The Power of Abstract Mathematics in Computer Science

Compositional Model Checking via Monoidal Categories

From (Abstract) Category Theory to (Concrete) Fast and Scalable Algorithms

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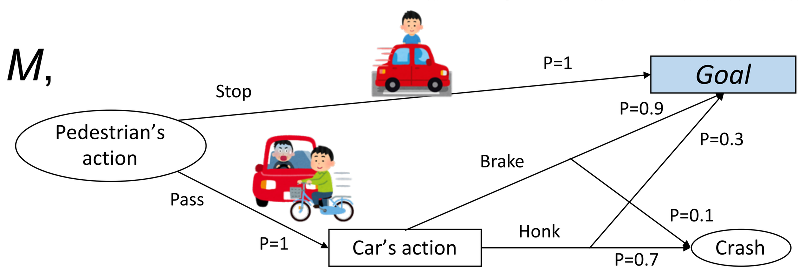
Compositional Probabilistic Model Checking with String Diagrams of MDPs, Proc. CAV 2023. See also [Watanabe+, TACAS'24 & CAV'24]

Abstract. We present the first MDP model-checking algorithm that is fully compostional and automatic. It is a rare example of abstract category theory leading to concrete algorithmic merits.

Model Checking: Graph-Based Automated Analysis

We study the following problem (**probabilistic model checking**):

- given a **Markov decision process (MDP)** M,
- compute its **optimal expected reward**.
  - "expected": an action leads to a randomized next state
  - "optimal": for the best choice of actions (*strategy*)



An important problem, tacked also in reinforcement learning:

- finding the optimal strategy ➔ **decision making**
- precise computation of exp. reward ➔ **quality guarantee**

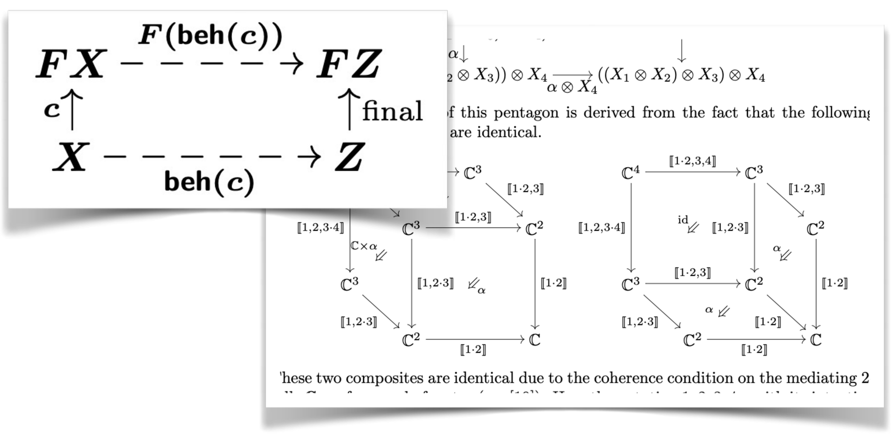
A topic heavily studied in the formal verification community (e.g. CAV).
Baier, C., Katoen, J.: Principles of model checking. MIT Press (2008)

Scalability is a challenge—sometimes an MDP has ~10^8 states (or more). "State explosion." **We want a scalable algorithm**.

Category Theory: an Abstract Language of Modern Math

Category theory is a language of modern mathematics
(Mac Lane, Eilenberg, Grothendieck, ...).

- It describes abstract structures using arrows,
- thus identifying essential similarities in different fields (e.g. algebra and geometry)



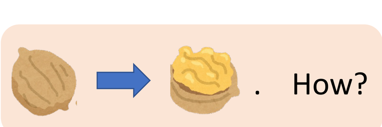
Use of category theory has been pursued in **computer science**, too
Functional programming and logic (Lawvere, Lambek, ..., cf. Haskell)
Coalgebra and process theory (Jacobs, Rutten, ...)

- "Get the setting right, then the problem solves itself"

However, the use of category theory has been mostly as a **theoretical backend**

- It helps to come up with a theory (definitions, correctness theorems)
- But **concrete algorithmic benefits have been rare**

Grothendieck's pursuit of simplicity and elegance: Nut opening example

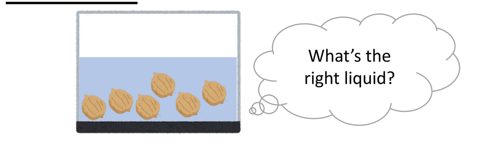


Alexander Grothendieck (1928-2014)
Leading figure of modern algebraic geometry
Categories, sheaves, ...
Fields medal, 1966

Method 1

✅ Quicker to get to a solution
❌ The solution is specific, not transportable. It does not scale

Method 2



❌ Takes time
✅ The solution, once found, is generic, widely applicable (other kind of nuts, shellfish, ...), scalable (use a bigger pool)

We pursue this with **category theory** (as Grothendieck did)

→ Once we establish the **right environment** (~ **formulation, definition, language**), a problem solves by itself

Compositionality: Algebraic "Divide-and-Conquer" for Scalability

Compositionality is a paradigm eagerly pursued in computer science.
It comes with performance advantage as well as mathematical blessing

The solution of a composition is obtained by ...

$$\mathcal{S}(\mathcal{A} \star \mathcal{B}) = \mathcal{S}(\mathcal{A}) \star \mathcal{S}(\mathcal{B})$$

first solving the components $\mathcal{A}$, $\mathcal{B}$ and then composing their solutions

Composition of *systems* (seqComp, parComp, sum, ...) — Composition of *semantics*

Conceptual Value

- "**Divide-and-Conquer**": simplifies a problem into smaller subproblems
- $\mathcal{S}(\mathcal{A})$, $\mathcal{S}(\mathcal{B})$ are **summaries** of components $\mathcal{A}$, $\mathcal{B}$. Unnecessary details get abstracted away

Performance Advantage

- Clear adv. when there are **duplicates** (reuse $\mathcal{S}(\mathcal{A})$!)

$$\mathcal{S}(\mathcal{A} \star \cdots \star \mathcal{A}) = \mathcal{S}(\mathcal{A}) \star \cdots \star \mathcal{S}(\mathcal{A})$$

- (In some cases you don't need duplicates, e.g. mergesort)

Mathematical Blessing

- Compositionality means that the solution

$$\mathcal{S}\colon \mathbb{M} \longrightarrow \mathbb{S}$$

is a **homomorphism**, preserving the operation ★

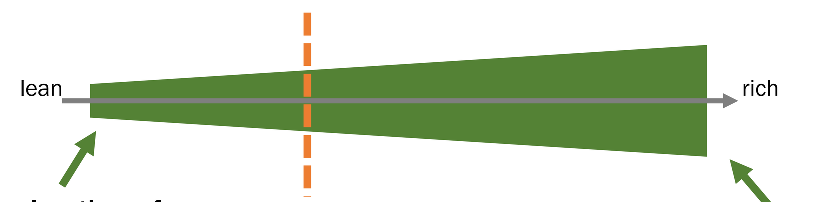
"Semantical Granularity" as a Challenge

Compositionality is not easy to achieve, though...
one has to strike the right "**granularity of semantics**"

$$\mathcal{S}(\mathcal{A} \star \mathcal{B}) = \mathcal{S}(\mathcal{A}) \star \mathcal{S}(\mathcal{B})$$

Imagine: estimating the output of a team of two

Q. What information should the summaries $\mathcal{S}(\mathcal{A})$, $\mathcal{S}(\mathcal{B})$ carry?



the **original notion of solution** (only) →
- enough when $\mathcal{A}$, $\mathcal{B}$ operate separately;
- not enough otherwise

find minimal enrichment that addresses interaction

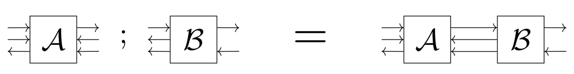
the **component itself** → no suppressed details, no performance advantage

Monoidal Categories Come to the Rescue

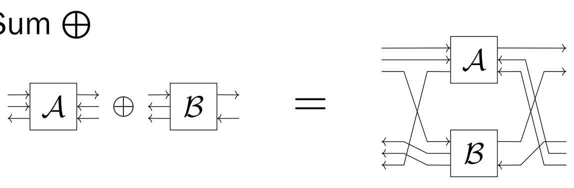
We use **string diagrams** for composing MDPs. They come from monoidal categories

String Diagram of MDPs

- Sequential composition ;



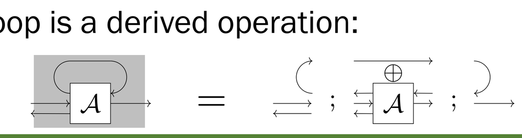
- Sum ⊕



- and some "constants" , , , ...

➔ **planar composition** of MDPs (mostly *sequential* composition; not parallel)

- Loop is a derived operation:



Background: Monoidal Categories

- Well-established field of category theory (Mac Lane, Kelly, Joyal, Street, ...)
- Used for many applications:
  quantum field theory (Khavanov, ...),
  quant. computation (Abramsky, Coecke, Vicary, Heunen, ...),
  linguistics (Sadrzadeh, Coecke, ...),
  signal flow diagrams (Bonchi, Sobocinski, Zanasi, ...), ...
- String diagrams as an *internal language* for monoidal categories [Joyal & Street, Adv. Math. 1991]
  - nicely expressive (planar composition, see left)
  - comes with a rich metatheory (see later)
- In particular, we use string diagrams for **compact closed categories** (compCCs).
  - Dual object A^* for each A
  - Example: fin.-dim. vector spaces, MDPs, ...

Use of monoidal categories comes with two notable benefits.

Benefit 1: Settling "Granularity of Semantics"

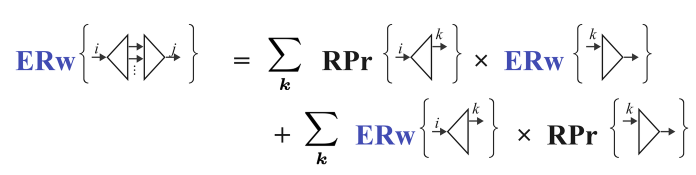
- Compositional solution is a homomorphism

$$\mathcal{S}\colon \mathbb{M} \longrightarrow \mathbb{S}$$

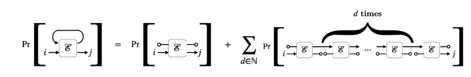
- The domain $\mathbb{M}$ is a compact closed category (compCC) of MDPs
  ➔ **We need $\mathbb{S}$ to be a compCC too!**
  - Minimally enrich the original solution domain (optimal expected reward) till it becomes a compCC
- Turns out: it suffices to additionally compute **reachability probabilities**
  - "decomposition equalities" (right)

- *Reachability probabilities* decompose into themselves:

- Similar decomposition of *expected rewards* requires more data:

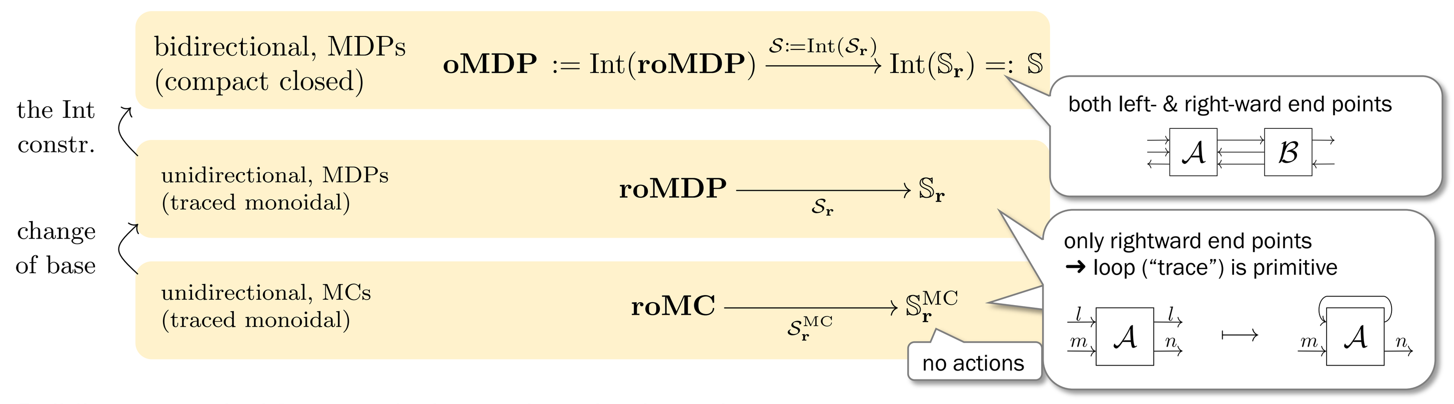


- The same for loops



- (NB: no actions so far, this is about Markov chains → next slide)

Benefit 2: Framework Upgrades for Free (uni-dir. MC ➔ uni-dir. MDP ➔ bi-dir. MDP)



- Building semantical frameworks is easy in a simple setting (*rightward-open MCs*; no action, end points are all rightward)
- Upgrades are **for free**, exploiting the categorical metatheory change of base (Eilenberg, Kelly, ...), the Int construction (Joyal, Street, Verity)

Algorithmic Outcomes

- The first MDP model checking algorithm that is fully compositional and automatic
  inspired by [Junges+, CAV'22] (not fully compositional), [Kwiatkowska+, Inf. Comp. '13] (not fully automatic)
- Can be **arbitrarily faster** than non-compositional algorithms (increase duplicates!);
  we observed max. x 6000 speed-up
- Many large systems are built compostionally
  ➔ practical potential

benchmark	$\lvert Q\rvert$	$\lvert E\rvert$	exec. time [s] DI-high	DI-mid	DI-low
Patrol1	10^8	10^8	21	42	83
Patrol2	10^8	10^8	23	48	90
Patrol3	10^9	10^9	22	43	89
Patrol4	10^9	10^9	30	60	121
Wholesale1	10^8	$2 \cdot 10^8$	130	260	394
Wholesale2	10^8	$2 \cdot 10^8$	92	179	274
Wholesale3	$2 \cdot 10^8$	$4 \cdot 10^8$	6	12	23
Wholesale4	$2 \cdot 10^8$	$4 \cdot 10^8$	129	260	393

benchmark	$\lvert Q\rvert$	$\lvert E\rvert$	exec. time [s] FZ-none	FZ-int.	FZ-all (PRISM)
Packets1	$2.5 \cdot 10^5$	$5 \cdot 10^5$	TO	1	65
Packets2	$2.5 \cdot 10^5$	$5 \cdot 10^5$	TO	3	64
Packets3	$2.5 \cdot 10^5$	$5 \cdot 10^5$	TO	1	56
Packets4	$2.5 \cdot 10^5$	$5 \cdot 10^5$	TO	3	56
Patrol5	10^8	10^8	22	22	TO
Wholesale5	$5 \cdot 10^7$	10^8	TO	14	TO

$\lvert Q\rvert$ is the number of positions; $\lvert E\rvert$ is the number of transitions (only counting action branching, not probabilistic branching); execution time is the average of five runs, in sec.; timeout (TO) is 1200 sec.

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