Chapter 7

Several non-trivial issues in Bose-Einstein condensation

This chapter reviews several important but subtle issues of Bose-Einstein condensation that are often overlooked in the standard argument. We will discuss such issues as condensate fragmentation, population fluctuations and phase-locking, dimensionality, and dynamical effects due to finite condensate lifetime.

7.1 Condensate fragmentation

The standard argument of BEC, presented in Chapters 3 and 4, leaves one central question unanswered: Why do the condensate particles accumulate in a single state? Why don't they share between several states that are degenerate or nearly degenerate, so that it makes no difference in the thermodynamic limit? The answer is non-trivial: it is the exchange interaction energy (Fock term) that makes condensate fragmentation costly [1].

Let us consider the interaction Hamiltonian (See eq.(3.6))

$$\hat{\mathcal{H}}_I = \frac{1}{2V} \sum_{p,p',q} V_q \hat{a}_p^+ \hat{a}_{p'}^+ \hat{a}_{p'-q} \hat{a}_{p+q}. \tag{7.1}$$

for two cases:

Case 1: Single state condensation

If all N particles are condensed in the lowest energy state

$$|\psi_0\rangle = \frac{1}{\sqrt{N!}} \left(\hat{a}_0^+\right)^N |0\rangle = |N\rangle, \tag{7.2}$$

the corresponding interaction energy is

$$\begin{aligned} E_0 \equiv \left\langle \psi_0 | \hat{\mathcal{H}}_I | \psi_0 \right\rangle &= \frac{V_0}{2V} \langle N | \hat{a}_0^{+2} \hat{a}_0^2 | N \rangle \\ &= \frac{V_0}{2V} N(N-1) \\ &\simeq \frac{V_0}{2V} N^2. \end{aligned} \tag{7.3}$$

Case 2: Two state condensation
On the other hand, if the condensate is fragmented into two states 1 and 2, with populations N_1 and N_2 $(N_1 + N_2 = N)$

$$|\psi_{12}\rangle = \frac{1}{\sqrt{N_1!N_2!}} \left(\hat{a}_1^{+}\right)^{N_1} \left(\hat{a}_2^{+}\right)^{N_2} |0\rangle = |N_1\rangle_1 |N_2\rangle_2, \tag{7.4}$$

where the kinetic energies of both states are close to that of the ground state 0 and thus the difference in the kinetic energies between (7.2) and (7.4) is negligible. The interaction energy involves all possible contractions of operators, which consists of the Hartree (or direct) terms with (i)$p = p' = p_1, q = 0$, (ii)$p = p' = p_2, q = 0$, (iii)$p = p_1, p' = p_2, q = 0$, (iv)$p = p_2, p' = p_1, q = 0$, and Fock (or exchange) terms with (v)$p = p_1, p' = p_2, q = p_2 - p_1$, (vi)$p = p_2, p' = p_1, q = p_1 - p_2$. Hence

$$\begin{aligned} E_{12} \equiv \left\langle \psi_{12} | \hat{\mathcal{H}}_I | \psi_{12} \right\rangle &= \left(\underbrace{\frac{1}{2} V_0 N_1^2 + \frac{1}{2} V_0 N_2^2 + V_0 N_1 N_2}_{\text{Hartree term}} + \underbrace{V_q N_1 N_2}_{\text{Fock term}} \right) / V \\ &\simeq \frac{1}{2V} V_0 N^2 + \frac{1}{V} V_q N_1 N_2. \end{aligned} \tag{7.5}$$

Since $V_0 \simeq V_q > 0$ (repulsive interaction), condensate fragmentation costs a macroscopic exchange energy. Genuine Bose-Einstein condensation is not an ideal gas effect but is due to exchange interaction (Fock energy).

It may be questioned that the use of the particle number state as the ground state is not justified. Indeed, the repulsive interaction leads to a quantum depletion, i.e. $N_0 < N$ even at $T = 0$, so that the ground state is complicated. The above argument against condensate fragmentation nevertheless remains true, as it relies on the comparison of two situations with the same amount of quantum depletion. The exchange interaction energy is reduced slightly by quantum depletion but it remains substantial.

If a Bose system is dynamical due to a finite particle lifetime, however, the above thermodynamic argument does not apply. Instead, condensate fragmentation is often unavoidable. We will come back to this problem in the next chapter.

7.2 Population fluctuations and phase locking

We gain the exchange interaction energy by a non-fragmented condensate as discussed in the previous section. Actually we can further reduce the exchange interaction energy by allowing the population fluctuations and, in return, introducing the localized phase into the condensate [1]. Let us introduce a Glauber's coherent state [2] for the ground state:

$$|\psi_0\rangle = e^{\phi \hat{a}_0^{+} - \phi^* \hat{a}_0} |0\rangle = e^{\phi \hat{a}_0^{+}} |0\rangle \equiv |\phi\rangle, \tag{7.6}$$

where ϕ is a complex number excitation amplitude whose phase $\theta = \arg(\phi)$ is stabilized to a specific value. The population and phase of the coherent state have finite fluctuations

$$\left\langle \Delta \hat{N}^2 \right\rangle = \left\langle \hat{N} \right\rangle = |\phi|^2, \tag{7.7}$$

$$\left\langle \Delta\hat{\theta}^2 \right\rangle = \frac{1}{4\left\langle \hat{N} \right\rangle} = \frac{1}{4|\phi|^2}. \tag{7.8}$$

$\hat{N}$ and $\hat{\theta}$ are the canonically conjugate observables, and thus the phase is stabilized, $\left\langle \Delta\hat{\theta}^2 \right\rangle \ll 1$, only at a cost of the increased population fluctuation $\left\langle \Delta\hat{N}^2 \right\rangle \gg 1$. We compare such a coherent state with the particle number eigenstate

$$|N\rangle = \int_0^{2\pi} d\theta e^{-iN\theta}|\phi(\theta)\rangle. \tag{7.9}$$

Equation (7.9) shows that the particle number state $|N\rangle$ is constructed as a coherent superposition of coherent states with different eigenvalues as shown in Fig. 7.1(a). The constructive and destructive interferences result in the fixed particle number but the phase is completely spread out. Similarly, the coherent state is expanded by the coherent superposition of the particle number eigenstates:

$$|\phi\rangle = \sum_N \frac{e^{-|\phi|^2/2}\phi^N}{\sqrt{N!}}|N\rangle. \tag{7.10}$$

As shown in Fig. 7.1(b), the constructive and destructive interferences among different particle number eigenstates result in the stabilized phase but the finite particle number noise. The interaction energy among the condensate is the same for the two states, (7.9) and (7.10), since both states have identical average particle number $\left\langle \hat{N} \right\rangle = N$.

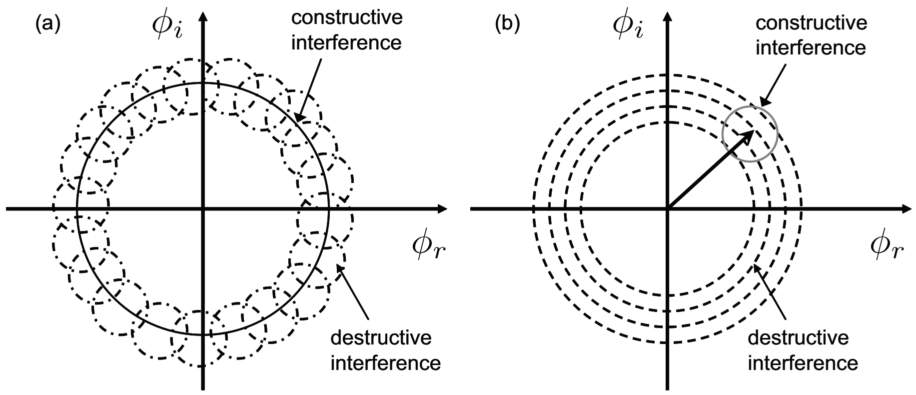

Figure 7.1: (a) A particle number eigenstate $|N\rangle_0$ constructed by the coherent superposition of coherent states. (b) A coherent state $|\phi\rangle_0$ constructed by the coherent superposition of particle number eigenstates.

The situation is different when we consider the quantum depletion (Bogoliubov) term of the Hamiltonian:

$$\hat{\mathcal{H}}_B = \frac{1}{2V}\sum_q V_q \hat{a}_0^+\hat{a}_0^+\hat{a}_q\hat{a}_{-q} + h.c., \tag{7.11}$$

which allows virtual excitations of two particles out of the condensate. In order to take such a quantum depletion into account, let us consider a variational state:

$$|\psi_0\rangle \quad = \quad e^{\phi\hat{a}_0^+ + \sum_q \lambda_q \hat{a}_q^+\hat{a}_{-q}^+}|0\rangle$$

$$= \quad |\phi\rangle_0 \otimes \sum_q \left[|0\rangle_q |0\rangle_{-q} + \lambda_q |1\rangle_q |1\rangle_{-q} + \cdots\right], \tag{7.12}$$

where a variational parameter λ_q is determined so that the interaction energy is minimized. The modulus $|\lambda_q|$ is determined as

$$\begin{aligned} |\lambda_q|^2 \simeq N_q &= \frac{q^2/2m + V_0}{2\varepsilon(q)} - \frac{1}{2} \\ &\simeq \frac{\sqrt{mgn}}{2q} \\ &= \frac{1}{2\sqrt{2}k\xi} \end{aligned} \tag{7.13}$$

The Bogolibov interaction energy is given by

$$E_B \equiv \left\langle \psi_0 | \hat{\mathcal{H}}_B | \psi_0 \right\rangle = \sum_q \frac{V_q}{2V} \left(\phi_0^{*2} \lambda_q + c.c.\right), \tag{7.14}$$

where we used $\langle\phi|\hat{a}_0^+ = \phi^* \langle\phi|$ and $\hat{a}_q \hat{a}_{-q} |1\rangle_q |1\rangle_{-q} = |0\rangle_q |0\rangle_{-q}$, and higher order terms such as $|2\rangle_q |2\rangle_{-q}, |3\rangle_q |3\rangle_{-q} \cdots$ are neglected. If we express the complex excitation amplitudes as $\phi = |\phi| e^{i\theta_0}$ and $\lambda_q = |\lambda_q| e^{i\theta_q}$, (7.14) becomes

$$E_B = \sum_q \frac{V_q}{V} |\phi|^2 |\lambda_q| \cos\left(2\theta_0 - \theta_q\right). \tag{7.15}$$

The Bogoliubov interaction energy is minimum when $2\theta_0 - \theta_q = \pi$. It is energetically favorable that the condensate has a well-defined phase and the excitations are phase-locked to the condensate with a 180° phase difference. The reduced energy is macroscopic, $\frac{V_q}{V}|\phi|^2|\lambda_q| \sim g n_0 \sqrt{N_q}$, where $n_0 = |\phi|^2/V$ and N_q is the average population of the excitation modes (see Chapter 4). From (7.13), N_q is on the order of one if $k\xi \lesssim 1$. In fact, the quantum depletion is bound to occur due to the Bogoliubov Hamiltonian. Then, the phase stabilization of the condensate is prefered and the phase locking $2\theta_0 - \theta_q = \pi$ is implemented simultaneously. This argument does not answer a following question: If (7.15) is negative and proportioned to $|\lambda_q|$, does the system perfer the continuous grow of the excitations at a cost of substantial quantum depletion of the condensate? Actually, the π-phase difference between the condensate and the excitations guarantees this does not happen and the quantum depletion is kept a minimum level that the quantum mechanics allows. The situation is analogous to the parametric deamplification in quantum optics [3].

Population fluctuations and phase stabilization of the condensate, which are caused by phase locking between the condensate and excitations, are genuine signatures of spontaneous symmetry breaking [4], which is distinct from the standard picture of Bose-Einstein condensation of a non-interacting ideal gas. The stabilized phase is responsible for superfluidity, in which a superfluid current is generated by the gradient of the phase as discussed in Chapter 6.

7.3 Dimensionality

Bose-Einstein condensation does not occur in uniform, infinite 1D and 2D systems since the energy density of states $\rho(\varepsilon)$ does not vanish in the limit of $\varepsilon \to 0$ [5, 6] (see also Chapter3). However, BEC can be restored in 1D and 2D systems of an appropriate confining potential is implemented. The energy density of states in a d-dimensional uniform system with a system size L is

$$\rho(\varepsilon) = \Omega_d \left(\frac{L}{2\pi}\right)^d \frac{1}{2} \left(\frac{2m}{\hbar^2}\right)^{\frac{d}{2}} \varepsilon^{\frac{d}{2}-1}, \tag{7.16}$$

$$\Omega_d = \begin{cases} 1(d=1) \\ 2\pi(d=2) \\ 4\pi(d=3) \end{cases}.$$

If there is a confirming potential $V(r)$ as shown in Fig. 7.2, a system size L becomes energy-dependent.

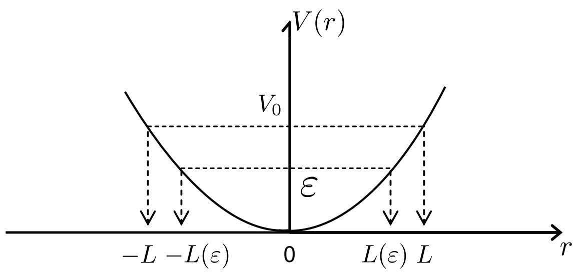

Figure 7.2: A confining potential $V(r)$ introduces an energy-dependent system size L(ε).

For a particular form of the potential $V(r) = V_0 \left(\frac{|x|}{L}\right)^\eta$, the system size at an energy ε is given by $L(\varepsilon) = L\left(\frac{\varepsilon}{V_0}\right)^{1/\eta}$. Therefore, the energy density of states is proportioned to $\rho(\varepsilon) \propto L^d \varepsilon^{\frac{d}{2}-1} \propto \varepsilon^{\frac{d}{\eta}+\frac{d}{2}-1}$. The new exponent $\frac{d}{\eta} + \frac{d}{2} - 1$ can be made positive, so that $\rho(\varepsilon \to 0) = 0$, by properly choosing the confining potential profile η for arbitrary dimensions. That is, the BEC is restored for $0 < \eta < 2$ in a one-dimensional system and for $\eta > 0$ in a two-dimensional system.

7.3.1 An ideal Bose gas in one-dimensional systems

A. General case
Let us assume a symmetric one-dimensional confirming potential

$$V(x) = V_0 \left(\frac{|x|}{L}\right)^\eta, \tag{7.17}$$

for which the system size for a particle with a kinetic energy ε is given by

$$l(\varepsilon) = L\left(\frac{\varepsilon}{V_0}\right)^{1/\eta}. \tag{7.18}$$

The energy density of states for such a system is then calculated as [7]

$$\begin{aligned} \rho(\varepsilon) &= \frac{\sqrt{2m}}{h}\int_{-l(\varepsilon)}^{l(\varepsilon)} \frac{dx}{\sqrt{\varepsilon - V(x)}} \\ &= \frac{\sqrt{2m}}{h} L \frac{\varepsilon^{\frac{1}{\eta}-\frac{1}{2}}}{V_0^{\frac{1}{\eta}}} F(\eta), \end{aligned} \tag{7.19}$$

where

$$F(\eta) = \int \frac{y^{\frac{1-\eta}{\eta}}}{\sqrt{1-y}} dy. \tag{7.20}$$

Note that $\rho(\varepsilon) \to 0$ in the limit of $\varepsilon \to 0$ if $\eta < 2$.

Thermal equilibrium distribution of total N particles in such a one-dimensional system is

$$\begin{aligned} N &= N_0 + \int_0^\infty \rho(\varepsilon) \frac{1}{e^{(\varepsilon-\mu)/k_B T} - 1} d\varepsilon \\ &= N_0 + \frac{\sqrt{2m}}{h} L \frac{F(\eta)}{V_0^{1/\eta}} (k_B T)^{\frac{1}{\eta}+\frac{1}{2}} g_1\left(\eta, \frac{\mu}{k_B T}\right). \end{aligned} \tag{7.21}$$

Here $g_1\left(\eta, \frac{\mu}{k_B T}\right)$ is a one-dimensional Bose function defined by

$$g_1(\eta, x) = \int_0^\infty \frac{y^{-\frac{1}{\eta}-\frac{1}{2}}}{e^{(y-x)} - 1} dy, \tag{7.22}$$

which has a finite value for $x = 0$ only if $\eta < 2$. In this case, N remains finite at $\mu = 0$ and $T \neq 0$. Thus, BEC is recovered. The BEC critical temperature T_c can be obtained by assuming $N_0 = 0$ and $\mu = 0$ in (7.21),

$$k_B T_c = \left[\frac{h}{2mL} \frac{V_0^{1/\eta}}{F(\eta)} \frac{1}{g_1(\eta, 0)} N\right]^{2\eta/(2+\eta)}. \tag{7.23}$$

A one-dimensional Bose gas features a BEC phase transition if the external potential is more "confining" than a parabolic potential, i.e. $0 < \eta < 2$.

B. Parabolic confining potential

If a confining potential is parabolic,

$$V(x) = \frac{1}{2} m\omega^2 x^2, \tag{7.24}$$

the eigen-energy of the trapped mode is

$$E = \hbar\omega \left(n_x + 1/2\right), \tag{7.25}$$

where n_x is the quantum number. Since we are interested in the BEC in such a parabolic potential, the following condition is satisfied:

$$k_B T \gg \{-\mu, \hbar\omega\}\,. \tag{7.26}$$

Then, the total number of particles is split into the condensate and excited states:

$$N = N_0 + \frac{k_B T}{\hbar\omega}\sum_{n_x=1}^{M}\frac{1}{n_x - \mu/\hbar\omega} + \sum_{n_x=M+1}^{\infty}\frac{1}{\exp\left[\left(\hbar\omega n_x - \mu\right)/k_B T\right] - 1}. \tag{7.27}$$

The first term of R.H.S. in (7.27) is the condensate population in the ground state with $n_x = 0$. The second term is the population in the excited states in the energy range,

$$1 \ll M \ll \frac{k_B T}{\hbar\omega}. \tag{7.28}$$

If we use the expression for the chemical potential,

$$\mu \simeq -k_B T/N_0, \tag{7.29}$$

(7.27) is reduced to

$$N \simeq N_0 + \frac{k_B T}{\hbar\omega} ln \frac{k_B T}{\hbar\omega}. \tag{7.30}$$

At a critical temperature $T_c, N_0 = 0$ and $N \simeq k_B T_c/\hbar\omega$ so that we obtain

$$\begin{aligned} N &\simeq \frac{k_B T_c}{\hbar\omega} ln \frac{k_B T_c}{\hbar\omega} \\ &\simeq \frac{k_B T_c}{\hbar\omega} ln N \\ &\sim O\left(\frac{k_B T_c}{\hbar\omega}\right), \end{aligned} \tag{7.31}$$

or

$$k_B T_c \sim O\left(N\hbar\omega\right). \tag{7.32}$$

At a temperature below T_c, the fractional condensate is expressed as

$$\begin{aligned} \frac{N_0}{N} &\simeq 1 - \frac{1}{N}\frac{k_B T_c}{\hbar\omega} ln\left(\frac{k_B T}{\hbar\omega}\right) \\ &\sim 1 - \frac{T}{T_c}. \end{aligned} \tag{7.33}$$

Here we used $N\hbar\omega \simeq k_B T_c$ and $ln\left(\frac{k_B T_c}{\hbar\omega}\right) \simeq 1$. The particle density at BEC threshold is

$$\begin{aligned} n_{1D} &= \frac{N}{2L_{\text{eff}}} \\ &\simeq \sqrt{\frac{m k_B T_c}{8\hbar^2}}, \end{aligned} \tag{7.34}$$

where the effective trap size L_{eff} at BEC threshold is determined through the relation:

$$k_B T_c = \frac{1}{2} m\omega^2 L_{\text{eff}}. \tag{7.35}$$

Since the thermal de Broglie wavelength at BEC threshold is

$$\lambda_{T_c} = \sqrt{\frac{2\pi\hbar^2}{mk_B T_c}} \tag{7.36}$$

the phase space density at BEC threshold satisfies the following quantum degeneracy relation:

$$n_{1D}\lambda_{T_c} = \sqrt{\frac{\pi}{4}} \simeq 1. \tag{7.37}$$

7.3.2 An interacting Bose gas in one-dimensional systems

A. Long range order in one-dimensional systems

The phase correlation function of the order parameter in an interacting Bose gas is written as

$$\chi_{(s)} = \frac{2mc}{n_{1D}} \int \left(N_p + \frac{1}{2}\right) \frac{e^{ip\cdot s/\hbar}}{p} \cdot \frac{dp}{2\pi\hbar}, \tag{7.38}$$

where $s = |x - x'|$ is the distance between two spatial points, and the population of the Bogoliubov quasi-particles at momentum p is given by

$$N_p = \frac{1}{e^{cp/k_B T} - 1}. \tag{7.39}$$

At low temperatures, the quantum depletion is dominant, i.e. N_p is much smaller than 1/2. In this limit, (7.38) is reduced to

$$\chi_{(s)} = \frac{mc}{n_{1D}} \int_0^{\hbar/\xi} \frac{e^{ips/\hbar\omega}}{p} \frac{dp}{2\pi\hbar}. \tag{7.40}$$

The first-order spatial coherence for a large s value is now evaluated as

$$\begin{aligned} g^{(1)}(s) &= \exp\left[\chi(s) - \chi(0)\right] \\ &\simeq \left(\frac{\xi}{s}\right)^{\nu}, \end{aligned} \tag{7.41}$$

where $\nu = \frac{mc}{2\pi\hbar n_{1D}}$, which is much smaller than one for a large n_{1D} value. Therefore, the phase coherence extends up to a macroscopic distance $s \gg \xi$ for an interacting Bose gas in a uniform one-dimension system.

At high temperatures, the thermal depletion is dominant, i.e. N_p is much greater than 1/2. In this limit, $N_p \simeq k_B T/cp$ and the first-order spatial coherence is expressed as

$$g^{(1)}(s) = \exp\left(-s/r_0\right), \tag{7.42}$$

where

$$r_0 = 2n_{1D}\hbar^2/kBTm. \tag{7.43}$$

This result indicates that the spatial coherence length r_0 is much larger than the average inter-particle distance $d = 1/n_{1D}$ is the quantum degeneracy condition,

$$k_B T < \frac{\hbar^2}{m} n_{1D}^2, \tag{7.44}$$

is satisfied.

B. Tonks-Girardeau gas

The interaction energy per particle is expressed as $I = gn$, while the kinetic energy of a particle confined solitary is given by

$$K = \frac{\hbar^2}{m}\left(\frac{1}{\bar{r}}\right)^2 = \frac{\hbar^2}{m} n^2, \tag{7.45}$$

where $\bar{r} = \frac{1}{n}$ is the average distance between particles and n is the 1D density (see Fig. 7.3). The ratio of the interaction energy to the kinetic energy is defined by

$$\gamma = \frac{I}{K} = \frac{mg}{\hbar^2 n}. \tag{7.46}$$

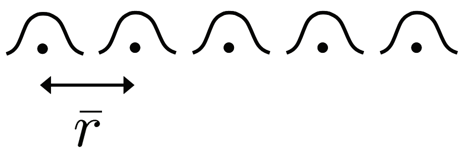

Figure 7.3: A Tonks-Girardeau gas in a one-dimensional system.

When $\gamma \ll 1$, the healing length $\xi = \frac{\hbar}{\sqrt{mgn}}$ is much lager than $\bar{r} = 1/n$, so that such a system tries to minimize the kinetic energy. This is a BEC regime or weakly interacting regime. When $\gamma \gg 1$, ξ is much smaller than $\bar{r}$ so that such a system tries to minimize the interaction energy. This is a Tonks-Giraudeau regime or strong interacting regime. Bosons cannot penetrate each other and behave as fermions.

In a Tonks-Giraudeau gas, the chemical potential is given by

$$\mu = \frac{\hbar^2}{2m} k_{\text{eff}}^2 = \frac{\hbar^2}{2m} (\pi n_{1D})^2, \tag{7.47}$$

where $k_{\text{eff}} = \pi/\bar{r} = \pi n_{1D}$ is the effective wavenumber consumed to confine a particle in a spacial region of $\bar{r}$. The sound velocity is expressed as

$$c = \sqrt{\frac{n_{1D}}{m}\frac{\partial \mu}{\partial n_{1D}}} = \frac{\pi \hbar n_{1D}}{m}, \tag{7.48}$$

while the healing length is reduced to

$$\xi = \frac{\hbar}{mc} = \frac{1}{\pi n_{1D}} = \frac{\bar{r}}{\pi}. \tag{7.49}$$

The first-order coherence function of the interacting 1D Bose gas is given by (7.41), i.e. $g^{(1)}(s) = (\xi/s)^{\nu}$, where the exponent is

$$\nu = \frac{mc}{2\pi\hbar n_{1D}} = \frac{1}{2}. \tag{7.50}$$

The inverse-Fourier transform of $g^{(1)}(s)$ results in the particle distribution in the excited states:

$$n(p) \simeq \frac{N}{\sqrt{2\pi} p_F} \left(\frac{p_F}{p}\right)^{1/2}. \tag{7.51}$$

The Tonks-Giraudeau gas is characterized by the inverse-square-root dependence of $g^{(1)}(s)$ and $n(p)$.

7.3.3 An ideal Bose gas in finite two-dimensional systems

A. General case

Next let us assume an isotopic two-dimensional confirming potential

$$V(r) = V_0 \left(\frac{r}{a}\right)^{\eta}, \tag{7.52}$$

for which the normalized system size for a particle with kinetic energy ε is given by

$$r^* \equiv \frac{r(\varepsilon)}{a} = \left(\frac{\varepsilon}{V_0}\right)^{1/\eta}. \tag{7.53}$$

Thus, the energy density of states is calculated by [7]

$$\rho(\varepsilon) = \frac{2\pi m}{h^2} \int_0^{r(\varepsilon)} 2\pi r dr = \frac{2\pi^2 m a^2}{h^2} \left(\frac{\varepsilon}{V_0}\right)^{\frac{2}{\eta}}. \tag{7.54}$$

Thermal equilibrium distribution of total N particles in such a two-dimensional system is

$$N = N_0 + \frac{2\pi^2 m a^2}{h^2 V_0^{2/\eta}} (k_B T)^{\frac{2}{\eta}+1} g_2\left(\eta, \frac{\mu}{k_B T}\right), \tag{7.55}$$

where a two-dimensional Bose function is defined by

$$g_2(\eta, x) = \int_0^{\infty} \frac{y^{2/\eta}}{e^{(y-x)} - 1} dy. \tag{7.56}$$

Note that $g_2(\eta, 0)$ has a finite value for all positive values of η. The BEC critical temperature is obtained by substituting $N_0 = 0$ and $\mu = 0$ in (7.55)

$$k_B T_c = \left[\frac{h^2 V_0^{2/\eta} N}{2\pi^2 m a^2 g_2(\eta, 0)}\right]^{\frac{2}{2+\eta}}, \tag{7.57}$$

which has a maximum value at $\eta = 2$ (parabolic potential). A two-dimensional Bose gas features a BEC phase transition as far as a trap potential ($\eta > 0$) is implemented.

The confining potential of the form, given by (7.52), does not describe a finite trap with an infinite and sudden barrier. The BEC critical temperature in such finite 2D and 1D systems are discussed in reference [8].

As mentioned already in Chapter 3, the BEC is a phenomenon of macroscopic population in a ground state at a temperature satisfying $k_BT_c \geq k_BT \gg \varepsilon_1 - \varepsilon_0$, where ε_0 and ε_1 are the kinetic energies of the ground state and first excited state. If confining potential is too strong and the system size is too small to satisfy the above condition, then the concept of BEC is irrelevant. The discussion of this section applies when the two conditions are simultaneously satisfied: 1) a confining potential is strong enough, so that $\rho(\varepsilon \rightarrow 0) = 0$ in the limit of $\varepsilon \rightarrow 0$, and 2) a confining potential is not too strong, so that $\varepsilon_1 - \varepsilon_0 \ll k_BT_c$.

B. Parabolic confining potential

When the confining potential is isotopically parabolic, i.e. $\eta = 2$ in (7.52), we can write the potential energy and quantized energy as

$$V(r) = \frac{1}{2}m\omega^2(x^2 + y^2), \tag{7.58}$$

$$E = \hbar\omega\left(n_x + n_y + 1\right). \tag{7.59}$$

Thermal equilibrium distribution of N particles in such a system is expressed as

$$N = N_0 + \int_0^\infty d\varepsilon\rho(\varepsilon)\frac{d\varepsilon}{e^{(\varepsilon-\mu)/k_BT} - 1}, \tag{7.60}$$

where the energy density of states is given by

$$\rho(\varepsilon) = \varepsilon/(\hbar\omega)^2. \tag{7.61}$$

Using (7.61) in the second term of R.H.S of (7.52) together with $\mu \simeq 0$, the energy integral is reduced to $\sim \frac{\pi^2}{6}\left(\frac{k_BT}{\hbar\omega}\right)^2$. At a BEC threshold, we can assume $N_0 = 0$ so that we obtain

$$k_BT_c \simeq \sqrt{\frac{6N}{\pi^2}}\hbar\omega. \tag{7.62}$$

The particle density at BEC threshold is written as

$$\begin{aligned} n_{2D} &\equiv \frac{N}{S} = \frac{\left(\frac{k_BT_c}{\hbar\omega}\right)^2 \cdot \frac{\pi^2}{6}}{\left(\frac{2\pi k_BT_c}{m\omega^2}\right)} \\ &= \frac{\pi}{12} \cdot \frac{k_BT_cm}{\hbar^2}. \end{aligned} \tag{7.63}$$

If we recall the thermal de Broglie wavelength is $\lambda_T = \sqrt{\frac{2\pi\hbar^2}{mk_BT_c}}$, the phace space density at BEC threshold is reduced to

$$n_{2D}\lambda_T^2 = \frac{\pi^2}{6} \sim 1.5, \tag{7.64}$$

and we recover the quantum degeneracy condition once again.

At $T < T_c$, the total number of particles in the excited states becomes smaller than the total number of particles by $N \times \left(\frac{T}{T_c}\right)^2$. Using this relation to (7.60), we have the fractional condensate

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^2. \tag{7.65}$$

This result is compared to $\frac{N_0}{N} = 1 - \frac{T}{T_c}$ for a 1D harmonic potential and $\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3$ for a 3D harmonic potential.

7.3.4 Berezinskii-Kousterlitz-Thouless (BKT) phase transition

A. BKT phase transition temperature

The energy cost of a quantized vortex in a 3D superfield,

$$E_v = L\pi\rho_{3s}\left(\frac{\hbar}{m}\right)^2 ln\left(\frac{R}{r_c}\right), \tag{7.66}$$

is macroscopic since it is proportional to the length L of the cylinder. Here ρ_{3s} is the 3D superfield mass density. Therefore, thermal creation of quantized vorcities is not possible in a cold 3D Bose gas, but it is possible in a cold 2D Bose gas. Certain of a quantized vortex is thermodynamically profitable if the total free energy of the system would be decreased by the appearance of a quantized vortex. The free energy is expressed as

$$F_v = E_v - TS, \tag{7.67}$$

where the vortex energy cost and the entropy are given by

$$E_v = \pi\rho_{2s}\left(\frac{\hbar}{m}^2\right) ln\left(\frac{R}{r_c}\right), \tag{7.68}$$

$$S = k_B ln\left(\frac{R^2}{r_c^2}\right). \tag{7.69}$$

Here $r_c \simeq \xi$ is the core size of the vortex, ρ_{2s} is the 2D superfluid mass density and R^2/r_c^2 is the number of possible states for the creation of one vortex.

The condition of $E_v < 0$ produces the BKT phase transition temperature:

$$T \geq \frac{\pi}{2k_B}\rho_{2s}\left(\frac{\hbar}{m}\right)^2 = T_{BKT}, \tag{7.70}$$

or

$$k_B T_{BKT} = \frac{\pi}{2} n_{2s} \frac{\hbar^2}{m}. \tag{7.71}$$

The thermal de Broglie wavelength at BKT phase transition temperature is

$$\lambda_{T,BKT} = \sqrt{\frac{2\pi\hbar^2}{m k_B T_{BKT}}} = \sqrt{\frac{4}{n_{2s}}}, \tag{7.72}$$

or

$$n_{2s}\lambda_{T,BKT}^2 = 4. \tag{7.73}$$

When $T < T_{BKT}$, the threshold creation of quantized vortices is suppressed and a long-range order is restored.

B. First-order spatial coherence function

If the order parameter of the BKT phase is expressed as $\psi r = \sqrt{n(r)}e^{iS(r)}$, the phase correlation function is given by

$$\begin{aligned} \chi_{(s)} &= \langle S(s)S(0)\rangle \\ &= \frac{mc^2}{\rho_{2s}}\int N_p \cdot \frac{e^{ip\cdot s/\hbar}}{p}\frac{d^2p}{(2\pi\hbar)^2}, \end{aligned} \tag{7.74}$$

where

$$N_p \simeq \frac{1}{e^{cp/k_BT}-1} \simeq \frac{k_BT}{cp}. \tag{7.75}$$

By introducing a lower cut-off momentum $\hbar/s$ and higher cut-off momentum k_BT/c, we can evaluate the integral (7.74) as

$$\chi_{(s)} = \frac{k_BTm^2}{\rho_{2s}}\int_{\hbar/s}^{k_BT/c}\frac{e^{ips/\hbar}}{p^2}\cdot\frac{d^2p}{(2\pi\hbar)^2}, \tag{7.76}$$

and

$$\begin{aligned} \chi_{(0)} - \chi_{(s)} &= \frac{k_BTm^2}{\rho_{2s}}\int_{\hbar/s}^{k_BT/c}\frac{1-\cos(ps/\hbar)}{p^2}\cdot\frac{d^2p}{(2\pi\hbar)^2} \\ &= \frac{k_BTm^2}{2\pi\hbar^2\rho_{2s}}ln\left(\frac{s}{s_T}\right). \end{aligned} \tag{7.77}$$

Here S_T is a characteristic length defined by

$$S_T = \frac{\hbar_c}{k_BT} = \frac{\lambda_T^2}{2\pi\xi}. \tag{7.78}$$

The first-order spatial coherence function is obtained as

$$\begin{aligned} g^{(1)}(s) &\simeq e^{-[\chi(0)-\chi(s)]} \\ &\simeq \left(\frac{s_T}{s}\right)^{\nu}, \end{aligned} \tag{7.79}$$

where

$$\nu = \frac{k_BTm}{2\pi\hbar^2\rho_{2s}} = \frac{1}{4}\left(\frac{T}{T_{BKT}}\right). \tag{7.80}$$

We have obtained a similar power-law dependence as a 1D case, but the origin of the power-law decay is the thermal excitation in a 2D case, while it is due to the particle-particle interaction in a 1D case.

7.4 Dynamical condensation with a finite lifetime

In a Bose-Einstein condensation experiment, bosons are usually trapped in a confining potential in order to maintain a required critical density and some sort of cooling mechanism is implemented in order to realize a required critical temperature. A particle trap has a finite lifetime of the order of a second for atomic BEC and a picosecond to nanosecond for exciton-polariton BEC. A cooling mechanism also accompanies a particle loss process, such as evaporation of atoms or photon emission of exciton-polaritons. Thus a phenomenon of BEC must be experimentally probed during a time window between the condensate formation and the condensate decay. If this time window is very wide, the BEC is indistinguishable from a thermal equilibrium phenomenon of a closed system. On the other hand, if this time window is very short, the condensation phenomenon is necessarily dynamical and should be considered as the property of an open dissipative system. The former is true for most atomic BEC experiments and the latter applies to exciton-polariton BEC experiments.

A standard Gross-Pitaevskii equation is not adequate for describing the second case. Instead, such a system is treated by an open dissipative Gross-Pitaevskii equation, which is a coupled equation for the condensate order parameter $\psi_0(r,t)$ and the reservoir population $n_R(r,t)$:

$$i\hbar\frac{d}{dt}\psi_0(r,t) = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(r) - \frac{i\hbar}{2}\left[\gamma_c - Rn_R(r,t)\right] + g_c|\psi_0(r,t)|^2 + g_R n_R(r,t)\right\}\psi_0(r,t), \tag{7.81}$$

$$\frac{d}{dt}n_R(r,t) = P(r,t) - \gamma_R n_R(r,t) - Rn_R(r,t)|\psi_0(r,t)|^2. \tag{7.82}$$

In (7.81), $V_{\text{ext}}(r)$ is an external trap potential, γ_c is the condensate particle loss rate, $Rn_R(r,t)$ is the gain rate for the condensate due to bosonic final state stimulation, and g_c and g_R are the condensate-condensate and condensate-reservoir repulsive interactions ($g_c, g_R > 0$). As mentioned already, $g_R = 2g_c$ due to the additional exchange interaction term which exists only in the condensate-reservoir interaction. The reservoir density $n_R(r,t)$ is controlled by the external pump rate (particle injection rate) $P(r,t)$, the spontaneous decay rate γ_R and the stimulated decay rate $R|\psi_0(r,t)|^2$. There is no phase coherence in the reservoir mode, so that the simple rate equation for the density $n_R(r,t)$ is enough for describing such a dynamical system. The c-number order parameter $\psi_0(r,t)$ does not include the excitations which introduce the amplitude and phase fluctuations for the condensate. If necessary, they can be calculated by a perturbation technique already used in Chapter 4.

If the pump rate $P(r,t)$ is independent of time, both condensate and reservoir are expected to establish steady state solutions. However, those study state solutions are distinct from thermal equilibrium solutions that we have discussed so far. The numerical results in the condensate particle distribution shown in Fig. 7.4 feature such an example. Even though we concluded in sec.7.2 that the condensate fragmentation is suppressed by the exchange interaction energy (Fock term), clear signatures of condensate fragmentation are demonstrated in the numerical results shown in Fig. 7.4(a) and (b), that is, two or three states are simultaneously occupied by the macroscopic population. This is the so-called spatial hole burning effect which is well-known in laser physics. When the ground state has a large population, the simulated decay rate $R|\psi_o(r)|^2$ in (7.82) is maximum at the condensate center and lower at the condensate peripheral. Consequently, the reservoir density $n_R(r)$ has a population dip at the center (hole burning). The unusual spatial distribution of $n_R(r)$ gives a considerately higher gain $Rn_R(r)$ for higher energy states, so that they capture a macroscopic population and creates condensate fragmentation. Before they cool down to the ground state, they decay as photons and captured by detectors. If there is no reservoir density, $n_R(r) = 0$, and a condensate lifetime is infinitely long, such condensate fragmentation is not formed.

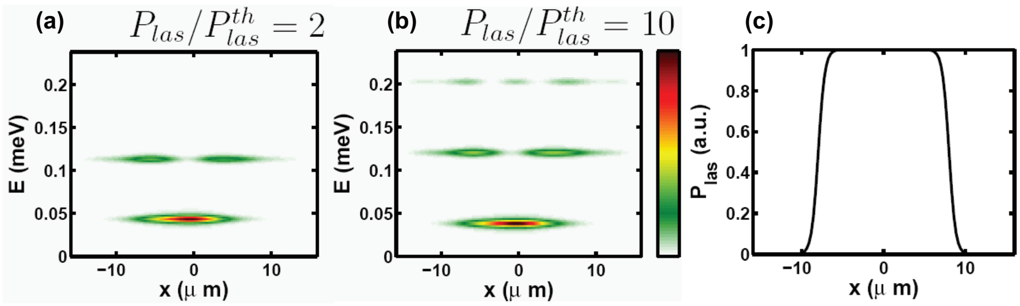

Figure 7.4: (a)(b) Real space particle distribution at the pump rates, $p/p_{th} = 2$ and 10 vs. the condensate energy. (c) Real space pump distribution.

Another remarkable feature shown in Fig. 7.4 is a gain-induced trapping mechanism. In the numerical model, it is assumed that an external trap potential is negligible, $V_{\rm ext}(r) \sim 0$. Nevertheless, clear trapped mode structures appear. This is because the pump profile $P(r)$ has a spatial dependence as shown in 7.4(c) and condensate particles can be trapped by the imaginary part, $i\frac{\hbar}{2}Rn_R(r)$, of the trap potential energy rather than the real part, $V_{\rm ext}(r)$, in this case.

The above example suggests that the properties of a dynamical condensate described by the open dissipative Gross-Pitaevskii equation, (7.81) and (7.82), are different from those of an equilibrium BEC. Probably the most subtle issue among them is the quantum fluctuations of the amplitude and phase of the condensate order parameter. We will discuss the point in the next chapter.

Bibliography

[1] P. Nozieres. *Bose-Einstein Condensation*, page 15. Cambridge University Press, Cambridge, 1995.

[2] Roy J. Glauber. The quantum theory of optical coherence. *Phys. Rev.*, 130(6):2529–2539, 1963.

[3] Y. Yamamoto and A. Imamoglu. *Mesoscopic Quantum Optics*. Wiley-Interscience, 1999.

[4] Y. Nambu. Quasi-particles and gauge invariance in the theory of superconductivity. *Phys. Rev.*, 117:648–663, 1960.

[5] P. C. Hohenberg. Existence of long-range order in one and two dimensions. *Phys. Rev.*, 158:383–386, 1967.

[6] N. D. Mermin and H. Wagner. Absence of ferromagnetism or antiferromagnetism in one- or 2- dimensional isotropic Heisenberg models. *Phys. Rev. Lett.*, 17:1133, 1966.

[7] V. Bagnato and D. Kleppner. Bose-Einstein condensation in low-dimensional traps. *Phys. Rev. A*, 44:7439–7441, 1991.

[8] D. S. Petrov, D. M. Gangardt, and G. V. Shlyapnikov. Low-dimensional trapped gasses. *J. Phys. IV.*, 116:5–44, 2004.