Chapter 4

Bogoliubov theory of the weakly interacting Bose gas

In the presence of BEC, the ideal Bose gas has a constant pressure against variation of volume, so that the system features infinite compressibility. This pathological feature originates from the absence of particle-particle interaction. Therefore, it is not surprising that interactions between particles affect the properties of the gas in a dramatic way, even for very dilute samples. This problem was first attacked by Bogoliubov using a new perturbation technique [1]. The theory provides the theoretical framework of modern approaches to BEC in dilute gases and is the main focus of this chapter.

4.1 Dilute and cold gases

In a dilute gas, the range of the inter-particle forces, r_0, is much smaller than the average distance d between particles:

$$r_0 \ll d = n^{-\frac{1}{3}} = \left(\frac{N}{V}\right)^{-\frac{1}{3}}. \tag{4.1}$$

The interaction between particles is non-zero only when the two particles are separated within the characteristic distance r_0. Figure 4.1(a) shows a representative two-body scattering potential $V(r)$ as a function of particle-particle distance r. The Fourier transform of $V(r)$ is written as

$$V(p) = \int V(r) e^{-ip\cdot r/\hbar} dr, \tag{4.2}$$

and the scattering amplitude $V(p)$ is independent of the momentum p as far as $p \ll \hbar/r_0$ but it rolls off when p becomes large compared to $\hbar/r_0$, as shown in Fig. 4.1(b). The condition (4.1) allows one to consider only two-body interaction, while configurations with three or more particles interacting simultaneously can be safely neglected. If this approximation is valid, such a system is called a dilute gas. A second important consequence of (4.1) is that the use of the asymptotic expression for the wavefunction of their relative motion is justified when the scattering amplitude is evaluated and that all the properties of the system can be expressed in terms of a single parameter, called a scattering length, as explained below.

We shall also assume the temperature of the dilute gas is so low that the momentum distribution of the thermal components $p \sim \sqrt{2mk_BT}$ is much smaller than the characteristic momentum $p_c = \hbar/r_0$ determined by r_0:

$$T \ll \frac{\hbar^2}{2mk_B r_0^2}. \tag{4.3}$$

The condition (4.3) suggests that the scattering amplitude becomes independent of particle energy as well as of scattering angle. The scattering amplitude is safely approximated by its low-energy value $V_0 = \int V(r)dr$, which is determined by the s-wave scattering length a through [2]:

$$V_0 = \frac{4\pi\hbar^2}{m}a. \tag{4.4}$$

In conclusion, the s-wave scattering length a characterizes all the interaction effects of the dilute and cold gas. The diluteness condition in terms of the scattering length is summarized by

$$|a|n^{\frac{1}{3}} \ll 1, \tag{4.5}$$

which should always be satisfied in order to apply the perturbation theory of Bogoliubov.

4.2 Hamiltonian of the weakly interacting Bose gas and the lowest-order approximation

The Hamiltonian of the system is expressed in terms of the field operators $\hat{\psi}$:

$$\hat{\mathcal{H}} = \int \left(\frac{\hbar^2}{2m} \nabla\hat{\psi}^+(r)\nabla\hat{\psi}(r) \right) dr + \frac{1}{2}\int \hat{\psi}^+(r)\hat{\psi}^+(r')V(r'-r)\hat{\psi}(r)\hat{\psi}(r')dr'dr, \tag{4.6}$$

where $V(r'-r)$ is the two-body scattering potential. For a uniform gas occupying a volume L^3, the field operator $\hat{\psi}$ can be expanded by the plane waves:

$$\hat{\psi}(r) = \frac{1}{\sqrt{L^3}} \sum_p \hat{a}_p e^{ip\cdot r/\hbar}, \tag{4.7}$$

where $\hat{a}_p$ is the annihilation operator for a single particle state of a plane wave with momentum p. By substituting (4.7) into (4.6), one obtains

$$\hat{\mathcal{H}} = \sum_p \frac{p^2}{2m}\hat{a}_p^+\hat{a}_p + \frac{1}{2L^3}\sum_{p_1,p_2,q} V_q \hat{a}^+_{p1+q}\hat{a}^+_{p2-q}\hat{a}_{p_1}\hat{a}_{p_2}. \tag{4.8}$$

Here $V_q = \int V(r)\exp\left(-iq\cdot r/\hbar\right)dr$ is the Fourier transform of the two-body scattering potential.

In real systems $V(r)$ always contains a short-range term, as shown in Fig. 4.1(a) which makes it difficult to solve the Schrödinger equation at the microscopic level. However, in virtue of the above assumptions on the dilute and cold gases, one can conclude that the actual form of $V(r)$ is not important for describing the macroscopic properties of the gas, as far as the assumed fictitious potential $V_{\rm eff}(r)$ gives the correct value for the low momentum value of its Fourier transform $V_{q \ll \hbar/r_0}$. It is therefore convenient to replace the actual potential $V(r)$ with an effective, smooth potential $V_{\text{eff}}(r)$ (see Fig. 4.1). Since the macroscopic properties of the system depend on $V_{q=0} = V_0 = \int V_{\text{eff}}(r)dr$ (or the s-wave scattering length a), this procedure will provide the correct answer to this complicated many-body problem as far as the system is dilute and cold.

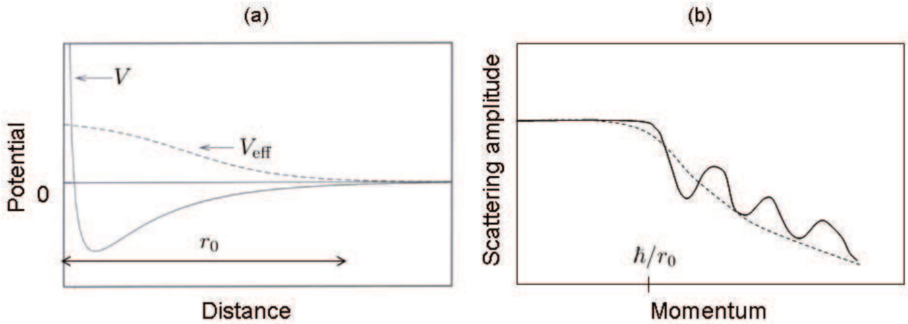

Figure 4.1: (a) The actual two-body scattering potential $V(r)$ and the effective, smooth potential $V_{\text{eff}}(r)$. The average range r_0 of both potentials, at the right edge of this figure, is much smaller than the average inter-particle distance $n^{-1/3}$. (b) The Fourier transform of $V(r)$ and $V_{\text{eff}}(r)$.

Replacing V_q with V_0 in (4.8), one obtains the new Hamiltonian

$$\hat{\mathcal{H}} = \sum_p \frac{p^2}{2m} \hat{a}_p^+ \hat{a}_p + \frac{V_0}{2V} \sum_{p_1,p_2,q} \hat{a}_{p_1+q}^+ \hat{a}_{p_2-q}^+ \hat{a}_{p_1} \hat{a}_{p_2}. \tag{4.9}$$

Since the lowest energy state is occupied by macroscopic number of particles in BEC, one can neglect the quantum fluctuation of this state and replace the operator $\hat{a}_0$ with a c-number:

$$\hat{a}_0 \equiv \sqrt{N_0}. \tag{4.10}$$

In a dilute and cold gas, the occupation number of the excited states with $p \neq 0$ is small. In the lowest-order approximation, one can neglect all terms with $p \neq 0$ in (4.9) and the ground state energy takes the form

$$E_0 = \frac{V_0}{2L^3} N_0^2 \simeq \frac{V_0}{2L^3} N^2, \tag{4.11}$$

where V_0 can be expressed in terms of the $s-$wave scattering length a using the Born approximation [2]

$$V_0 = \frac{4\pi\hbar^2 a}{m} = g. \tag{4.12}$$

According to (4.11), contrary to the BEC of an ideal Bose gas, the pressure of the BEC of a weakly-interacting Bose gas does not vanish at $T = 0$:

$$P \equiv -\frac{\partial E_0}{\partial L^3} = \frac{g}{2} n^2, \tag{4.13}$$

where $n = N/L^3$ is the particle density. Accordingly, the compressibility remains finite:

$$\frac{\partial n}{\partial P} = \frac{1}{gn}. \tag{4.14}$$

Using the hydrodynamic relation, one obtains the relation between the sound velocity c and the compressibility [3]:

$$\begin{aligned} \frac{1}{mc^2} &= \frac{\partial n}{\partial P} \\ \text{or} \quad c &= \sqrt{\frac{gn}{m}}. \end{aligned} \tag{4.15}$$

In the next section we will obtain the same result for c by analyzing the dispersion relation for an excitation spectrum, namely, Bogoliubov linear dispersion in the low momentum limit [1].

The thermodynamic stability requires that the compressibility $\frac{\partial n}{\partial P}$ is positive, i.e. $g > 0$ or $a > 0$. This leads to the important conclusion that a dilute uniform BEC can stably exist only if the system has a "repulsive" interaction.

Finally, the chemical potential is given by

$$\mu \equiv \frac{\partial E_0}{\partial N} = gn = mc^2, \tag{4.16}$$

which is always positive even at $T = 0$. This is an energy required to add one particle into the condensate or an energy gained by subtracting one particle from the condensate.

4.3 Bogoliubov approximation and excitation spectrum

If we split the operator $\hat{a}_0$ for the ground state and $\hat{a}_p$ for the excited states in the interaction term of (4.9), the Hamiltonian can be decomposed to:

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{V_0}{2L^3}\hat{a}_0^+\hat{a}_0^+\hat{a}_0\hat{a}_0 + \sum_p \frac{p^2}{2m}\hat{a}_p^+\hat{a}_p \\ &+ \frac{V_0}{2L^3}\sum_{p\neq 0}\left(4\hat{a}_0^+\hat{a}_p\hat{a}_0\hat{a}_p + \hat{a}_p^+\hat{a}_{-p}^+\hat{a}_0\hat{a}_0 + \hat{a}_0^+\hat{a}_0^+\hat{a}_p\hat{a}_{-p}\right). \end{aligned} \tag{4.17}$$

The momentum conservation retains only quadratic terms in $\hat{a}_p$ with $p \neq 0$. In particular, a factor of 4 in front of the third term of R.H.S. of (4.17) corresponds to the cases of i) $p_1 = 0, p_2 = p, q = 0$, ii) $p_1 = 0, p_2 = p, q = p$, iii) $p_1 = p, p_2 = 0, q = 0$, and iv) $p_1 = p, p_2 = 0, q = -p$ in (4.9). The two cases (i) and (iii) do not involve the transfer of a momentum between two particles, so that these two terms are called 'direct terms". The two other cases (ii) and (iv) involve the transfer of a momentum between two particles, so that they are called 'exchange terms". Identical quantum particles are indistinguishable, so we must take into account all those direct and exchange terms.

We can replace $\hat{a}_0^+$ and $\hat{a}_0$ with $\sqrt{N}$ in the third term of (4.17) as we have done previously, but in the first term we have to work with higher accuracy by using the normalization relation, $\hat{a}_0^+\hat{a}_0 + \sum_{p\neq 0}\hat{a}_p^+\hat{a}_p = N$, or

$$\hat{a}_0^+\hat{a}_0^+\hat{a}_0\hat{a}_0 \simeq N^2 - 2N\sum_{p\neq 0}\hat{a}_p^+\hat{a}_p. \tag{4.18}$$

Substitution of (4.12) and (4.18) into (4.17) yields the following expression for the Hamiltonian:

$$\hat{\mathcal{H}} = \frac{1}{2}gnN + \sum_p \frac{p^2}{2m}\hat{a}_p^+\hat{a}_p + \frac{1}{2}gn\sum_{p\neq 0}\left(2\hat{a}_p^+\hat{a}_p + \hat{a}_p^+\hat{a}_{-p}^+ + \hat{a}_p\hat{a}_{-p}\right). \tag{4.19}$$

The third term of this Hamiltonian represents the self-energy of the excited states due to the interaction, simultaneous creation of the excited states at momenta p and $-p$, and simultaneous annihilation of the excited states, respectively.

Equation (4.19) can be diagonalized by the linear transformation

$$\hat{a}_p = u_p\hat{b}_p + v_{-p}\hat{b}_{-p}^+, \hat{a}_p^+ = u_p\hat{b}_p^+ + v_{-p}\hat{b}_{-p}, \tag{4.20}$$

known as the Bogoliubov transformation. The two parameters, u_p and v_{-p}, are determined uniquely by the following requirements. The new operators, $\hat{b}_p$ and $\hat{b}_p^+$, are assumed to obey the same Bosonic commutation relation as the real particle operators, $\hat{a}_p$ and $\hat{a}_p^+$:

$$\left[\hat{b}_p, \hat{b}_{p'}^+\right] = \hat{b}_p\hat{b}_{p'}^+ - \hat{b}_{p'}^+\hat{b}_p = \delta_{pp'}. \tag{4.21}$$

This commutation relation imposes the constraint for the two parameters u_p and v_{-p}:

$$u_p^2 - v_{-p}^2 = 1, \tag{4.22}$$

or equivalently

$$u_p = \cosh(\alpha_p), v_{-p} = \sinh(\alpha_p). \tag{4.23}$$

The value of α_p must be chosen in order to make the coefficients of the non-diagonal terms $\hat{b}_p^+\hat{b}_{-p}^+$ and $\hat{b}_p\hat{b}_{-p}$ in (4.19) vanish. This condition is rewritten as

$$\frac{gn}{2}\left(u_p^2 + v_{-p}^2\right) + \left(\frac{p^2}{2m} + gn\right)u_p v_{-p} = 0. \tag{4.24}$$

Using the relations $\cosh(2\alpha) = \cosh^2(\alpha) + \sinh^2(\alpha)$ and $\sinh(2\alpha) = 2\cosh(\alpha)\sinh(\alpha)$, (4.24) is identical to

$$\coth(2\alpha_p) = -\frac{p^2/2m + gn}{gn}, \tag{4.25}$$

from which the two coefficients are determined uniquely as

$$u_p, v_{-p} = \pm\left(\frac{p^2/2m + gn}{2\varepsilon(p)} \pm \frac{1}{2}\right)^{1/2}, \tag{4.26}$$

where

$$\varepsilon(p) = \left[\frac{gn}{m}p^2 + \left(\frac{p^2}{2m}\right)^2\right]^{1/2}, \tag{4.27}$$

is the famous dispersion law of the Bogoliubov excitation spectrum. By substituting the Bogolibov transformation (4.20) with the expressions (4.26) for the two coefficients u_p and v_{-p}, the Hamiltonian (4.19) is finally diagonalized:

$$\mathcal{H} = E_0 + \sum_{p \neq 0} \varepsilon(p) \hat{b}_p^+ \hat{b}_p, \tag{4.28}$$

where

$$E_0 = \frac{1}{2} gnN + \frac{1}{2} \sum_{p \neq 0} \left[\varepsilon(p) - gn - \frac{p^2}{2m} + \frac{m(gn)^2}{p^2} \right], \tag{4.29}$$

is the ground state energy calculated to the higher-order of approximation.

Results (4.27)-(4.29) have a deep physical meaning. The original system of interacting particles can be described by the Hamiltonian for non-interacting quasi-particles (collective excitations) having a dispersion law $\varepsilon(p)$. In this picture, a real particle $\hat{a}_p$ is described as the superposition of the forward propagating many quasi-particles $u_p \hat{b}_p$ and the backward propagating many quasi-particles $v_{-p} \hat{b}_{-p}^+$(see (4.20)). Note that when a momentum is small, $p \ll \sqrt{mgn}$, $|u_p| \simeq |v_{-p}| \gg 1$ and $\hat{a}_p \sim u_p \left(\hat{b}_p - \hat{b}_{-p}^+ \right)$. However, when a momentum becomes large, $p \gg \sqrt{mgn}$, $|u_p| \simeq 1$ and $|v_{-p}| \simeq 0$ and the quasi-particle $\hat{b}_p$ becomes indistinguishable from the real particle $\hat{a}_p$, i.e. $\hat{a}_p \sim \hat{b}_p$.

The ground state of the interacting Bose gas system at $T = 0$ is now defined as the vacuum state for the Bogoliubov quasi-particle:

$$\hat{b}_p |0\rangle = 0 \quad {}^{\forall} p \neq 0. \tag{4.30}$$

The ground state energy in the lowest-order calculation is $E_0 \simeq \frac{1}{2} gnN$ as discussed already. The first-order correction to that is calculated by (4.29) and given by [3, 4]

$$E_0 = \frac{1}{2} gnN \left[1 + \frac{128}{15\sqrt{\pi}} \left(na^3 \right)^{1/2} \right], \tag{4.31}$$

and the chemical potential in the first order correction is

$$\mu = \frac{\partial E_0}{\partial N} = gn \left[1 + \frac{32}{3\sqrt{\pi}} \left(na^3 \right)^{1/2} \right]. \tag{4.32}$$

Even though (4.11) or (4.31) is often referred to as the ground state energy of the interacting Bose gas system, the true ground state of such a system is a solid as pointed out already in Chapter 2. The gas phase in the quantum degenerate regime is actually a metastable state and (4.11) or (4.31) is the energy of this metastable ground state.

4.4 Excitation spectrum

4.4.1 Gross-Pitaevskii equation

In order to calculate the excitation spectrum, we can replace the field operator (4.7) by the classical fields not only for the ground state at $p = 0$ but also for the excited states at $p \neq 0$. By including the explicit time dependence, the corresponding classical field is expressed as

$$\psi(r,t) = \psi_0(r)e^{-i\mu t} + \psi_k e^{i[k\cdot r-(\mu+\omega)t]} + \psi_{-k}e^{-i[k\cdot r-(\mu-\omega)t]}, \tag{4.33}$$

where ψ_0 is the order parameter, μ is the chemical potential, ψ_k and ψ_{-k} are the excitation amplitudes of the forward and backward propagating plane waves corresponding to the excited states with wave numbers $\pm k$. The positive and negative frequency deviations from the chemical potential, $\mu+\omega$ and $\mu-\omega$ for the two waves ψ_k and ψ_{-k}, are introduced to satisfy the energy conservation in the simultaneous creation or annihilation of the two particles in the excited states at k and $-k$, as shown in Fig. 4.2(see (4.19).

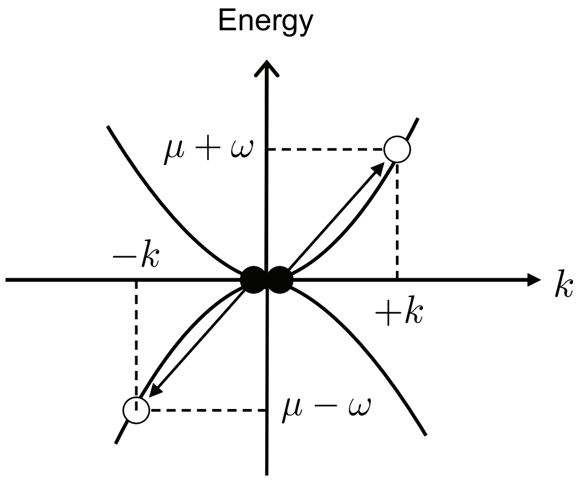

Figure 4.2: (a) Energy and momentum conservation in two-particle scattering process.

The Heisenberg equation of motion $i\hbar\frac{d}{dt}\hat{\psi}(r,t) = \left[\hat{\psi}(r,t),\hat{\mathcal{H}}\right]$ for the field operator under the above approximation (4.33) is written as

$$i\hbar\frac{d}{dt}\psi(r,t) = \left[-\frac{\hbar}{2m}\nabla^2 + g|\psi(r,t)|^2\right]\psi(r,t). \tag{4.34}$$

This form of the equation for the classical field is called a Gross-Pitaevskii equation and a key working equation for the classical analysis of the BEC of a weakly interacting Bose gas.

4.4.2 Bogoliubov excitation spectrum

If we substitute (4.33) into (4.34) and compare the $e^{-i\mu t}$ terms in both sides of the equation, we obtain

$$\mu = g\psi_0^2. \tag{4.35}$$

By comparing $e^{i[k\cdot r-(\mu+\omega)t]}$ terms in both sides of the equation, we have

$$(\mu+\omega)\psi_k = \frac{\hbar k^2}{2m}\psi_k + \mu\left(2\psi_k + \psi_{-k}\right). \tag{4.36}$$

Similarly, the comparison of $e^{-i[k\cdot r-(\mu-\omega)t]}$ terms in both sides of the equation leads to

$$(\mu-\omega)\psi_{-k} = \frac{\hbar k^2}{2m}\psi_{-k} + \mu\left(2\psi_{-k}+\psi_k\right). \tag{4.37}$$

Equations (4.36) and (4.37) can be summarized as

$$[A]\begin{pmatrix}\psi_k\\ \psi_{-k}\end{pmatrix} = \begin{pmatrix}\frac{\hbar k^2}{2m}+\mu & \mu\\ -\mu & \frac{\hbar k^2}{2m}-\mu\end{pmatrix}\begin{pmatrix}\psi_k\\ \psi_{-k}\end{pmatrix} = \omega\begin{pmatrix}\psi_k\\ \psi_{-k}\end{pmatrix}. \tag{4.38}$$

This is the eigenvalue equation for the eigenstates (ψ_k, ψ_{-k}) of the linear operator (matrix) $[A]$ with an eigen-frequency ω. In order to have a non-trivial solution for (4.38), i.e. the non-zero solutions for ψ_k and ψ_{-k}, the following determinant must be identically equal to zero:

$$Det\,[A-\omega I] = 0 \tag{4.39}$$

The eigen-frequency ω can be obtained by solving the characteristic equation (4.39) as

$$\omega = \pm\sqrt{\left(\frac{\hbar k^2}{2m}\right)\left[\left(\frac{\hbar k^2}{2m}\right)+2\mu\right]}. \tag{4.40}$$

As shown in Fig. 4.3, the excitation spectrum at each wavenumber k is split into the positive and negative branches. The eigen-state in the positive branch with an eigen-frequency $\omega_+ = \sqrt{\left(\frac{\hbar k^2}{2m}\right)\left[\left(\frac{\hbar k^2}{2m}+\mu\right)\right]}$ consists of the two plane waves $\psi_k^{(+)}$ and $\psi_{-k}^{(+)}$, which satisfies

$$\left(\frac{\hbar k^2}{2m}+\mu\right)\psi_k^{(+)} + \mu\psi_{-k}^{(+)} = \omega_+\psi_k^{(+)}, \tag{4.41}$$

or

$$\psi_k^{(+)} = \frac{-\mu}{\frac{\hbar k^2}{2m}+\mu-\omega_+}\psi_{-k}^{(+)} = -\frac{\mu}{\Delta}\psi_{-k}^{(+)}. \tag{4.42}$$

Here $\Delta = \frac{\hbar k^2}{2m}+\mu-\omega_+$ is a positive quantity (see Fig. 4.3). Combined with the normalization condition, $\psi_k^{(+)2}+\psi_{-k}^{(+)2}=1, \psi_k^{(+)}$ and $\psi_{-k}^{(+)}$ are expressed as

$$\psi_k^{(+)} = \frac{\mu}{\sqrt{\mu^2+\Delta^2}}, \tag{4.43}$$

$$\psi_{-k}^{(+)} = \frac{-\Delta}{\sqrt{\mu^2+\Delta^2}}. \tag{4.44}$$

At small wavenumbers, $\Delta \simeq \mu$ so that the eigen-state in the positive branch is reduced to

$$\begin{aligned}\psi_{r,t}^{(+)} &= \frac{1}{\sqrt{2}}\left\{e^{i[k\cdot r-(\mu+\omega_+)t]} - e^{-i[k\cdot r-(\mu+\omega_+)t]}\right\}\\ &= i\sqrt{2}\sin(k\cdot r)e^{-i(\mu+\omega_+)t}.\end{aligned} \tag{4.45}$$

Since $\psi_0(r)$ is a real-number order parameter, the pure imaginary in (4.45) suggests that the positive branch of the excitation spectrum represents the spatial phase modulation of the condensate.

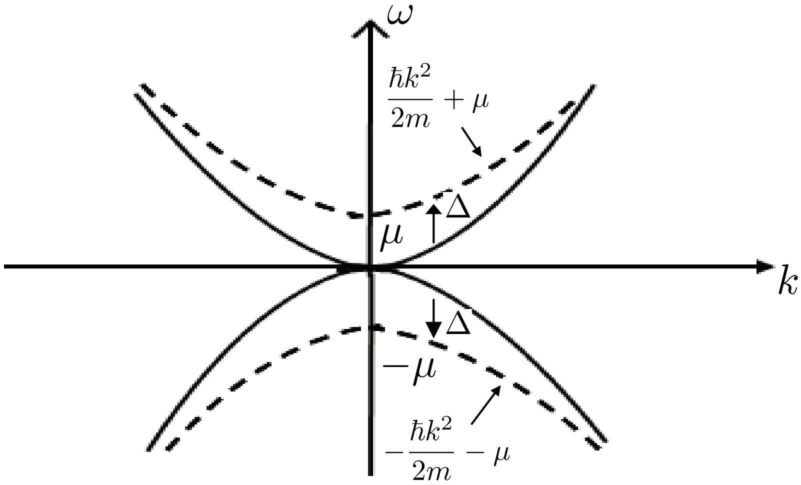

Figure 4.3: The Bogoliubov excitation spectrum.

At large wavenumbers, $\Delta \simeq 0$ so that the eigen-state in the positive branch is reduced to the simple plane wave,

$$\psi^{(+)}_{r,t} = e^{i[k\cdot r-(\mu+\omega_+)t]}. \tag{4.46}$$

The eigen-state in the negative branch with an eigen-frequency $\omega = -\sqrt{\left(\frac{\hbar k^2}{2m}\right)\left[\left(\frac{\hbar k^2}{2m}\right)+2\mu\right]}$ consists of the two plane waves $\psi^{(-)}_k$ and $\psi^{(-)}_{-k}$, which satisfies

$$\psi^{(-)}_k = \frac{-\mu}{\frac{\hbar k^2}{2m}+\mu-\omega_-} = \frac{\mu}{\Delta}\psi^{(-)}_{-k}. \tag{4.47}$$

Combined with the normalization condition, $\psi^{(-)2}_k + \psi^{(-)2}_{-k} = 1$, $\psi^{(-)}_k$ and $\psi^{(-)}_{-k}$ are expressed as

$$\psi^{(-)}_k = \frac{\mu}{\sqrt{\mu^2+\Delta^2}}, \tag{4.48}$$

$$\psi^{(-)}_{-k} = \frac{\Delta}{\sqrt{\mu^2+\Delta^2}}. \tag{4.49}$$

At small wavenumbers, $\Delta \simeq \mu$ so that the eigen-state in the negative branch is reduced to

$$\begin{aligned}\psi^{(-)}(r,t) &= \frac{1}{\sqrt{2}}\left\{e^{i[k\cdot r-(\mu+\omega_-)t]} + e^{-i[k\cdot r-(\mu+\omega_-)t]}\right\} \\ &= \sqrt{2}\cos(k\cdot r)e^{-i(\mu+\omega_-)t}.\end{aligned} \tag{4.50}$$

The negative branch of the excitation spectrum corresponds to the spatial amplitude modulation of the condensate. At large wavenumbers, $\Delta = 0$ so that the eigen-state in the negative branch is also reduced to the simple plane wave,

$$\psi^{(-)}r,t = e^{i[k\cdot r-(\mu+\omega_-)t]}. \tag{4.51}$$

4.4.3 Sound velocity, healing length and energy shift

For small momenta $p \ll mc = \sqrt{mgn}$, the Bogoliubov dispersion law (4.27) is well approximated by the phonon-like linear dispersion form

$$\varepsilon(p) = cp, \tag{4.52}$$

where $c = \sqrt{gn/m}$ is the sound velocity. This result agrees with our previous result (4.15) derived from the equation of state (4.13). According to the Bogoliubov theory, the long wavelength (low momentum) excitations of an interacting Bose gas are the sound waves. These excitations can also be regarded as the Nambu-Goldstone modes associated with breaking of gauge symmetry caused by Bose-Einstein condensation. In this low momentum regime, a real particle is represented by the coherent superposition of forward and backward propagating many quasi-particles: $\hat{a}_p = u_p\hat{b}_p + v_{-p}\hat{b}^+_{-p}$, where $|u_p| \sim |v_{-p}| \sim \sqrt{mc/2p} \gg 1$.

In the opposite limit $p \gg mc$, the Bogoliubov dispersion law (4.27) is reduced to the free-particle form:

$$\varepsilon(p) = \frac{p^2}{2m} + gn. \tag{4.53}$$

This is consistent with the above argument in (4.4.1). The (additional) interaction energy gn in (4.53) is traced back to the third term of the Hamiltonian (4.19). The interaction energy between the two particles in the condensate is calculated by the first term of (4.19) and is equal to gn. On the other hand, the interaction energy between one particle in the condensate and another particle in the excitation spectrum is equal to $2gn$. A factor of two comes from the equal contributions of the direct and exchange terms for identical bosons. In this large momentum regime, a forward propagating real particle is almost identical to a forward propagating quasi-particle: $\hat{a}_p \sim \hat{b}_p\,(u_p \simeq 1, v_{-p} \simeq 0)$.

The transition from the phonon regime to the free particle regime takes place near the point where $\frac{p^2}{2m} = gn$ is satisfied. If we use $p = \hbar/\xi$ in this equation, the characteristic length ξ can be written as

$$\xi = \sqrt{\frac{\hbar^2}{2mgn}} = \frac{1}{\sqrt{2}}\frac{\hbar}{mc}, \tag{4.54}$$

which is often referred to as "healing length". This is the length scale in which the density and phase fluctuations in the condensate are removed by the interaction between condensed particles. The healing length plays an important role in our discussion on superfluidity in the next chapter. The Bogoliubov dispersion law (4.27) is schematically shown in Fig. 4.4(a).

The first observation of the Bogoliubov excitation spectrum was reported in 1998, using the two photon Bragg scattering spectroscopy technique in atomic BEC [5]. Fig. 4.4(b) shows the experimental results which show a remarkable agreement with the theory. Figure 4.5 shows the observed excitation energy vs. (in-plane) momentum for an excitation-polariton condensate [6]. An angle-resolved energy spectroscopy for the leakage photons from the semiconductor microcavity provides directly the dispersion law of the excitation spectrum. The white solid line represents the quadratic dispersion law of a single exciton-polariton (normal state) far below the BEC critical density, while the black solid line shows the displaced quadratic dispersion law by the energy μ (chemical potential). The

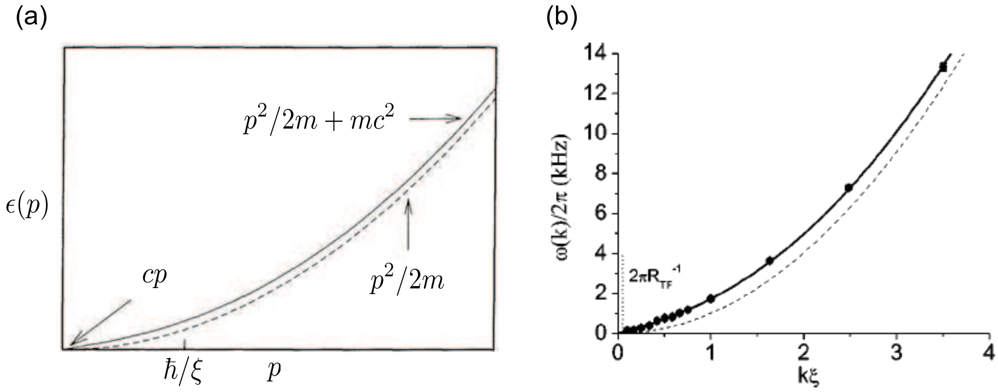

Figure 4.4: (a) Bogoliubov dispersion of elementary excitations. The transition between the phonon mode ($\varepsilon(p) = cp$) and the free particle mode ($\varepsilon(p) = p^2/2m + mc^2$) takes place at $p \sim \hbar/\xi$. Energy is given in units of mc^2. (b) Measured Bogoliubov excitation spectrum for atomic BEC [5].

black solid line in Fig. 4.5 corresponds to the dashed line in Fig. 4.4(a). The Bogoliubov excitation spectrum is indicated by the pink solid line in Fig. 4.5, which shows again a remarkable agreement with the experimental results.

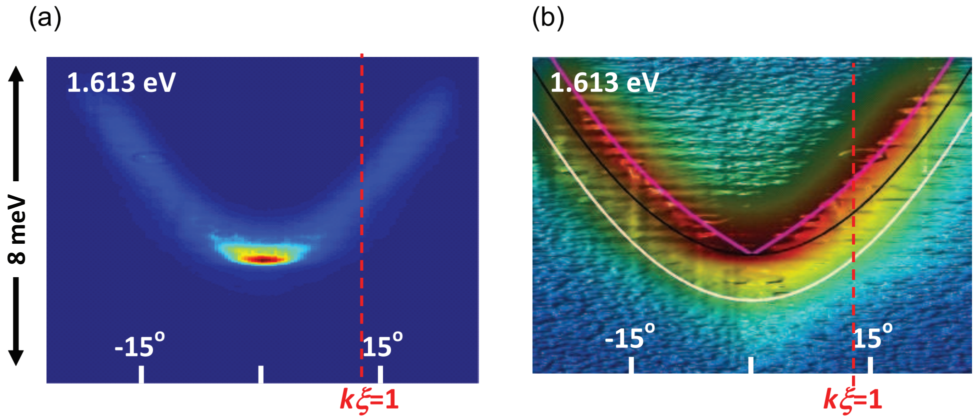

Figure 4.5: The observed luminescence (leakage photons) from an exciton-polariton condensate in a linear plot of intensity (left panel) and in a logarithmic plot of intensity (right panel). The white, black and pink solid lines the dispersion laws: $\frac{p^2}{2m} - \mu$, $\frac{p^2}{2m}$ and $\varepsilon(p)$ given by (4.27)[6].

The Bogoliubov dispersion law (4.27) can be rewritten in a normalized form:

$$\varepsilon(p)/U = \sqrt{(k\xi)^2 \left[(k\xi)^2 + 2\right]}, \tag{4.55}$$

where $U = gn$ is the interaction energy (or the chemical potential) and k is the in-plane wave number. One of the unique features of the Bogoliubov excitation spectrum is its universal scaling law, i.e. the relation between $\varepsilon(p)/U$ and $k\xi$ is independent of the interaction strength, mass of particles and particle density. This universal scaling law was confirmed experimentally as shown in Fig. 4.6[6].

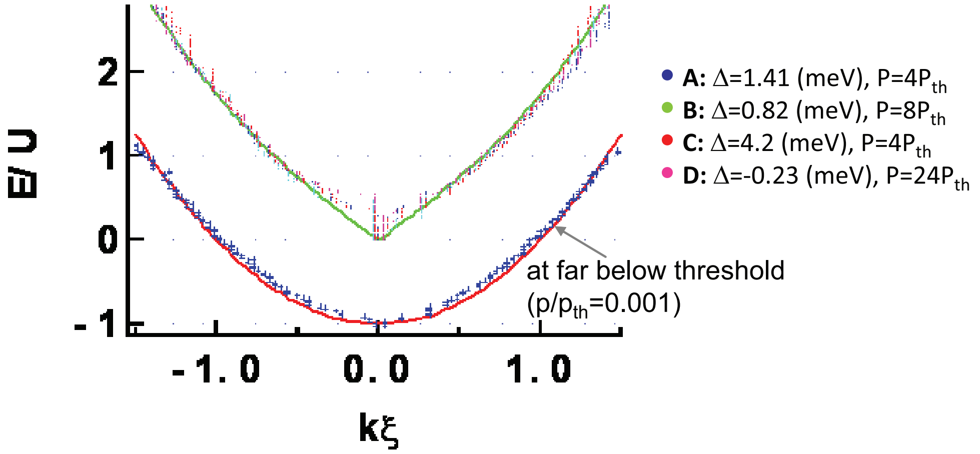

Figure 4.6: The normalized energy $\varepsilon(p)/U$ vs. normalized wavenumber $k\xi$ for four exciton-polariton condensates with different values of U, m and g [6].

The slope of the linear dispersion at small momentum regime provides the sound velocity, $c = \frac{d\varepsilon(p)}{dp}$. The estimated sound velocity for the atomic BEC shown in Fig. 4.4(b) is on the order of $\sim$ 1cm/sec, while that for the exciton-polariton BEC shown in Fig. 4.5 is on the order of $\sim 10^8$cm/sec. The eight orders of magnitude difference between the two sound velocities comes from the difference of the mass, $m_{\text{atom}} \sim 10^9 m_{\text{polariton}}$, and that of the interaction strength, $U_{\text{atom}} \sim 10^{-6} U_{\text{polariton}}$.

4.5 Quantum and thermal depletion

The average number of quasi-particles N_p carrying a momentum p must obey the Bose-Einstein distribution with the chemical potential $\mu = 0$

$$N_p \equiv \langle \hat{b}_p^+ \hat{b}_p \rangle = \frac{1}{\exp(\beta\varepsilon(p)) - 1}. \tag{4.56}$$

This should not be confused with the average number $\langle \hat{a}_p^+ \hat{a}_p \rangle$ of real particles. Using Bogoliubov transformation (4.20), one obtains

$$N_p \equiv \langle \hat{a}_p^+ \hat{a}_p \rangle = |v_{-p}|^2 + |u_p|^2 \langle \hat{b}_p^+ \hat{b}_p \rangle + |v_{-p}|^2 \langle \hat{b}_{-p}^+ \hat{b}_{-p} \rangle, \tag{4.57}$$

where $\langle \hat{b}_p \hat{b}_{-p} \rangle = \langle \hat{b}_p \rangle \langle \hat{b}_{-p} \rangle = 0$ and $\langle \hat{b}_p^+ \hat{b}_{-p}^+ \rangle = \langle \hat{b}_p^+ \rangle \langle \hat{b}_{-p}^+ \rangle = 0$ for thermal states at $p \neq 0$ are used. Thus, the number of particles in the condensate can be calculated by

$$N_0 \equiv N - \sum_{p \neq 0} N_p = N - \frac{V}{(2\pi\hbar)^3} \int dp \left[|v_{-p}|^2 + \frac{|u_p|^2 + |v_{-p}|^2}{\exp(\beta\varepsilon(p)) - 1} \right], \tag{4.58}$$

where (4.56) is used. At absolute zero temperature $T = 0$, there is a finite leakage of particles into the excitation spectrum given by $|v_{-p}|^2$ even though the population of quasi-particles is zero, $\langle \hat{b}_p^+ \hat{b}_p \rangle = 0$. This fundamental leakage of particles from the condensate is referred to quantum depletion. At higher temperatures, the thermal population of quasi-particles, the second term in the integrand in (4.58), dominates. This extra leakage of particles from the condensate is called thermal depletion.

Using (4.26), one can obtain the real particle occupation number at small momentum regime due to quantum depletion:

$$N_p^{(q)} = \frac{p^2/2m + gn}{2\varepsilon(p)} - \frac{1}{2} \xrightarrow[p \ll \hbar/\xi]{} \frac{\sqrt{mgn}}{2p}. \tag{4.59}$$

The population increases with a $1/p$ dependence when the momentum is decreased. This infrared divergency is a general property exhibited by BEC at $T = 0$. On the other hand, the quantum depletion disappears according to the law of $\sim 1/p^4$ when the momentum becomes larger than the characteristic momentum $\hbar/\xi$.

In order to calculate the real particle occupation number due to thermal depletion in the same low momentum (low energy) regime, we can use the low-energy expansion of the Bose-Einstein distribution (4.56), $\langle \hat{b}_p^+ \hat{b}_p \rangle \simeq k_B T/\varepsilon(p)$, and the relation $|u_p|^2 + |v_{-p}|^2 \simeq gn/\varepsilon(p)$ in (4.57). The results is

$$N_p^{(th)} \xrightarrow[p \ll \hbar/\xi]{} \frac{k_B T gn}{\varepsilon(p)^2} \simeq \frac{mk_B T}{p^2}, \tag{4.60}$$

which features a stronger infrared divergency compared to the quantum depletion and is a factor 2 smaller than the one exhibited by the ideal Bose gas at $T \neq 0$ (see Chapter 3). The thermal depletion disappears according to the law of $\sim e^{-\beta\varepsilon(p)}$ at $p \gg \hbar/\xi$.

Figure 4.7 shows the $1/p^2$ dependence of the populations in the Bogoliubov excitation spectrum at $p \lesssim \hbar/\xi$ for an exciton-polariton condensate [6]. This result reveals the exciton-polariton condensate is depleted thermally rather than quantum mechanically. In fact, the measured gas temperature suggests that the thermal energy $k_B T$ exceeds the energy scale $\sqrt{\frac{p^2}{2m} \cdot U}$ in this system, so that $N_p^{(th)}$ should dominate over $N_p^{(q)}$.

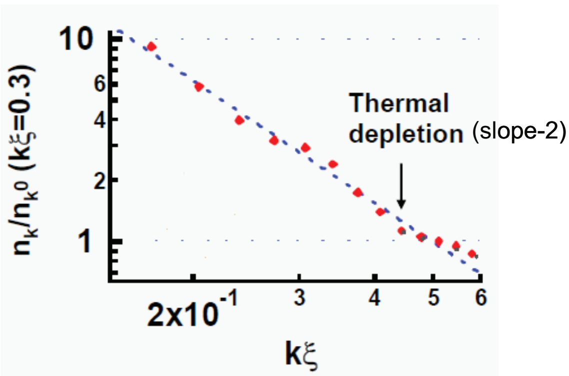

Figure 4.7: The normalized population vs. normalized wavenumber in an exciton-polariton condensate.

Integration of (4.58) at $T = 0$ yields

$$N_0 = N \left[1 - \frac{8}{3\sqrt{\pi}} \left(na^3 \right)^{1/2} \right], \tag{4.61}$$

which reveals that the quantum depletion is determined by the same parameter $\sqrt{na^3}$ characterizing the higher-order corrections to the ground state energy (4.31) and the chemical potential (4.32).

4.6 Spatial coherence

Using the momentum distribution of real particles in the excitation spectrum at $T = 0$, (4.59), one can calculate the first-order spatial coherence function:

$$\begin{aligned} g^{(1)}(s) &= \frac{N_0}{N} + \frac{1}{N}\frac{L^3}{(2\pi\hbar)^3}\int dpn(p)e^{-ip\cdot s/\hbar} \\ &= \frac{N_0}{N} + \frac{1}{N}\frac{L^3}{(2\pi\hbar)^3}\int dp|v_{-p}|^2 e^{-ip\cdot s/\hbar} \\ &= \frac{N_0}{N} + \frac{1}{N}\frac{L^3}{\xi^3}D\left(\frac{s}{\xi}\right), \end{aligned} \tag{4.62}$$

where we have defined the dimensionless function

$$D\left(s/\xi\right) = \frac{1}{2}\frac{1}{(2\pi)^3}\int dk\left(\frac{1+k^2}{\sqrt{k^4+2k^2}} - 1\right)e^{-i(s/\xi)\cdot k}. \tag{4.63}$$

Here $k = p\xi/\hbar$. For $s/\xi \gg 1$, the function $D\left(s/\xi\right)$ behaves like $\frac{\xi^2}{\sqrt{32}\pi^2 s^2}$. Equation (4.62) shows that the first-order spatial coherence decreases in the characteristic length scale ξ for an interacting Bose gas at $T = 0$.

$$g^{(1)}(s) = \frac{N_0}{N} + \frac{1}{N}\frac{L^3}{\sqrt{32}\pi^2\xi}\cdot\frac{1}{s^2}. \tag{4.64}$$

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