{A}}_\mu(t)\hat{A}_\nu(t) + \hat{A}_\mu(t)\dot{\hat{A}}_\nu(t)\right\rangle \\
&= \left\langle \hat{D}_\mu(t)\hat{A}_\nu(t)\right\rangle + \left\langle \hat{F}_\mu(t)\hat{A}_\nu(t)\right\rangle \\
&\quad + \left\langle \hat{A}_\mu(t)\hat{D}_\nu(t)\right\rangle + \left\langle \hat{A}_\mu(t)\hat{F}_\nu(t)\right\rangle.
\end{aligned} \tag{8.24}$$

The second and fourth terms of the above equation are already evaluated in (8.16) and equal to $D_{\mu\nu}$. Therefore, the diffusion coefficient $D_{\mu\nu}$ can be given by

$$2D_{\mu\nu} = -\left\langle \hat{D}_\mu(t)\hat{A}_\nu(t)\right\rangle - \left\langle \hat{A}_\mu(t)\hat{D}_\nu(t)\right\rangle, \tag{8.25}$$

where we assume the steady state condition, $\frac{d}{dt}\left\langle \hat{A}_\mu(t)\hat{A}_\nu(t)\right\rangle = 0$. Equation (8.25) is called the Einstein relation between the drift and diffusion coefficients.

The Einstein relation allows us to derive the Heisenberg-Langevin equation and the two-time correlation function of the noise operators if we already know the corresponding classical equation of motion.

8.2 Heisenberg-Langevin equations for dynamical condensates

8.2.1 Derivation of the equations

The classical equations of motion for a dynamical condensate are an open dissipative Gross-Pitaevskii equation (7.30) for the condensate order parameter and a rate equation (7.31) for the pump reservoir population. Please note that the pump reservoir is different from the external reservoir treated in Sec. 8.1. The conceptual diagram for the particle flow is shown in Fig. 8.2. Before we quantize those equations, we separate the condensate

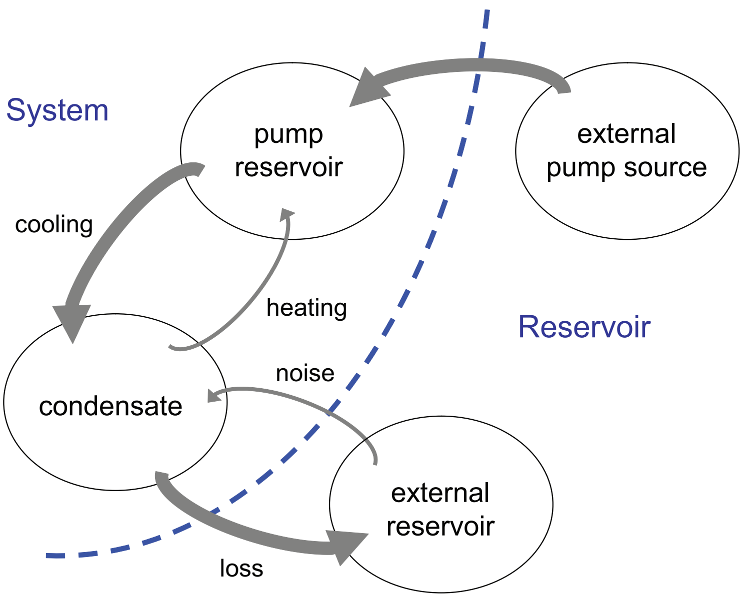

Figure 8.2: The dynamical condensate coupled to the pump reservoir and the external reservoir.

order parameter into the product of the excitation amplitude and the spatial wavefunction:

$$\psi_0(r,t) = a(t)u_0(r) = A(t)e^{-i\omega_0 t}u_0(r). \tag{8.26}$$

Here the ground state spatial wavefunction $u_0(r)$ and the oscillation frequency ω_0 (or the chemical potential $\hbar\omega_0$) are determined by the balance between the kinetic energy $-\frac{\hbar^2}{2m}\nabla^2$, the confining potential $V_{\text{ext}}(r)$ and the particle-particle interaction. We neglect the condensate-condensate and condensate-reservoir interaction terms except for their contribution to $u_0(r)$ and ω, i.e. $g_c = g_R = 0$ for simplicity. Similarly we can split the pump reservoir population into the product of the time dependent and space dependent parts:

$$n_R(r,t) = n(t)v_R(r). \tag{8.27}$$

The normalization conditions for $u_0(r)$ and $v_R(r)$ are given by

$$\int |u_0(r)|^2 dr = \int |v_R(r)|^2 dr = 1. \tag{8.28}$$

The classical equation of motions for $A(t)$ and $n(t)$ are now rewritten as

$$\frac{d}{dt}A(t) = -\frac{1}{2}\left[\gamma_c - \Gamma n(t)\right]A(t), \tag{8.29}$$

$$\frac{d}{dt}n(t) = p(t) - \gamma_R n(t) - \Gamma n(t)|A(t)|^2. \tag{8.30}$$

Here $p(t)$ is the (time dependent) pump rate and $\Gamma = R\int u_0(r)v_R(r)dr$ is the effective gain coefficient. The Heisenberg-Langevin equation can be obtained by replacing $A(t), n(t)$ and $N_c = |A(t)|^2$ by the operators $\hat{A}(t), \hat{n}(t)$ and $\hat{A}^+(t)\hat{A}(t)+1$, respectively, and adding the noise operators:

$$\frac{d}{dt}\hat{A}(t) = -\frac{1}{2}\left[\gamma_c - \Gamma\hat{n}(t)\right]\hat{A}(t) + \hat{F}_A(t), \tag{8.31}$$

$$\frac{d}{dt}\hat{n}(t) = p(t) - \gamma_R\hat{n}(t) - \Gamma\hat{n}(t)\left[\hat{A}^+(t)\hat{A}(t)+1\right] + \hat{F}_n(t). \tag{8.32}$$

The two-time correlation functions of the noise operators can be obtained using the Einstein relation:

$$\left\langle \hat{F}_A^+(t)\hat{F}_A(t')\right\rangle = \delta(t-t')\left[\gamma_c\left\langle \hat{N}_R\right\rangle + \Gamma\left\langle \hat{n}\right\rangle\left\langle \hat{A}^+\hat{A}\right\rangle\right], \tag{8.33}$$

$$\left\langle \hat{F}_A(t)\hat{F}_A^+(t')\right\rangle = \delta(t-t')\left[\gamma_c\left(\left\langle \hat{N}_R\right\rangle+1\right) + \Gamma\left\langle \hat{n}\right\rangle\left(\left\langle \hat{A}^+\hat{A}\right\rangle+1\right)\right], \tag{8.34}$$

$$\left\langle \hat{F}_n(t)\hat{F}_n(t')\right\rangle = \delta(t-t')2\left[p + \gamma_R\left\langle \hat{n}\right\rangle + \Gamma\left\langle \hat{n}\right\rangle\left(\left\langle \hat{A}^+\hat{A}\right\rangle+1\right)\right]. \tag{8.35}$$

The Heisenberg-Langevin equation for the condensate population $\hat{N}_c \equiv \hat{A}^+\hat{A}$ can be derived for (8.31),

$$\frac{d}{dt}\hat{N}_c(t) = -\gamma_c\hat{N}_c(t) + \Gamma\hat{n}(t)\left[\hat{N}_c(t)+1\right] + \hat{F}_N(t). \tag{8.36}$$

Here the two-time correlation functions for the new noise operator are given by

$$\left\langle \hat{F}_N(t)\hat{F}_N(t')\right\rangle = \delta(t-t')2\left[\gamma_c\left\langle \hat{N}_c\right\rangle + \Gamma\left\langle \hat{n}\right\rangle\left(\left\langle \hat{N}_c\right\rangle+1\right)\right], \tag{8.37}$$

$$\left\langle \hat{F}_N(t)\hat{F}_n(t')\right\rangle = -\delta(t-t')2\Gamma\left\langle \hat{n}\right\rangle\left(\left\langle \hat{N}_c\right\rangle+1\right). \tag{8.38}$$

Equations (8.32) and (8.36) constitute the quantum mechanical rate equations for the condensate population operator $\hat{N}_c(t)$ and the reservoir population operator $\hat{n}(t)$, where the two corresponding noise operators are negatively correlated as shown in (8.38). We will see the important consequence of this fact in the next section.

8.2.2 Linearization and noise spectrum

When the pump rate is well above the critical (quantum degeneracy) point (see Fig. 8.1), the condensate forms an order parameter with well-stabilized amplitude and phase except for small fluctuations, so that we can expand the excitation amplitude as

$$\hat{A}(t) = \left[A_c + \Delta\hat{A}(t)\right] e^{-i\Delta\hat{\phi}(t)}, \tag{8.39}$$

when A_c is a c-number average amplitude while $\Delta\hat{A}(t)$ and $\Delta\hat{\phi}(t)$ are the amplitude and phase noise operators. Using (8.39) we can also linearize the condensate population operator as

$$\hat{N}_c(t) = A_c^2 + 2A_c\Delta\hat{A}(t) = N_c + \Delta\hat{N}(t). \tag{8.40}$$

The pump reservoir population operator can be similarly linearized

$$\hat{n}(t) = n_R + \Delta\hat{n}(t). \tag{8.41}$$

If we substitute (8.40) and (8.41) into (8.32) and (8.36), take the ensemble averages and assume $\frac{d}{dt}\left\langle \hat{N}_c \right\rangle = \frac{d}{dt}\langle \hat{n} \rangle = 0$, we have the following steady state solutions

$$\gamma_c = \Gamma n_R, \tag{8.42}$$

$$p = \Gamma n_R A_c^2 = \gamma_c A_c^2, \tag{8.43}$$

where we assumed $\gamma_R \ll \Gamma A_c^2$, i.e. the pump reservoir population is mostly depleted by the stimulated scattering into the condensate. The physical meaning of these relations is clear. Equation (8.42) indicates that the loss rate of the condensate particles must be balanced by the gain rate (stimulated scattering rate) from the pump reservoir particles. Equation (8.43) shows that all injected particles per second, p, is extracted as the leakage condensate particles per second, $\gamma_c A_c^2$, which suggests the quantum efficiency is equal to one at well above condensation threshold.

The equations of motion for the fluctuation operators $\Delta\hat{A}$, $\Delta\hat{\phi}$ and $\Delta\hat{n}$ can be obtained by using (8.39), (8.41), (8.42) and (8.43) in the Heisenberg-Langevin equations,

$$\frac{d}{dt}\Delta\hat{A}(t) = \frac{1}{2A_c\tau_{st}}\Delta\hat{n}(t) + \frac{1}{2}\left[\hat{F}_A(t)e^{i\Delta\hat{\phi}(t)} + e^{-i\Delta\hat{\phi}(t)}\hat{F}_A^+(t)\right], \tag{8.44}$$

$$\frac{d}{dt}\Delta\hat{\phi}(t) = \frac{i}{2A_c}\left[\hat{F}_A(t)e^{i\Delta\hat{\phi}(t)} - e^{-i\Delta\hat{\phi}(t)}\hat{F}_A^+(t)\right], \tag{8.45}$$

$$\frac{d}{dt}\Delta\hat{n}(t) = -\left(\frac{1}{\tau_{st}} + \frac{1}{\tau_{sp}}\right)\Delta\hat{n}(t) - 2\gamma_c A_c\Delta\hat{A}(t) + \hat{F}_n(t). \tag{8.46}$$

Here $\tau_{sp} = \frac{1}{\gamma_R}$ is the spontaneous decay time of the pump reservoir population and $\tau_{st} = \frac{1}{\Gamma A_c^2}$ is the stimulated decay time. The Fourier analysis of (8.44)-(8.46) provides the amplitude and phase noise spectra as well as the pump reservoir population spectrum. At well above threshold, the amplitude spectrum is reduced to the Lorentzian

$$S_{\Delta A}(\Omega) = \frac{\gamma_c}{\Omega^2 + \gamma_c^2}. \tag{8.47}$$

Using the Parseval theorem $\left\langle \Delta\hat{A}^2 \right\rangle = \int_0^\infty \frac{d\Omega}{2\pi} S_{\Delta A}(\Omega)$, the variance of the amplitude noise operator $\Delta\hat{A}$ (or the population noise operator $\Delta\hat{N}$) is equal to

$$\left\langle \Delta\hat{A}^2 \right\rangle = \frac{1}{4}, \tag{8.48}$$

$$\left\langle \Delta\hat{N}^2 \right\rangle = 4A_c^2 \left\langle \Delta\hat{A}^2 \right\rangle = N_c. \tag{8.49}$$

The amplitude noise is equal to that of a coherent state and the condensate particle statistics obey the Poisson distribution. This result supports posteriori the assumption and argument on the phase-locking between the condensate and excitations that are presented in the previous chapter. The finite amplitude noise $\left\langle \Delta\hat{A}^2 \right\rangle = 1/4$ and particle number noise $\left\langle \Delta\hat{N}^2 \right\rangle = N_c$ in the condensate are traced back to the zero-point fluctuation (vacuum field) injected from outside of the trap, which appears as the diffusion coefficient γ_c in eq. (8.34), and the pump fluctuation, which appears as the diffusion coefficient p in (8.35). Their contributions are equal at well above threshold. We assumed that no particles are injected from the external reservoir outside the trap, $\left\langle \hat{N}_R \right\rangle = 0$, and the pump rate obeys the Poisson point process with full shot noise. All the other noise sources are canceled out due to the negative correlation between the condensate noise source and the pump reservoir noise source, represented by (8.38)

At well above threshold, the phase noise spectrum is given by

$$S_{\Delta\phi}(\Omega) = \frac{\gamma_c}{\Omega^2 A_c^2}. \tag{8.50}$$

The phase noise spectrum is inversely proportional to squared frequency, which is a characteristic of the random walk phase diffusion and called the Wiener-Levy process [7]. The coupled equations (8.44) and (8.46) for $\Delta\hat{A}(t)$ and $\Delta\hat{n}(t)$ provide the stabilization force for the amplitude which counteracts the noise driving forces $\hat{F}_A$ and $\hat{F}_n$. As a result of this competition, the amplitude noise becomes identical to that of the coherent state. The equation of motion (8.45) for $\Delta\hat{\phi}(t)$ does not have such a restoring force and consequently the phase diffuses freely:

$$\left\langle \Delta\hat{\phi}^2(t) \right\rangle = 2D_\phi t, \tag{8.51}$$

where the diffusion constant is given by

$$2D_\phi = \frac{\gamma_c}{2A_c^2}. \tag{8.52}$$

The spectral lineshape of the condensate is calculated by the correlation function,

$$\begin{aligned} I(\omega) &= \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \left\langle \hat{A}^+(t)\hat{A}(0) \right\rangle \\ &\propto \int_0^{\infty} d\tau e^{-i\omega\tau} e^{-\left\langle \Delta\hat{\phi}(\tau)^2 \right\rangle}. \end{aligned} \tag{8.53}$$

Because of the above phase diffusion process with an exponential decay, the condensate energy (or frequency) is broadened to a Lorentzian shape with a full width at half-maximum (FWHM) of $2D_\phi$ [7].

8.2.3 Excess particle number noise and higher-order coherence functions

The Heisenberg-Langevin equations (8.32) and (8.36) assume that the stimulated and spontaneous cooling rate into the condensate, $\Gamma\hat{n}(t)\left[\hat{N}_c(t)+1\right]$ is proportional to the pump reservoir population $\hat{n}(t)$. In order to satisfy the energy conservation, the difference in the pump reservoir particle energy and the condensate particle energy must be absorbed in a third physical system. In the case of an exciton-polariton condensate, this energy conservation is satisfied by phonon emission [8]. In fact, this is not a sole cooling channel and there is an extra cooling mechanism, four wave mixing, in which two pump reservoir particles scatter into one condensate particle and one hot particle with the momentum and energy conservation satisfied. This second cooling mechanism is a dominating channel over the first one and results in the heating of the pump reservoir particles when a quantum degeneracy point is exceeded and the stimulated four wave mixing is switched on. As a result of the heating of the pump reservoir particles, the clumping of the pump reservoir population above the condensate threshold is not perfect but remains mild as shown in Fig. 8.3[8].

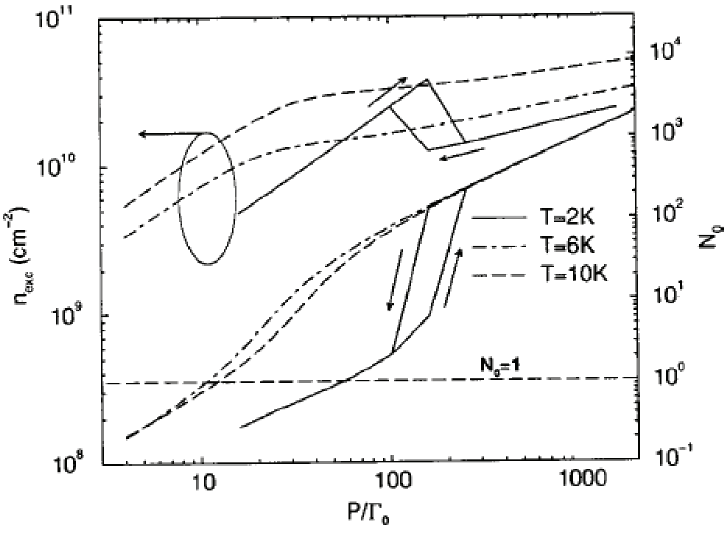

Figure 8.3: Pump reservoir population n_R (left) and condensate population N_c (right) vs. normalized pump rate p/γ_c [8].

This imperfect gain clumping means that the negative correlation between $\hat{F}_N(t)$ and $\hat{F}_n(t)$ in (8.38) never cancel the noise associated with the stimulated and spontaneous cooling processes completely. Because of the imperfect gain clumping, the particle number noise $\left\langle\Delta\hat{N}_c^2\right\rangle$ is below that of the thermal state, $\left\langle\hat{N}_c\right\rangle\left(\left\langle\hat{N}_c\right\rangle+1\right)$, but considerably higher than the Poisson limit, $\left\langle\hat{N}_c\right\rangle$, as shown in Fig. 8.4 even at well above the condensate threshold [8].

The excess particle number noise is experimentally characterized by the measurement

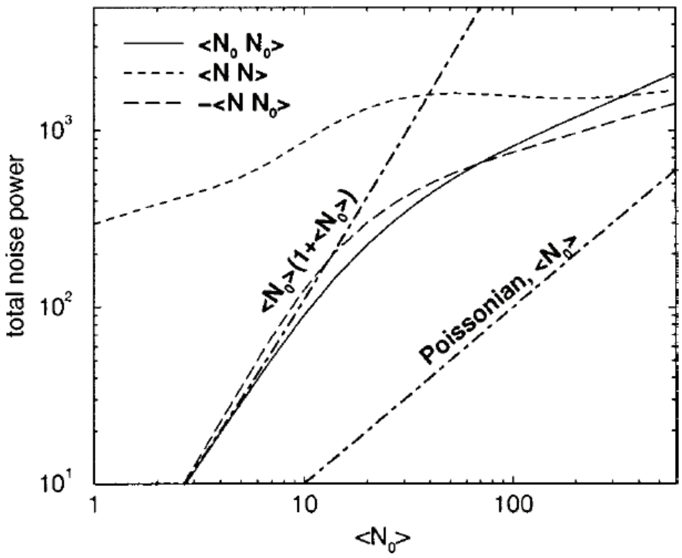

Figure 8.4: The particle number noise $\left\langle \Delta\hat{N}_c^2 \right\rangle$ vs. the average particle number $\left\langle \hat{N}_c \right\rangle$ of an exciton-polariton condensate [8].

of higher-order coherence functions $g^{(n)}(0)$ defined by [9]

$$g^{(n)}(0) = \frac{\langle \hat{\psi}_1^+(0)\hat{\psi}_2^+(0)\cdots\hat{\psi}_n^+(0)\hat{\psi}_n(0)\cdots\hat{\psi}_1(0)\rangle}{\langle \hat{\psi}_1^+(0)\hat{\psi}_1(0)\rangle\langle \hat{\psi}_2^+(0)\hat{\psi}_2(0)\rangle\cdots\langle \hat{\psi}_n^+(0)\hat{\psi}_n(0)\rangle}, \tag{8.54}$$

where $\hat{\psi}_i(0)$ is the field operator in front of a particle detector i at time $t = 0$. The advantage of measuring $g^{(n)}(0)$ rather than $\langle\Delta\hat{N}_c^2\rangle$ is that $g^{(n)}(0)$ is independent of particle loss between the trap and the particle detector, while the direct measurement of $\langle\Delta\hat{N}_c^2\rangle$ suffers from the evolution of the noise power according to

$$\langle\Delta\hat{N}^2\rangle = L^2\langle\Delta\hat{N}_c^2\rangle + L(1-L)\langle\hat{N}_c\rangle, \tag{8.55}$$

where L is the particle loss including the quantum efficiency of the particle detector [10]. Figure 8.5 shows the measured $g^{(2)}(0)$ and $g^{(3)}(0)$ vs. the pump rate p/p_{th} for an exciton-polariton condensate [11]. The experimental results agree well with the theoretical predictions at well above the condensate threshold. The discrepancy between the experimental and theoretical results near the threshold is an experimental artifact: the response time of the particle detector is larger than the correlation time of the particle number noise so that the real $g^{(2)}(0)$ and $g^{(3)}(0)$ values of the condensate are washed out by the integration effect of measurement.

If the condensate is in a coherent state, the particle number statistics obey a Poisson distribution and the coherence functions are coherent in all orders:

$$g^{(n)}(0) = 1 \quad \text{(coherent state)}. \tag{8.56}$$

On the other hand, if the condensate is in a thermal state, the particle number statistics features a super-Poisson distribution and the coherence functions are

$$g^{(n)}(0) = n! \quad \text{(thermal state)}. \tag{8.57}$$

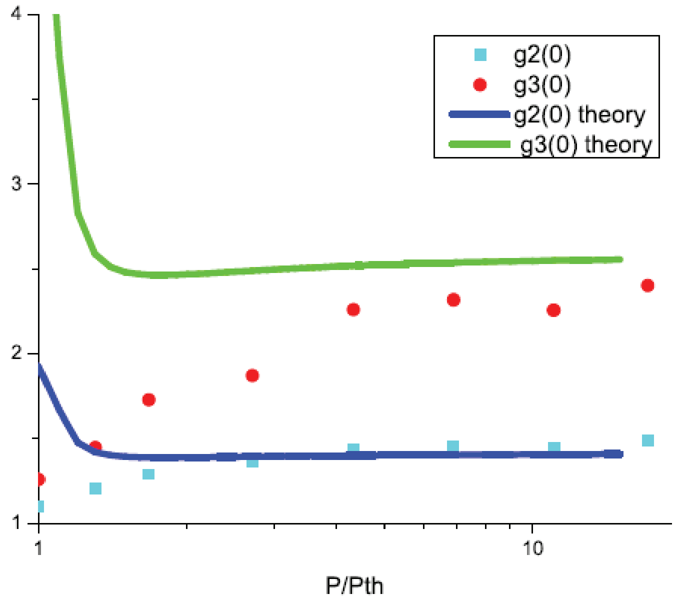

Figure 8.5: The theoretical and experimental second- and third-order coherence functions $g^{(2)}(0)$ and $g^{(3)}(0)$ of an exciton-polariton condensate [11].

The theoretical and experimented results, $g^{(2)}(0) \sim 1.4$ and $g^{(3)}(0) \sim 2.6$, shown in Fig. 8.5 indicate the condensate is not a simple thermal state but not close to a coherent state, either. The results are compatible with the results shown in Fig. 8.4.

8.2.4 Excess phase noise and spectral linewidth

So far we have neglected the self-phase modulation term, $g_c|\psi_0(r,t)|^2\psi_0(r,t)$, and the cross-phase modulation term, $g_R n_R(r,t)\psi_0(r,t)$, in the open dissipative Gross-Pitaevskii equation. If we keep these two terms in the quantization of the Gross-Pitaevskii equation, the R.H.S. of the Heisenberg-Langevin equation (8.45) is supplemented by these two terms:

$$\frac{d}{dt}\Delta\hat{\phi}(t) = \frac{i}{2A_c}\left[\hat{F}_A(t)e^{i\Delta\hat{\phi}(t)} - e^{-i\Delta\hat{\phi}(t)}\hat{F}_A^{+}(t)\right] + \frac{3g_c}{\hbar}\Delta\hat{N}(t) + \frac{g_R}{\hbar}\Delta\hat{n}_R(t). \qquad (8.58)$$

The phase is not only modulated by the intrinsic Langevin noise source $\hat{F}_A(t)$ but also by the condensate population noise $\Delta\hat{N}(t)$ and the pump reservoir population noise $\Delta\hat{n_R}(t)$ through the condensate-condensate and condensate-pump reservoir repulsive interactions. As shown in Fig. 8.4, both $\left\langle \Delta\hat{N}^2 \right\rangle$ and $\left\langle \Delta\hat{N}_R^2 \right\rangle$ increase with the pump rate, while the intrinsic phase diffusion noise decrease with the pump rate (see (8.52)). Such an expected behavior is demonstrated in Fig. 8.6, where the spectral linewidth decreases according to the Schawlow-Townes linewidth at just above the threshold but increases due to the population fluctuations $\Delta\hat{N}(t)$ and $\Delta\hat{n}_R(t)$ at well above the threshold.

The unique behavior of the spectral linewidth of a dynamical condensate was experimentally confirmed as shown in Fig. 8.7[12]. In this particular experiment, the injected pump reservoir particles (exciton-polaritons) are spin unpolarized (a) and polarized (b). When the pump reservoir has a mixture of up-spin and down-spin species, the cooling

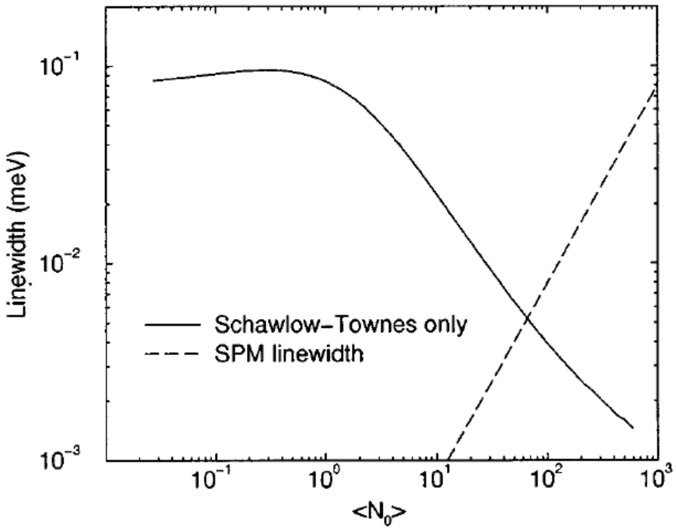

Figure 8.6: The spectral linewidth vs. condensate population for an exciton-polariton condensate [8].

process is more efficient so that the gain clumping becomes more perfect and the particle number noise is well suppressed. Consequently, the linewidth broadening at well above the condensate threshold is rather mild (Fig. 8.7(a)). On the other hand, when the pump reservoir has single spin species, the cooling process is less efficient so that the gain clumping becomes less perfect and the particle number noise is not well suppressed. As a result of this, the line broadening at well above the condensate threshold is severe (Fig. 8.7(b)).

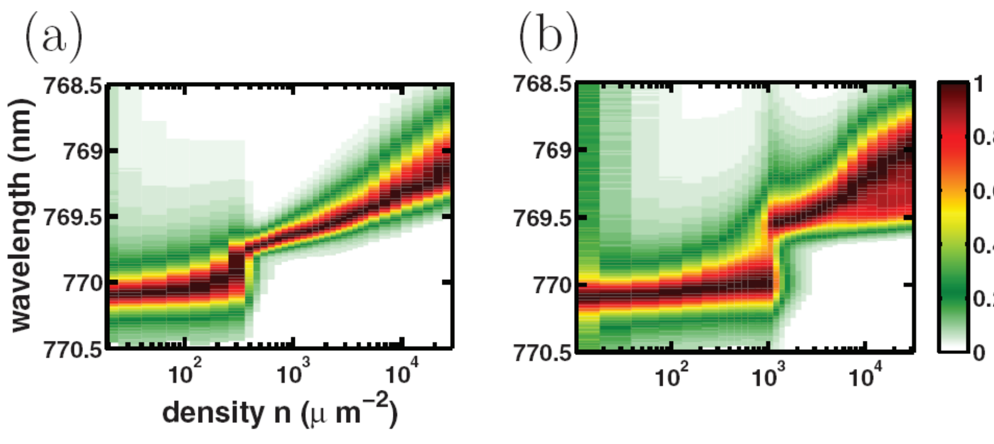

Figure 8.7: (a) The measured spectra near zero momentum ($|k_x| < 0.55\ \mu\text{m}^{-1}$) for linearly polarized pumping ($\theta_p = 90°, \theta_d = 0°$) as a function of polariton density. (b) Same spectra for left-circularly polarized pump and right-circularly polarized detection [12].

8.2.5 Input-output formalism

When we probe the quantum statistical properties of a dynamical condensate, our detector interacts with the output field leaked out of the trap rather than internal field inside the trap. This fact requires us to take one more step of calculation. The result of such theoretical analysis is somewhat surprising: the quantum statistics of the output field is different from those of the internal field, which is truly a quantum mechanical effect.

The output field operator $\hat{r}(t)$ is related to the internal field operator $\hat{A}(t)$ and the incident vacuum field $\hat{f}(t)$ by [5, 10]

$$\hat{r}(t) = \sqrt{\gamma_c}\hat{A}(t) - \hat{f}(t), \tag{8.59}$$

where $\langle \hat{r}^+\hat{r} \rangle$ and $\left\langle \hat{f}^+\hat{f} \right\rangle$ are normalized to the average particle flux per second while $\left\langle \hat{A}^+\hat{A} \right\rangle$ is normalized to the dimension-less particle number. We can express the output field operator as $\hat{r}(t) = \hat{A}_e(t)e^{-i\Delta\hat{\phi}_e(t)}$. At well above threshold, the spectrum of the amplitude noise $\Delta\hat{A}_e(t)$ is reduced to the white noise

$$S_{\Delta A_e}(\Omega) = \frac{1}{2}. \tag{8.60}$$

A shown in Fig.8.8(a), the amplitude noise spectrum at frequencies lower than γ_c stems from the pump noise, while that at frequencies higher than γ_c is traced back to the incident vacuum field fluctuation. We can calculate the spectrum of the particle flux operator $\hat{N}_e = \hat{r}^+\hat{r}$ by using the linearization, $\hat{N}_e \simeq \gamma_c A_c^2 + 2\sqrt{\gamma_c}A_c\Delta\hat{A}_e = N_e + \Delta\hat{N}_e$

$$S_{\Delta N_e}(\Omega) = 4\gamma_c A_c^2 S_{\Delta A_e}(\Omega) = 2N_e. \tag{8.61}$$

This is the full shot noise without any cut-off frequency. This means that if we integrate the output particles for an arbitrary time interval τ, the statistics is always Poissonian.

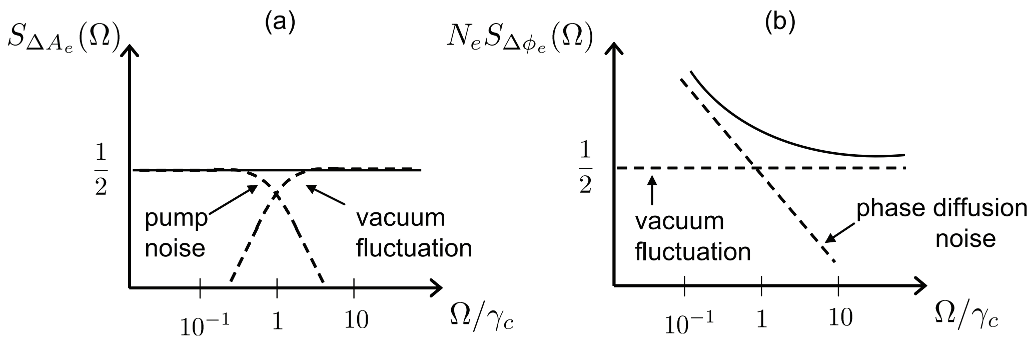

Figure 8.8: (a) The amplitude noise spectrum and (b) phase noise spectrum of the output particle field from the matter-wave laser.

At well above threshold, the spectrum of the phase noise $\Delta\hat{\phi}_e(t)$ is given by

$$N_e S_{\Delta\phi_e}(\Omega) = \frac{1}{2}\left(1 + \frac{2\gamma_c^2}{\Omega^2}\right), \tag{8.62}$$

where the first term of R.H.S. of (8.62) is due to the incident vacuum field fluctuation and the second term is contributed by the random walk phase diffusion noise studied in the previous section. In the sprit of the linearization, i.e. the small fluctuations around the large average value, the phase noise can be calculated using (8.62). At frequencies well above γ_c, the product of $S_{\Delta A_e}(\Omega)$ and $N_e S_{\Delta \phi_e}(\Omega)$ is equal to 1/4, which is the spectral minimum uncertainty product and the unique property of a broadband coherent state [10]. At frequencies well below γ_c, the random-walk phase diffusion noise enhances the phase noise so that the minimum uncertainty product is not satisfied.

Bibliography

[1] A. Imamoglu, R. J. Ram, S. Pau, and Y. Yamamoto. Nonequilibrium condensates and lasers without inversion: Exciton-polariton lasers. *Phys. Rev. A*, 53:4250–4253, 1996.

[2] H. Deng, G. Weihs, G. Snoke, J. Bloch, and Y. Yamamoto. Polariton lasing vs. photon lasing in a semiconductor microcavity. *Proc. Natl. Acad. Sci.*, 100:15318–15323, 2003.

[3] H. Deng, G. Weihs, C. Santori, J. Bloch, and Y. Yamamoto. Condensation of semiconductor microcavity exciton polaritons. *Science*, 298:199–202, 2002.

[4] R. L. Stratonovich. *Nonlinear nonequilibrium thermodynamics I. Linear and nonlinear fluctuation-dissipation theorems.* Springer-Verlag, Berlin, 1992.

[5] C. W. Gardiner and P. Zoller. *Quantum Noise.* Springer-Verlag, Heidelberg, 1999.

[6] F. Rief. *Fundamentals of Statistical and Thermal Physics.* McGraw-Hill Science, New York, 1965.

[7] M. Sargent, M. O. Scully, and W. E. Lamb. *Laser Physics.* Addison-Wesley, New York, 1974.

[8] F. Tassone and Y. Yamamoto. Lasing and squeezing of composite bosons in a semiconductor microcavity. *Phys. Rev. A*, 62:063809, 2000.

[9] Roy J. Glauber. The quantum theory of optical coherence. *Phys. Rev.*, 130(6):2529–2539, 1963.

[10] Y. Yamamoto and A. Imamoglu. *Mesoscopic Quantum Optics.* Wiley-Interscience, 1999.

[11] T. Horikiri, P. Schwendimann, A. Quattropani, S. Hofling, a. Forchel, and Y. Yamamoto. Higher order coherence of exciton-polariton condensates. *Phys. Rev. B*, cond-mat/0910.2978v2, 2010. (to be published).

[12] G. Roumpos, C-W. Lai, T. C. H. Liew, Y. G. Rubo, A. V. Kavokin, and Y. Yamamoto. Signature of the microcavity exciton˘polariton relaxation mechanism in the polarization of emitted light. *Phys. Rev. B*, 79:195310, 2009.