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Test of purity by LOCC

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ABSTRACT

Given *n*-copies of an unknown bipartite (possiblly mixed) state, our task is to test whether the state is a pure state of not. If one is allowed to use global operations, optimal one-sided error test is the projection onto the symmetric subspace, obviously. Is it possible to approximate the globally optimal measurement by LOCC when *n* is large?

KEYWORDS
statistical test, entanglement, LOCC, group representation

1 Introduction

Given n-copies $|\phi\rangle^{\otimes n}$ of an unknown bipartite (possiblly mixed) state $|\phi\rangle \in \mathcal{H}\otimes\mathcal{H}_B$, consider test of whether the state is a pure state of not.

If one is allowed to use all quantum operations, optimal one-sided error test is the projection onto the symmetric subspace of $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$. Here, one-sided test means a test accepting the hypothesis with certainty in case it is true. Then, the POVM element [1] for accepting the hypothesis have to be larger than the projector onto the linear span of $\left\{|\phi\rangle^{\otimes n}; |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B\right\}$, which is the symmetric subspace. Therefore, to minimize the probability of accepting the hypothesis in case it is not ture, the POVM element has to be the projection onto the symmetric subspace.

In this paper, we restrict operations used for test to local operations and classical communications (LOCC). Obviously, the globally optimal test, being non-separable, cannot be performed by LOCC. Therefore, we consider possitility of approximating the globally optimal measurement by LOCC when n is large. This problem is an example of LOCC testing of the hypothesis constituted of continuous family of entangled states.

2 A standard form of an ensemble of identical bipartite pure states

Suppose we are given n-copies of unknown pure bipartite state $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Here we assume $\mathcal{H}_A \simeq \mathcal{H}_B \simeq \mathcal{H}$ and $\dim \mathcal{H} = d$. It is known that $|\phi\rangle^{\otimes n}$ has the standard form defined as follows.

Note $|\phi\rangle^{\otimes n}$ is invariant by the reordering of copies, or the action of the permutation σ in the set $\{1, \ldots n\}$ such that

$$\bigotimes_{i=1}^{n} |h_{i,A}\rangle|h_{i,B}\rangle \mapsto \bigotimes_{i=1}^{n} |h_{\sigma^{-1}(i),A}\rangle|h_{\sigma^{-1}(i),B}\rangle, \tag{1}$$

where $|h_{i,A}\rangle \in \mathcal{H}_A$ and $|h_{i,B}\rangle \in \mathcal{H}_B$. The action of the symmetric group gives a decomposition of the tensored space $\mathcal{H}^{\otimes n}$ [2],

$$\mathcal{H}^{\otimes n} = \bigoplus_{\lambda} \mathcal{W}_\lambda, \ \mathcal{W}_\lambda := \mathcal{U}_\lambda \otimes \mathcal{V}_\lambda,$$

where $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$ is an irreducible space of the tensor representation of SU(d), and the representation (1) of the symmetric group, respectively, and

$$\lambda = (\lambda_1, \ldots, \lambda_d), \quad \lambda_i \geq \lambda_{i+1} \geq 0, \sum_{i=1}^{d} \lambda_i = n$$

is called *Young index*, which $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$ uniquely corresponds to. We denote by $\mathcal{U}_{\lambda,A}$, $\mathcal{V}_{\lambda,A}$, and $\mathcal{U}_{\lambda,B}$, $\mathcal{V}_{\lambda,B}$ the irreducible component of $\mathcal{H}_A^{\otimes n}$ and $\mathcal{H}_B^{\otimes n}$, respectively. Also, $\mathcal{W}_{\lambda,A} := \mathcal{U}_{\lambda,A} \otimes \mathcal{V}_{\lambda,A}$, $\mathcal{W}_{\lambda,B} := \mathcal{U}_{\lambda,B} \otimes \mathcal{V}_{\lambda,B}$. Hereafter, the projector onto a subspace is represented by the same symbol as the subspace.

Due to [3], in terms of this decomposition, $|\phi\rangle^{\otimes n}$ can

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be written as

$$|\phi\rangle^{\otimes n} = \bigoplus_{\lambda} a_\lambda |\phi_\lambda\rangle |\Phi_\lambda\rangle, \tag{2}$$

where $|\phi_\lambda\rangle \in \mathcal{U}_{\lambda,A} \otimes \mathcal{U}_{\lambda,B}$, and $|\Phi_\lambda\rangle \in \mathcal{V}_{\lambda,A} \otimes \mathcal{V}_{\lambda,B}$. While a_λ and $|\phi_\lambda\rangle$ are dependent on $|\phi\rangle$, $|\Phi_\lambda\rangle$ is a maximally entangled state which does not depend on $|\phi\rangle$,

$$|\Phi_\lambda\rangle := \frac{1}{\sqrt{d_\lambda}} \sum_{i=1}^{d_\lambda} |f_i\rangle |f_i\rangle,$$

with $\{|f_i\rangle\}$ being an orthonormal complete basis of $\mathcal{V}_\lambda$, and $d_\lambda := \dim \mathcal{V}_\lambda$.

Observe that the linear span of all the vectors in the form of (2) is the symmetric subspace of $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$. Therefore, denoting the projector on this subspace by Π^n, we have

$$\Pi^n = \bigoplus_{\lambda} \mathbf{1}_{\mathcal{U}_{\lambda,A}} \otimes \mathbf{1}_{\mathcal{U}_{\lambda,B}} \otimes |\Phi_\lambda\rangle \langle \Phi_\lambda|. \tag{3}$$

3 An LOCC test of maximally entangled state

[4] treats the problem of testing whether the given state ρ is the d dimensional maximally entangled state

$$|\Phi\rangle := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |f_i\rangle |f_i\rangle$$

using LOCC, and found out a protocol whose probability of accepting the hypothesis equals

$$\frac{\langle \Phi| \rho |\Phi\rangle + \frac{1}{(d)^2}}{1 + \frac{1}{(d)^2}}. \tag{4}$$

This test is one-sided, i.e., accepts the hypothesis with probability 1 in case that it is true. Also, when d is very large, this approximately equals $\langle \Phi| \rho |\Phi\rangle$, which is the accepting probability of globally optimal one-sided test.

4 Protocol

Observe the globally optimal test, Π^n, is equivalent to the composition of the projector $\mathcal{W}_{\lambda,A} \otimes \mathcal{W}_{\lambda,B}$ followed by $\mathbf{1}_{\mathcal{U}_{\lambda,A}\otimes\mathcal{U}_{\lambda,B}} \otimes |\Phi_\lambda\rangle \langle \Phi_\lambda|$. While the former is implemented by an LOCC, the latter requires global operations. Hence, instead, we perform an asymptotically optimal LOCC test of the maximally entangled state in [4]. So, our protocol is:

(i) A and B applies the projective measurement $\{\mathcal{W}_{\lambda,A}\}_\lambda$ and $\{\mathcal{W}_{\lambda,B}\}_\lambda$, respectively.

(ii) Do the test for maximally entangled state in [4] to $\mathrm{tr}_{\mathcal{U}_{\lambda,A}} \rho_{n,\lambda}$, where $q^n_\lambda := \mathrm{tr}\, \rho^{\otimes n} \mathcal{W}_{\lambda,A} \otimes \mathcal{W}_{\lambda,B}$ and $\rho_{n,\lambda} := \frac{1}{q_\lambda} \mathcal{W}_{\lambda,A} \otimes \mathcal{W}_{\lambda,B} \rho^{\otimes n} \mathcal{W}_{\lambda,A} \otimes \mathcal{W}_{\lambda,B}$.

5 Performance of the protocol

In this subsection, it is proved that our protocol is asymptotically as good as globally optimal test, Π^n. If the given state is a pure state, obviously the acceptance probability P^n_{opt} of the test Π^n is 1. If the input is not a pure state, due to (9), we have

$$\begin{aligned} -\lim_{n\to\infty} \frac{1}{n} \log P^n_{opt} &= D((1, 0, \cdots, 0) \,||\, \boldsymbol{p}) \\ &= -\log p_1, \end{aligned}$$

where $\boldsymbol{p} := (p_1, \cdots, p_{d^2})$ is eigenvectors of the given state ρ. Also, by (3), when the given state is $\rho^{\otimes n}$,

$$\begin{aligned} P^n_{opt} &:= \sum_\lambda \mathrm{tr} \rho^{\otimes n} \mathbf{1}_{\mathcal{U}_{\lambda,A}} \otimes \mathbf{1}_{\mathcal{U}_{\lambda,B}} \otimes |\Phi_\lambda\rangle \langle \Phi_\lambda| \\ &= \sum_\lambda q^n_\lambda \mathrm{tr} \langle \Phi_\lambda| \rho_{n,\lambda} |\Phi_\lambda\rangle. \end{aligned}$$

Below, we will show our LOCC test is asymptotically equivalent to this globally optimal test. On the other hand, due to (4), our test will accept the input ρ_n with the probability

$$P^n_* := \sum_\lambda q^n_\lambda \frac{\mathrm{tr} \langle \Phi_\lambda| \rho_{n,\lambda} |\Phi_\lambda\rangle + \frac{1}{(d_\lambda)^2}}{1 + \frac{1}{(d_\lambda)^2}}.$$

If the given state ρ is a pure state,

$$P^n_* = \sum_\lambda q^n_\lambda \frac{1 + \frac{1}{(d_\lambda)^2}}{1 + \frac{1}{(d_\lambda)^2}} = 1.$$

Suppose ρ is not a pure state. Observe

$$\begin{aligned} P^n_* &\le \sum_\lambda q_\lambda \left(\mathrm{tr} \langle \Phi_\lambda| \rho_{n,\lambda} |\Phi_\lambda\rangle + \frac{1}{(d_\lambda)^2} \right) \\ &= P^n_{opt} + \sum_\lambda \frac{q^n_\lambda}{(d_\lambda)^2}, \end{aligned}$$

where, by virture of (5),

$$\begin{aligned} \sum_\lambda \frac{q^n_\lambda}{(d_\lambda)^2} &= \sum_\lambda \frac{\mathrm{tr}\, \rho^{\otimes n} \mathcal{W}_{\lambda,A} \otimes \mathcal{W}_{\lambda,B}}{(d_\lambda)^2} \\ &\le \sum_\lambda \frac{(p_1)^n (\dim \mathcal{W}_{\lambda,A})^2}{(d_\lambda)^2} \\ &= (p_1)^n \sum_\lambda (\dim \mathcal{U}_{\lambda,A})^2 \\ &= (p_1)^n \sum_\lambda \left(\frac{\prod_{i<j} \left(\lambda_i - \lambda_j - i + j \right)}{\prod_{i=1}^{d-1} (d-i)!} \right)^2 \\ &\le (p_1)^n (n+1)^d n^{d^2}. \end{aligned}$$

(Also, one may use the relation

$$\sum_\lambda (\dim \mathcal{W}_{\lambda,A})^2$$
$$= \dim\left(\text{symmetric subspace of } \left(\mathbb{C}^{d^2}\right)^{\otimes n}\right)$$
$$\leq (n+1)^{d^2}$$

)

Therefore, even if the state ρ is not a pure state,

$$-\lim_{n\to\infty}\frac{1}{n}\log P_*^n \geq \log p_1 = -\lim_{n\to\infty}\frac{1}{n}\log P_{opt}^n.$$

Since the other side of inequality is trivial, we have

$$-\lim_{n\to\infty}\frac{1}{n}\log P_*^n = -\lim_{n\to\infty}\frac{1}{n}\log P_{opt}^n.$$

Therefore, regardless of whether ρ is pure or not, our LOCC protocol very closely approximates the globally optimal protocol when n is large.

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Appendix

A Group representation theory

Lemma 1 *Let U_g and U'_g be an irreducible representation of G on the finite-dimensional space $\mathcal{H}$ and $\mathcal{H}'$, respectively. We further assume that U_g and U'_g are not equivalent. If a linear operator A in $\mathcal{H}\oplus\mathcal{H}'$ is invariant by the transform $A \to U_g \oplus U'_g A U_g^* \oplus U_g'^*$ for any g, $\mathcal{H}A\mathcal{H}' = 0$ [2].*

Lemma 2 *(Shur's lemma [2]) Let U_g be as defined in lemma 1. If a linear map A in $\mathcal{H}$ is invariant by the transform $A \to U_g A U_g^*$ for any g, $A = c\mathrm{Id}_{\mathcal{H}}$.*

B Representation of symmetric group and SU

Due to [2], we have

$$\dim \mathcal{U}_\lambda = \frac{\prod_{i<j}\left(l_i - l_j\right)}{\prod_{i=1}^{d-1}(d-i)!}, \tag{5}$$

$$d_\lambda = \dim \mathcal{V}_\lambda = \frac{n!}{\prod_{i=1}^{d}(\lambda_i + d - i)!}\prod_{i<j}\left(l_i - l_j\right), \tag{6}$$

with $l_i := \lambda_i + d - i$. It is easy to show

$$\log\dim \mathcal{U}_\lambda \leq d^2 \log n. \tag{7}$$

Below,

$$|\phi\rangle = \sum_{i=1}^{d}\sqrt{p_i}\,|e_i\rangle\,|e_i\rangle\,,\ \rho := \mathrm{Tr}_B|\phi\rangle\langle\phi|$$

where $\{|e_i\rangle\}_i$ is an orthonormal basis of $\mathcal{H}$. With $a_\lambda^\phi = \mathrm{Tr}\mathcal{W}_{\lambda,A}\rho^{\otimes n}$,

$$\left|\frac{\log d_\lambda}{n} - \mathrm{H}\left(\frac{\lambda}{n}\right)\right| \leq \frac{d^2+2d}{2n}\log(n+d), \tag{8}$$

$$\sum_{\frac{\lambda}{n}\in\mathrm{R}} a_\lambda^\phi \leq (n+1)^{d(d+1)/2}\exp\left\{-n\min_{\boldsymbol{q}\in\mathrm{R}}\mathrm{D}\left(\boldsymbol{q}\|\boldsymbol{p}\right)\right\}, \tag{9}$$

where R is an arbitrary closed subset [5].

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