Chapter 3

Bose-Einstein Condensation of An Ideal Gas

An ideal gas consisting of non-interacting Bose particles is a fictitious system since every realistic Bose gas shows some level of particle-particle interaction. Nevertheless, such a mathematical model provides the simplest example for the realization of Bose-Einstein condensation. This simple model, first studied by A. Einstein [1], correctly describes important basic properties of actual non-ideal (interacting) Bose gas. In particular, such basic concepts as BEC critical temperature T_c (or critical particle density n_c), condensate fraction N_0/N and the dimensionality issue will be obtained.

3.1 The ideal Bose gas in the canonical and grand canonical ensemble

Suppose an ideal gas of non-interacting particles with fixed particle number N is trapped in a box with a volume V and at equilibrium temperature T. We assume a particle system somehow establishes an equilibrium temperature in spite of the absence of interaction. Such a system can be characterized by the thermodynamic partition function of canonical ensemble

$$Z = \sum_R e^{-\beta E_R}, \tag{3.1}$$

where R stands for a macroscopic state of the gas and is uniquely specified by the occupation number n_i of each single particle state i: $\{n_0, n_1, \cdots\cdots\}$. $\beta = 1/k_B T$ is a temperature parameter. Then, the total energy of a macroscopic state R is given by only the kinetic energy:

$$E_R = \sum_i \varepsilon_i n_i, \tag{3.2}$$

where ε_i is the eigen-energy of the single particle state i and the occupation number n_i satisfies the normalization condition

$$N = \sum_i n_i. \tag{3.3}$$

The probability of finding the ideal gas in a specific macroscopic state R is given by the law of large numbers in statistical mechanics [2]:

$$P_R = \frac{e^{-\beta E_R}}{Z}. \tag{3.4}$$

Using (3.4), one can calculate the average particle number of a specific single particle state (s)

$$
\begin{aligned}
n_s &= \sum_R n_s P_R \\
&= \frac{\sum_{\{n_0, n_1 \cdots\}} n_s e^{-\beta(\varepsilon_0 n_0 + \varepsilon_1 n_1 + \cdots)}}{\sum_{\{n_0, n_1 \cdots\}} e^{-\beta(\varepsilon_0 n_0 + \varepsilon_1 n_1 + \cdots)}} \\
&= \frac{\sum_{n_s} n_s e^{-\beta \varepsilon_s n_s} \sum^{(s)} e^{-\beta(\varepsilon_0 n_0 + \varepsilon_1 n_1 + \cdots)}}{\sum_{n_s} e^{-\beta \varepsilon_s n_s} \sum^{(s)} e^{-\beta(\varepsilon_0 n_0 + \varepsilon_1 n_1 + \cdots)}}
\end{aligned}
\tag{3.5}
$$

Here $\sum^{(s)}$ stands for the summation over $\{n_0, n_1, \cdots\}$ except for a particular summation for the state s. If the state s does not have a particle, N particles must be distributed over the states other than s so that the partition function in this case is given by

$$Z_s(N) = \sum^{(s)} e^{-\beta(\varepsilon_0 n_0 + \varepsilon_1 n_1 + \cdots)}, \tag{3.6}$$

$$\sum_i^{(s)} n_i = N. \tag{3.7}$$

If the state s has one particle, the remaining $N - 1$ particles must be distributed over the states other than s and the partition function in this case is modified to

$$Z_s(N - 1) = \sum^{(s)} e^{-\beta(\varepsilon_0 n_0 + \varepsilon_1 n_1 + \cdots)}, \tag{3.8}$$

$$\sum_i^{(s)} n_i = N - 1. \tag{3.9}$$

If we substitute these expressions into (3.5), we have

$$n_s = \frac{0 \times Z_s(N) + e^{-\beta \varepsilon_s} Z_s(N - 1) + 2 e^{-2\beta \varepsilon_s} Z_s(N - 2) + \cdots}{Z_s(N) + e^{-\beta \varepsilon_s} Z_s(N - 1) + e^{-2\beta \varepsilon_s} Z_s(N - 2) + \cdots}. \tag{3.10}$$

In order to proceed further, one has to introduce the approximate expression for the partition functions $Z_s(N - \Delta N)$. For this purpose, calculating the logarithmic function $\log Z_s(N - \Delta N)$ is more appropriate than calculating $Z_s(N - \Delta N)$ itself, since the former is a much more slowly varying function of ΔN compared to the latter [2]. Then one can truncate the expansion of $\log Z_s(N - \Delta N)$ to the first order of ΔN:

$$\log Z_s(N - \Delta N) \simeq \log Z_s(N) + \left[\frac{\partial}{\partial N} \log Z_s(N) \right] (-\Delta N). \tag{3.11}$$

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The first-order expansion coefficient is considered to be a universal constant independent of s:

$$\alpha_s = \frac{\partial}{\partial N} \log Z_s(N) = \frac{\partial}{\partial N} \log Z(N) = \alpha. \qquad (3.12)$$

This is because $Z(N)$ is the summation over many microscopic states and so the variation of its logarithm with respect to the total number of particles should be insensitive as to weather or not the particular state s is omitted. From (3.11) and (3.12), one obtains

$$Z_s(N - \Delta N) \simeq Z_s(N)e^{-\alpha \Delta N}. \qquad (3.13)$$

Using (3.13) in (3.10), one can calculate the average occupation number of the state s as

$$
\begin{aligned}
n_s &= \frac{\sum_{n_s} n_s e^{-n_s(\beta \varepsilon_s + \alpha)}}{\sum_{n_s} e^{-n_s(\beta \varepsilon_s + \alpha)}} \\
&= -\frac{1}{\beta}\frac{\partial}{\partial \varepsilon_s} \log \left[\sum_s e^{-n_s(\beta \varepsilon_s + \alpha)} \right] \\
&= \frac{1}{e^{\beta \varepsilon_s + \alpha} - 1}.
\end{aligned}
\qquad (3.14)
$$

The first-order expansion coefficient α is now replaced by a new parameter, called the chemical potential defined by $\mu = -\alpha/\beta$. μ can be uniquely determined by the normalization condition (3.3):

$$N = \sum_i n_i = \sum_i \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1}. \qquad (3.15)$$

Equation (3.14) is referred to as Bose-Einstein distribution function, in which the average occupation number n_s is determined uniquely by the temperature parameter β, the eigen-energy of the single particle state ε_s and the chemical potential μ.

The thermodynamic partition function (3.1) was defined for the system with a fixed number of particles. By taking an advantage of the unique relationship (3.15) between the total number of particles N and the chemical potential μ, one can extend (3.1) to the system with a varying total number of particles. The new thermodynamic partition function can be written as

$$Z = \sum_{n_0} e^{-\beta(\varepsilon_0 - \mu)n_0} \sum_{n_1} e^{-\beta(\varepsilon_1 - \mu)n_1} \ldots \ldots. \qquad (3.16)$$

By considering the chemical potential μ as a free parameter, (3.16) can describe the system with all possible values of N and is referred to as the thermodynamic partition function of grand canonical ensemble.

The grand canonical potential can be defined by $\Omega = E - TS - \mu N$, where E and S are the total energy and the entropy of the system[2]. Then, Ω is related to Z via [2]

$$\Omega = -k_B T \ln Z = k_B T \sum_i \ln \left[1 - e^{-\beta(\varepsilon_i - \mu)} \right]. \qquad (3.17)$$

The total number of particles N can be evaluated from (3.17) as

$$N = -\frac{\partial \Omega}{\partial \mu} = \sum_i \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1}, \qquad (3.18)$$

as well as the total energy

$$E = \Omega - T\frac{\partial \Omega}{\partial T} - \mu\frac{\partial \Omega}{\partial \mu} = \sum_i \frac{\varepsilon_i}{e^{\beta(\varepsilon_i - \mu)} - 1}. \tag{3.19}$$

as they should be. If we recall the average occupation number of the state i is $n_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1}$, the expressions (3.18) and (3.19) can be easily interpreted. The entropy S is similarly calculated as

$$\frac{S}{k_B} \equiv \frac{1}{k_B}\left(-\frac{\partial \Omega}{\partial T}\right) = \sum_i \left[\log(1 + n_i) + n_i \log(1 + n_i)\right]. \tag{3.20}$$

This is the maximum entropy (randomness) if the average number of particles per state n_i is given. The probability distribution of each state is that of a thermal state, i.e. $p_i(n) = \frac{1}{1+n_i}\left(\frac{n_i}{1+n_i}\right)^n$ [2]. The first term of R.H.S in (3.20) dominates the entropy when the single particle state is occupied by many particles, $n_i \gg 1$, and is referred to as "wave entropy" [3]. The second term of R.H.S. in (3.20) dominates the entropy when the single particle state is occupied sparsely, $n_i \ll 1$, and is referred to as "particle entropy" [3].

3.2 BEC critical temperature

Equation (3.14) provides the important constraint for the chemical potential, i.e. $\mu < \varepsilon_0$, where ε_0 is the lowest energy eigenvalue of the single particle states. The violation of this inequality would result in a negative value of the occupation number for the states with energies smaller than μ. When μ approaches ε_0 from smaller values, the occupation number $n_0 = \frac{1}{e^{\beta(\varepsilon_0 - \mu)} - 1}$ becomes increasingly large. This is the physical picture behind Bose-Einstein condensation. One can split the total number of particles into

$$N = N_0 + N_{th}, \tag{3.21}$$

where

$$N_{th} = \sum_{i \neq 0} n_i(T, \mu), \tag{3.22}$$

is the number of particles in all excited states except for the lowest energy ground state. N_{th} is also referred to as the thermal component of the gas at temperature T and chemical potential μ. For a finite temperature T and a large volume V, N_{th} has a smooth behavior as a function of μ and reaches its maximum $N_c = N_{th}(T, \mu = \varepsilon_0)$ asymptotically at $\mu = \varepsilon_0$ as shown in Fig. 3.1. On the other hand, N_0 diverges when μ is approaching to ε_0. If the value of N_c is lager than N, (3.21) is always satisfied for values of μ considerably smaller than ε_0 and N_0 is negligible compared to N. This corresponds to the point (μ_1, N_1) in Fig. 3.1. If N_c is smaller than N, on the contrary, the occupation number N_0 of the ground state is substantial and thus it is expected the condensate is formed. This corresponds to the point (μ_2, N_2) in Fig. 3.1. Since the ground state population is given by $N_0 = \left[e^{\beta(\varepsilon_0 - \mu)} - 1\right]^{-1}$, the difference between the ground state energy and the chemical potential is substantially smaller than $k_B T$:

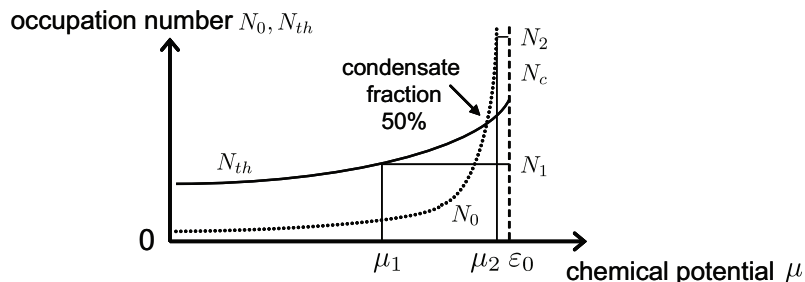$$\varepsilon_0 - \mu = k_B T \ln\left(1 + \frac{1}{N_0}\right) \simeq \frac{k_B T}{N_0}. \tag{3.23}$$

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Figure 3.1: The occupation number N_0 in the ground state and N_{th} in all excited states vs. chemical potential μ. If $N > N_c$, the system exhibits BEC.

The critical temperature T_c of BEC is operationally defined by the relation

$$N_{th}\left(T_c, \mu = \varepsilon_0\right) = N. \tag{3.24}$$

For a system with a finite volume V, the total number of single particle states in the energy range $k_B T$ measured from the ground state energy ε_0 is finite, and so $N_{th}\left(T_c, \mu = \varepsilon_0\right)$ stays a finite value. This means that (3.24) is always satisfied at finite temperatures and the BEC critical temperature exists irrespective of dimensionality. This is in sharp contrast to the opposite conclusions drawn for a uniform 2D or 1D system with infinite area or length [4]. We will discuss this point further in the next section.

3.3 BEC threshold in a uniform system

3.3.1 Energy density of states

The energy density of states for a free particle with a mass m in a $d-$dimensional system with a system size L is calculated by the standard method

$$\rho^{(d)}(\varepsilon) = \Omega_d \left(\frac{L}{2\pi}\right)^d \frac{1}{2} \left(\frac{2m}{\hbar^2}\right)^{\frac{d}{2}} \varepsilon^{\frac{d}{2}-1}, \tag{3.25}$$

where

$$\Omega_d = \begin{cases} 4\pi & (d = 3) \\ 2\pi & (d = 2) \\ 1 & (d = 1) \end{cases} \tag{3.26}$$

The normalized energy density of states is plotted for three-, two- and one-dimensional systems in Fig. 3.2. Note that $\rho^{(3)}(\varepsilon)$ vanishes when $\varepsilon \to 0$, while $\rho^{(2)}(\varepsilon)$ stays constant and $\rho^{(1)}(\varepsilon)$ diverges in the same limit. This difference in the asymptotic behaviors of $\rho^{(d)}(\varepsilon \to 0)$ results in the important consequence that a finite BEC critical temperature T_c exists only in a three-dimensional system [4]. However, as mentioned already, if a system size is finite, the non-zero BEC critical temperature T_c exists independent of dimensionality.

3.3.2 BEC critical temperature/density

The BEC critical temperature for a uniform 3D system is given by the condition that all particles accommodated in excited single particle states (except for a ground state) when
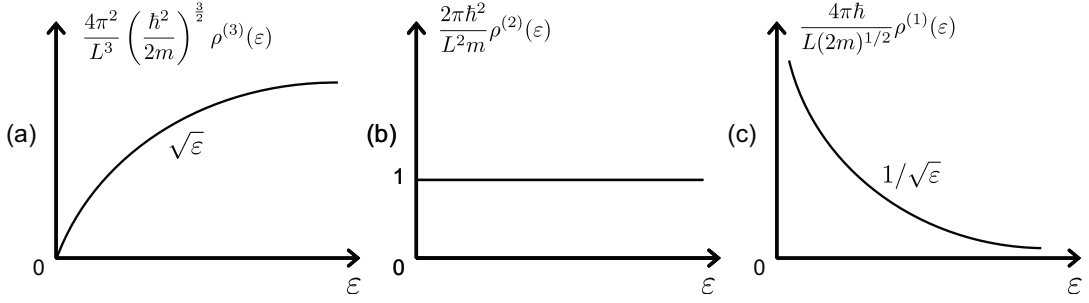
Figure 3.2: The normalized energy density of states for uniform 3D, 2D and 1D systems.

$\mu = \varepsilon_0 = 0$ are equal to the total number of particles in the system:

$$N = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\int_0^\infty \sqrt{\varepsilon}\frac{d\varepsilon}{e^{\beta\varepsilon}-1}.\tag{3.27}$$

We can evaluate the energy integral using the relation, $\frac{1}{e^x-1} = \sum_{n=1}^\infty e^{-nx}$, where $x = \beta\varepsilon$:

$$\begin{aligned}\int_0^\infty \sqrt{\varepsilon}\frac{d\varepsilon}{e^{\beta\varepsilon}-1} &= \beta^{-\frac{3}{2}}\sum_{n=1}^\infty\int_0^\infty e^{-nx}x^{1/2}dx\\&= \beta^{-\frac{3}{2}}\sum_{n=1}^\infty n^{-\frac{3}{2}}\int_0^\infty e^{-t}\sqrt{t}dt\\&= \beta^{-\frac{3}{2}}\zeta\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right),\end{aligned}\tag{3.28}$$

where $\zeta\left(\frac{3}{2}\right) = \sum_{n=1}^\infty n^{-\frac{3}{2}}$ and $\Gamma\left(\frac{3}{2}\right) = \int_0^\infty e^{-t}\sqrt{t}dt$. Substituting (3.28) into (3.27), one obtain the BEC critical density for a uniform 3D system,

$$\begin{aligned}n_c \equiv \frac{N_c}{V} &= \frac{1}{4\pi^2}\left(\frac{2m}{\beta\hbar^2}\right)^{\frac{3}{2}}\zeta\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)\\&\simeq 2.612\frac{1}{\lambda_{T_c}^3}.\end{aligned}\tag{3.29}$$

Here $\lambda_{T_c} = \sqrt{\frac{2\pi\hbar^2}{mk_BT_c}}$ is the thermal de Broglie wavelength at the critical temperature. Since $n_c^{-1/3}$ is the average distance between particles, the BEC critical condition (3.29) means that BEC occurs when the inter-particle distance becomes comparable to the thermal de Broglie wavelength of the particle at a given temperature. When the temperature is lower than T_c (or equivalently the density is larger than n_c), the mixture of the condensate at $\varepsilon_0 = 0$ (single particle ground state) and the thermal populations at $\varepsilon > 0$ (single particle excited states) is formed:

$$n = \frac{2.612}{\lambda_T^3} + n_0.\tag{3.30}$$

3.3.3 Condensate fraction

The BEC critical condition is often expressed in terms of the Bose function $g_p(Z)$ defined by

$$
\begin{aligned}
g_p(Z) &= \frac{1}{\Gamma(p)} \int_0^\infty dx \, x^{p-1} \frac{1}{Z^{-1}e^x - 1} \\
&= \sum_{l=1}^\infty \frac{Z^l}{l^p}.
\end{aligned} \tag{3.31}
$$

where $Z = e^{\beta\mu}$ is a fugacity and $\Gamma(p) = (p-1)!$. The energy integral for uniform 3D, 2D and 1D systems are then reduced to the Bose functions of $Z = 1$ and $p = \frac{3}{2}, p = 1$ and $p = 1/2$, respectively. Among them, only $g_{\frac{3}{2}}(1)$ converges, while $g_1(1)$ and $g_{1/2}(1)$ diverge, so that a finite critical temperature $T_c \neq 0$ exists only for a 3D system as far as a system is uniform and infinite [4].

At a critical temperature T_c in a uniform 3D system, all particles are in the thermal populations, i.e. distributed over single particle excited states at $\varepsilon > 0$:

$$
\frac{V}{\lambda_{T_c}^3} g_{\frac{3}{2}}(1) = N, \tag{3.32}
$$

while at lower temperatures than T_c, the thermal population is lower than (3.32)

$$
\frac{V}{\lambda_T^3} g_{\frac{3}{2}}(1) = N_T < N. \tag{3.33}
$$

Taking the ratio of (3.32) to (3.33), one obtains

$$
\frac{\lambda_T^3}{\lambda_{T_c}^3} = \frac{N}{N_T}. \tag{3.34}
$$

From this relation, the number of particles in the condensate is expressed as

$$
N_0 = N - N_T = N \left[1 - \left(\frac{T}{T_c} \right)^{\frac{3}{2}} \right]. \tag{3.35}
$$

Figure 3.3(a) shows the condensate fraction N_0/N vs. normalized temperature T/T_c. If an ideal gas is trapped in a 3D harmonic potential, the above argument must be modified to take into account the new boundary condition. The condensate fraction in this case is not given by (3.35) but given by $N_0/N = 1 - (T/T_c)^3$, which is confirmed by the experimental results shown in Fig. 3.3(b) [5].

3.3.4 Volume requirement for BEC

For a relatively small system, the energy difference between the first excited state and the ground state becomes an appreciable value, $\varepsilon_1 - \varepsilon_0 = \hbar^2/2mV^{2/3}$. If the thermal energy $k_B T$ becomes much smaller than $\varepsilon_1 - \varepsilon_0$, almost all particles occupy the ground state, i.e.
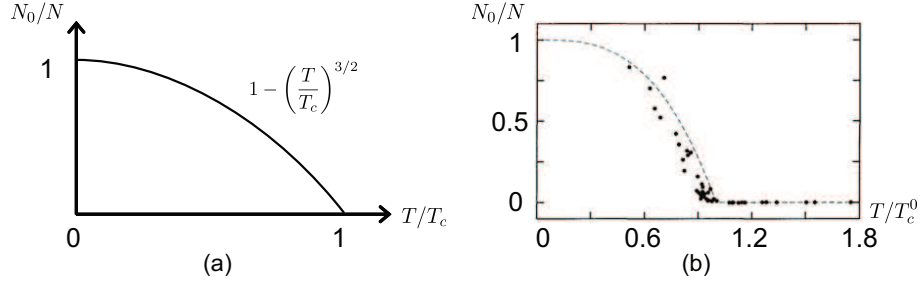
Figure 3.3: (a) The condensate fraction N_0/N vs. the normalized temperature T/T_c in a uniform 3D system. (b) The condensate fraction vs. the normalized temperature for an atomic BEC in a 3D harmonic trap [5]. The dashed line is given by $N_0/N = 1 - (T/T_c)^3$.

$N_T \ll N$ when $\hbar^2/2mV^{2/3} \gg k_BT$. This should not be considered as BEC. For a system with a large enough volume, the temperature range satisfying

$$\hbar^2/2mV^{2/3} \ll k_BT \ll k_BT_c, \tag{3.36}$$

can be found, where $N_T \ll N$ is realized due to the quantum statistical properties of Bose particles. From (3.36) we obtain the volume requirement for BEC as follows:

$$V \gg V_c = \left(\frac{2mk_BT_c}{\hbar^2}\right)^{\frac{3}{2}} \sim \lambda_{T_c}^3. \tag{3.37}$$

That is, the system volume must be much larger than the cube of thermal de Broglie wavelength. The use of the inequality $\hbar^2/2mV^{2/3} \ll k_BT$ is actually crucial in the above theory of BEC based on the energy density of states and the continuous energy integral rather than discrete sum over the single particle excited states.

The chemical potential approaches to the ground state energy, $\mu \to \varepsilon_0$, in the limit of $N_0 \to \infty$ as evidenced by (3.23). Setting $\varepsilon_0 - \mu = 0$ in the evaluation of N_T is then justified only if $\varepsilon_1 - \varepsilon_0 = \hbar^2/2mV^{2/3} \gg \varepsilon_0 - \mu \simeq k_BT/N_0$ is satisfied.

3.4 Thermodynamic functions of an ideal gas

The total energy of the system is given by

$$\begin{aligned} E &= \sum_i \frac{\varepsilon_i}{e^{[\beta(\varepsilon_i - \mu)]} - 1} \\ &= \frac{3}{2}k_BT \frac{V}{\lambda_T^3}g_{\frac{5}{2}}(Z), \end{aligned} \tag{3.38}$$

where $g_{\frac{5}{2}}(Z) = \frac{4}{3\sqrt{\pi}}\int_0^\infty dx x^{3/2}\frac{1}{Z^{-1}e^x - 1}$ corresponds to the Bose function (3.31) with $p = \frac{5}{2}$ and $Z = e^{\beta\mu}$. For $T < T_c, Z = 1$ and one has $g_{\frac{5}{2}}(1) = 1.342$. Then, the specific heat $C_V = \frac{\partial E}{\partial T}$ is obtained as

$$\frac{C_V}{k_BN} = \frac{15}{4}\frac{v}{\lambda_T^3}g_{\frac{5}{2}}(1), \tag{3.39}$$

for $T < T_c$, and

$$\frac{C_V}{k_B N} = \frac{15}{4}\frac{v}{\lambda_T^3}g_{\frac{5}{2}}(Z) - \frac{9}{4}\frac{g_{\frac{3}{2}}(Z)}{g_{\frac{1}{2}}(Z)}, \tag{3.40}$$

for $T > T_c$, where $v = 1/n = \frac{V}{N}$ is the specific volume. The specific heat has a typical cusp at T_c as shown in Fig. 3.4.
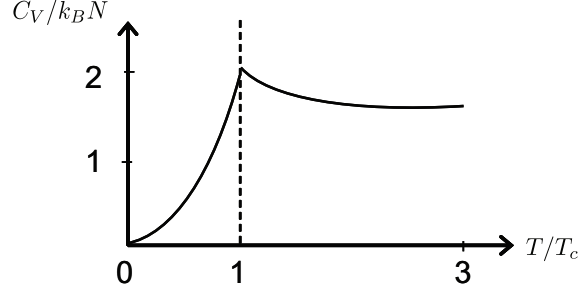

Figure 3.4: Specific heat of an ideal uniform Bose gas vs. temperature. For large temperatures, the curve approaches to the classical value of $3/2$.

For ideal gases in three dimensions, the thermodynamic relation $P = \frac{2}{3}\frac{E}{V}$ holds [2]. In BEC, the energy E increases linearly with the volume V, so that using (3.38) with $Z = 1$ for $T \leq T_c$, one obtains the equation of state for BEC,

$$P = \frac{k_B T}{\lambda_T^3}g_{\frac{5}{2}}(1). \tag{3.41}$$

The pressure of the gas does not depend on the volume in a BEC regime so that the compressibility of the BEC phase is infinite. This pathological feature is remedied by the inclusion of the two-body interactions (see Chapter 4). At $T > T_c$, the pressure is given by

$$P = \frac{k_B T}{\lambda_T^3}g_{\frac{5}{2}}(Z). \tag{3.42}$$

Since $Z = e^{\beta\mu}$ decreases toward zero with increasing the volume V, the pressure P also decreases with V. Figure 3.5 shows the equation of state of the ideal Bose gas. The phase transition line separating the BEC and normal phases is obtained by substituting (3.32) or equivalently $k_B T_c = \frac{2\pi\hbar^2}{m}\left[vg_{\frac{3}{2}}(1)\right]^{-2/3}$ into (3.41) as

$$Pv^{5/3} = \left(2\pi\hbar^2 m\right)g_{\frac{5}{2}}(1)/g_{\frac{3}{2}}(1)^{5/3}. \tag{3.43}$$

3.5   Spatial coherence function

The off-diagonal element of the one-body density operator (2.23), which is a key parameter for understanding the long-range order (or spatial coherence), is given by

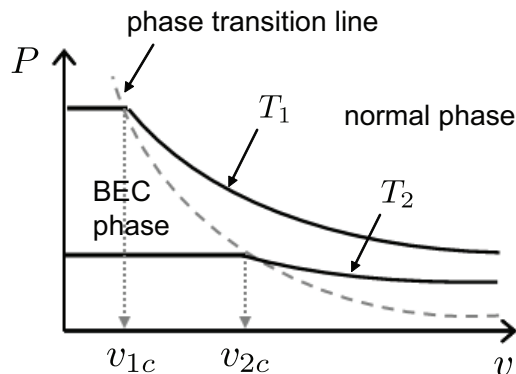$$g^{(1)}(r_1, r_2) = \frac{1}{N}\sum_i n_i\varphi_i^*(r_1)\varphi_i(r_2), \tag{3.44}$$

Figure 3.5: Pressure of the ideal Bose gas vs the specific volume $v = V/N$ for two temperatures $T_1 > T_2$.

for the ideal Bose gas. Here φ_i is the single particle wave function and n_i is the average occupation of the state i and given by (3.14). By separating the contribution of the condensate ($i = 0$) and that of the thermal component ($i \neq 0$), (3.44) can be rewritten as

$$g^{(1)}(r_1, r_2) = \frac{N_0}{N} + \frac{1}{N} \frac{V}{(2\pi\hbar)^3} \int dp \frac{e^{ip\cdot(r_1-r_2)/\hbar}}{e^{\beta(p^2/2m)} - 1}. \tag{3.45}$$

The corresponding momentum distribution is

$$n(p) = N_0 \delta(p) + \frac{V}{(2\pi\hbar)^3} \frac{1}{e^{\beta(p^2/2m)} - 1}. \tag{3.46}$$

The existence of the *delta* function in the momentum distribution for $T \leq T_c$ is the clear signature of BEC and is referred to as "momentum-space condensation".

It is interesting to see the behavior of the first-order spatial coherence function at large distances $s = |r_1 - r_2|$. This is determined by the small-momentum components of the thermal distribution (long-wavelength fluctuation). For a temperature below the BEC critical temperature $T < T_c$ one can use the low-momentum expansion:

$$\frac{1}{\exp(\beta p^2/2m) - 1} \sim \frac{2mk_B T}{p^2}, \tag{3.47}$$

where $p^2/2m \ll k_B T$ is used. Using (3.47) in (3.45), one obtains the $1/s$ behavior of $g^{(1)}(s)$,

$$g^{(1)}(s) = \frac{N_0}{N} + \frac{V}{N(2\pi)^3} \frac{Z}{\lambda_T^2} \cdot \frac{1}{s}, \tag{3.48}$$

where $Z = \exp(\beta\mu) \simeq 1$ is the fugacity and λ_T is the thermal de Broglie wavelength. The $1/p^2$ dependence in (3.47) and the $1/s$ dependence of (3.48) are the general consequences of BEC at a finite temperature.

At temperatures close to T_c, one can use the slightly different low-momentum expansion

$$\frac{1}{\exp[\beta(p^2/2m - \mu)] - 1} \sim \frac{Z 2mk_B T}{p^2 + p_c^2}, \tag{3.49}$$

where $p_c^2 = 2mk_BT(1-Z)$. Using this expression in (3.45) together with $N_0 = 0$, one obtains

$$g^{(1)}(s) = \frac{1}{N} \frac{V}{(2\pi)^3} \frac{Z}{\lambda_T^2} \frac{\exp\left[-\sqrt{4\pi(1-Z)}s/\lambda_T\right]}{s}. \tag{3.50}$$

The first-order coherence function decays exponentially, where λ_T and Z determine the decay constant.

At temperatures far above T_c, i.e. $\beta\mu \to -\infty$, the momentum distribution approaches the classical Maxwell-Boltzmann distribution $\exp\left[-p^2/2mk_BT\right]$ and the first order spatial coherence function(3.45) is reduced to

$$g^{(1)}(s) = \exp\left(-\frac{\pi s^2}{\lambda_T^2}\right). \tag{3.51}$$

The first-order coherence function decays by a gaussian law. For both cases, (3.50) and (3.51), the first-order spatial coherence function tends to zero within a distance approximately equal to the thermal de Broglie wavelength. However, at $T < T_c$, the first-order coherence function remains a finite value N_0/N at macroscopic distances due to BEC. Such behaviors of the cold Bose gas are shown in Fig.2.4.

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