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**Inner-iteration preconditioned block GMRES**

Algorithms for solving least squares problems.

Mainly least squares problems with many right hand sides. (Solving many problems at the same time.)

Precondition the problems to speed up CPU time.

**Table: CPU time of block BA-GMRES and IP Block BA-GMRES**

<table>
<thead>
<tr>
<th>p</th>
<th>Iter.(B=A^T)</th>
<th>CPU (s)</th>
<th>Iter.(IP)</th>
<th>CPU (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>547</td>
<td>0.4075</td>
<td>201</td>
<td>0.2535</td>
</tr>
<tr>
<td>2</td>
<td>285</td>
<td>0.6229</td>
<td>126</td>
<td>0.2025</td>
</tr>
<tr>
<td>3</td>
<td>195</td>
<td>0.3908</td>
<td>92</td>
<td>0.1707</td>
</tr>
<tr>
<td>4</td>
<td>149</td>
<td>0.3268</td>
<td>76</td>
<td>0.1751</td>
</tr>
<tr>
<td>5</td>
<td>121</td>
<td>0.2722</td>
<td>65</td>
<td>0.1728</td>
</tr>
<tr>
<td>6</td>
<td>102</td>
<td>0.2344</td>
<td>58</td>
<td>0.1728</td>
</tr>
<tr>
<td>7</td>
<td>88</td>
<td>0.2307</td>
<td>53</td>
<td>0.1842</td>
</tr>
</tbody>
</table>

**Algorithm**

**NR-SOR inner-iteration block BA-GMRES**

1. Choose $X_0 \in \mathbb{R}^{n \times p}$, $R_0 = C - AX_0$.
2. apply $t$ steps SOR to $A^T A w = A^T R_0$ to obtain $W_0 = P^t A^T R_0$, (NR-SOR),
3. $[W_1, R] = qr(W_0)$,
4. for $i = 1, 2, \ldots, k$ do
5. $U_i = AV_i$,
6. apply $t$ steps SOR to $A^T A w = A^T U_i$ to obtain $W_i = P^t A^T U_i$, (NR-SOR),
7. for $j = 1, 2, \ldots, i$ do
8. $H_{i,j} = V_j^T W_i$, $W_i = W_i - V_j H_{i,j}$,
9. end for
10. $[V_i, H_{i+1}] = qr(W_i)$,
11. Compute $Y_i \in \mathbb{R}^{n \times p}$ which minimizes $\| R_i \|_F = \| R - H_{i+1} \|_F$, $X_i = X_0 + [V_1, V_2, \ldots, V_i] Y_i$,
12. if $\| A^T R_0 \|_F < \epsilon \| A^T R_0 \|_F$ then stop
13. end if
14. end for

**Algorithm**

**Block NR-SOR**

1. Let $X_0$ be the initial solution and $R = C - AX^0$, $0 < \omega < 2$.
2. for $k = 1, 2, \ldots, l$ do
3. for $i = 1, 2, \ldots, n$ do
4. $\Delta_i^T = (\omega / \| a_i \|_2^2) R^T a_i$,
5. $X_i^{k+1} = X_i^k + \Delta_i^T$,
6. $R = R - a_i \Delta_i^T$ (rank-1 update).
7. end for
8. end for