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On Long-Term Optimal Production-Inventory Plan for a Closed Loop Supply Chain

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Abstract

Middle and long-term inventory-production plans have deserved huge interest of managers of supply chains since the 80's. However, from 90's to now, providing plans to the reverse channel of supply chains become also an important business practice. In this report, we are interested in developing long-term plans for closed-loop supply chains that allow managers building future production scenarios with minimum costs. In this way, we consider a stochastic quadratic problem subject to linear discrete-time inventory-production systems with probabilistic constraints. The objective of this problem is to determine long-term optimal production plan that allows managers attending demand for a single product. Products can be produced by manufacturing from a forward channel and/or by remanufacturing from a backward (i.e. reverse) channel. The demand fluctuation is a random variable, with monthly mean and standard deviation previously known. In its turn, the rate of return is deterministic, but with the average monthly rate determined from a percentage of the average levels of demand taken with a delay of some period. The random fluctuation of demand makes the serviceable inventory system in the forward channel a stochastic process. As a consequence, the variance of serviceable inventory grows over the periods of the planning horizon. It is shown that such a growing can make the stochastic problem infeasible. In order to mitigate such variability, a feedback gain that relates remanufacture rate to serviceable inventory level is provided from a minimum variance problem. As a result, an optimal long-term plan is developed from an equivalent Mean Value problem whose constraints are regulated by this gain. A simple example shows that optimal plans with gain have better performance than optimal plans that do not use a gain. Based on this example, we also evaluate the concept of the green company. In this case, we assume the premise that how less we discard more we are contributing to preserving the environment. Thus, we compare our optimal long-term solution with another one where the cost of discarding is three times more expensive; in the sequence, we make our conclusions about the benefits of being a green company and how much that can cost. At the end of the document, appendixes with additional information about the modeling process and other issues related to this study are provided.

Keywords: stochastic process, minimum variance, open-loop system, optimization, supply chain, reverse logistics, green company.

1. INTRODUCTION

Reverse logistics as it has been shown by Govindan et al (2015) is an important part of the supply chain process. Due to many reasons, particularly pulled by environment sustainability aspects, reverse logistics is a priority issue for the most part of industries, see Chin et al., (2015), Dekker et al., (2004), and Silva Filho (2012). Typical activities of planning, implementing and controlling the flow of material throughout the forward channel of the supply chain, is replicated throughout the reverse channel. Generically speaking, the main objective of reverse logistics is to move products from their final destination for the purpose of capturing value, or proper disposal. Operationally, however, reverse logistics can be understood as the process of recycling or remanufacturing used products in order to reduce waste.

Several reports related to reverse logistics issues are found in the literature; see Yang (2013). Part of them uses quantitative models to represent remanufacturing and recycling activities in the reverse channel; see Fleischmann (2001). A typology of quantitative models for reverse logistics based on three classes of problems is proposed by Fleischmann et al (1997). In short, quantitative models can be applied to the following classes: (i) reverse distribution problem that considers the collection and transportation of used products and packages; (ii) inventory control problem for systems with return flows that takes into account control mechanisms for collecting used products, remanufacturing and replacing them into the marketplace and (iii) production planning problem with reuse of parts and materials that consider the planning process of reusing items, parts, and products without remanufacturing.

The second class of problems of above typology is the main interest in this report. According to Fleischmann et al (1997), such problems can be decomposed into two distinct categories of problems, that is, repair problem in which failed items are replaced by spares; and product recovery problem in which used-products are remanufactured and then replaced into the marketplace. This last category considers production-inventory problems with special forward and backward systems; for instance see Zhao et al. (2016), Lee et al. (2017). Besides, random nature of the demand fluctuation makes the forward production-inventory system a stochastic process. As a consequence, this class of problem can be modeled as stochastic optimization problems; see Silva Filho and Andres (2016). In fact, stochastic systems are very common inventory-production problems because demand fluctuation is not precisely known in long-term. According to Fleischmann et al. (1997), the traditional classification of stochastic inventory systems based on discrete or continuous-time review models can be replicated for application in product recovery problems; see Silva Filho (2012) and Dobos (2003).

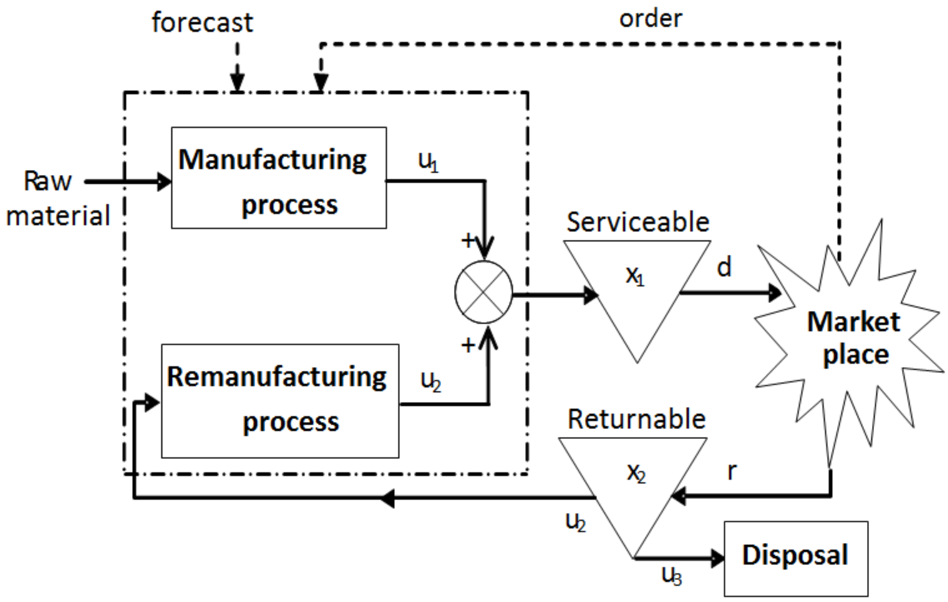

Figure 1. Forward and reverse channel of closed-loop supply chain

In this report, we revisit and formulate a chance-constraint, stochastic quadratic problem subject to linear, discrete-time inventory-production systems; see Silva Filho (2014). The objective of this problem is to develop a long-term production plan to meet a demand for a single product with a minimum cost. Forward and backward channels of a closed-loop supply chain are respectively considered to provide this optimal plan. The demand fluctuation is a random variable, having mean and standard-deviation known over periods of the planning horizon. Paralleling, the return rate of collecting used-product is assumed deterministic, but whose value depends on past fluctuating demand, that means that a product after being placed on the market can last some periods before being collected again from the marketplace. The random fluctuation of demand makes the serviceable inventory system in the forward channel a stochastic process. As a consequence, the variance of serviceable inventory grows over the periods of the planning horizon. It is shown that the growing of variance over periods can infeasible the solution of the stochastic problem. In order to mitigate such variability, a feedback gain, relating remanufacture rate to serviceable inventory level, is provided by solving a minimum variance problem. As a result, an optimal plan is developed from an equivalent Mean Value problem that has its chance-constraints regulated by this gain. The basic idea is to use long-term plans to create inventory-production scenarios that allow managers to analyze different possibilities of cost's reduction for their closed-loop supply chains. Through a simple use case, taken as an example, it is possible to show that optimal plans that use the gain have better performance than ones that do not.

We also in this report try to define the concept of a green company based on its ability to be sustainable with respect to the environment. This feature can be measured in different dimensions, such as the rate at which the company recycles its products or the rate of discarding, which are directly related to reverse logistics policy. Thus, considering our simple example we try to evaluate how costs for discarding should influence the optimal solution of the problem, giving important information for manager's decision-making.

Finally, at the end of the report, appendixes are found. They bring additional information about this study, such as: Appendix A presents a short discussion about chance-constraints and the impact of the free evolution of second statistic moments that become larger for each new period of the planning horizon. If we do not treat such a growing evolution of variance, the equivalent deterministic constraint, obtained from the transformation of stochastic problem, can fail to preserve their boundaries feasible over the periods; appendix B discusses the minimum variance problem and shows how an optimal gain can be determined and its relation with trade-off parameter; appendix C discusses the use of long-term plan in the building of scenarios that enable managers to make decision. In this appendix, we show by simulation scheme how the trade-off parameter computed in section 3 is a good approximation for optimal trade-off parameter. At last, in appendix D, an excel spreadsheet that contains all data used in this study is shown.

The remainder of this report is organized as follows: Section 2 discusses revising a discrete-time stochastic quadratic model with chance-constraints for representing a closed-loop supply chain problem; Section 3 shows how to solve this problem by means of a minimum variance problem to reduce the variability of serviceable inventory variable. An equivalent deterministic problem is used then to provide an optimal open-loop solution. Section 4 presents a simple example where open-loop optimal plans are provided and compared as a result of the solution of the problem with and without gain.

2. THE STOCHASTIC MODEL

Figure 1 illustrates the forward and reverse channels of a closed-loop supply chain. Note that there are two warehouses in this figure: the first one denotes the serviceable inventory unit, being used to storage manufactured and remanufactured products for attending the fluctuating demand; and the second denotes the returnable unit, being used to storage used-products after being collected from the marketplace. Figure 1 also shows that used-products storage into the returnable unit can be in sequence remanufactured or discarded.

It is worth emphasizing that the demand for the product must be met by the combination of manufactured and remanufactured products. Some features and properties of the system exhibited in Figure 1 are:

a. Demand “d” is a normal random variable that follows a stationary stochastic process.
b. Return rate “r” is assumed a deterministic variable whose value depends on the past fluctuation of monthly mean demand; there is a time-delay τassociated with return products from the market;
c. Both manufacturing and remanufacturing processes have an upper physical capacity of processing. Similarly, upper capacities of storage for serviceable and returnable inventories units are considered.
d. Used-products may be disposed of as soon as being collected. There are two main reasons to discard a used-product: the first one is a technical justification that occurs when the collected product is not appropriated to reuse anymore; and the second has a financial justification, in which

remanufacturing all products can significantly raise the inventory levels, and, as a consequence, increase the costs.

2.1. Inventory-Production System

The forward and backward inventory-production processes, illustrated in figure 1, can mathematically be modeled by a discrete-time stochastic system with two state variables that represent serviceable inventory levels and returnable inventory levels, and three control variables that are related to manufacturing, remanufacturing, and discard rates. These systems are described by the following two difference equations, which represent respectively inventory balance equations related to forward and reverse channel of the supply chain (Silva Filho, 2014).

$$x_1(k+1) = x_1(k)+u_1(k)+u_2(k)-d(k) \quad (1)$$

$$x_2(k+1) = x_2(k)-u_2(k)-u_3(k)+r_\tau(k) \quad (2)$$

where, for each period k, we have:

$x_1(k)$ = inventory level of serviceable unit;

$x_2(k)$ = inventory level of returnable unit;

$u_1(k)$ = production rate of manufacturing;

$u_2(k)$ = production rate of remanufacturing;

$u_3(k)$ = discard rate;

$d(k)$ = demand level (random);

$r_\tau(k)$ = return level (deterministic) .

Some comments related to the process (1)-(2): it is assumed that stationary demand is normally distributed with first $\hat{d}(k)$ and second $\sigma_d^2 > 0$ statistic moments known for each period k. To assume normality hypothesis for production-inventory processes is not usual in practice, but authors – as, for instance, Graves (1999) – justify the use in reason of its easy approximation by other discrete distributions. As a direct consequence of demand randomness, serviceable inventory level $x_1(k)$ is also a random variable. Due to the linearity of the balance equation (1), the variable $x_1(k)$ follows similar distribution to that one of demand d(k). Thus, $x_1(k)$ has a mean $\hat{x}_1(k)$ and variance $V_{x_1}(k)$ that evolves over periods. The return rate is defined as a function dependent on demand, that is, $r_\tau(k) = \eta \cdot \hat{d}(k-\tau)$; where $0 \le \eta \le 1$ denotes the percentage of used-products that return, after τ periods of delay. Note that r(k) is a deterministic variable, whose value for each period k is taken from past values of estimated demand, that is, $\hat{d}(k-\tau)$. Generically speaking, return rate is usually independent of demand, but for short life time products (e.g. refurbishing of wood pallets and toner), it is possible to consider a relative association between the amount of products sold into the marketplace and amount of products collected and submitted for remanufacturing or refurbishing.

2.2. Stochastic Nature of the Closed-loop Supply Chain

The variability of serviceable inventory variable $x_1(k)$ can be observed from the evolution of its second statistic moment over periods k $(\text{i.e.}, V_{x_1}(k))$.In open-loop operation of system (1), this variance increases over periods, that is, $V_{x_1}(k+1) \geq V_{x_1}(k), \forall k \geq 0$. Such variability is a result of the uncertainty about demand over the future periods. This implies that we can produce more and more new and remanufactured products. Since it is assumed that the price for remanufacturing is lower than the price for manufacturing new products, a good idea is to collect more used-products and remanufactured. Thus, this report tries to relate the supply of serviceable inventory storage with remanufacturing products, that is, to associate the inventory (state) variable $x_1(k)$ with remanufactured (control) variable $u_2(k)$. As a result, $u_2(k)$ becomes a random variable with mean and variance known over time. Note that system (2) should be considered a random process, which follows a similar distribution of variable $u_2(k)$. Besides, the linearity nature of systems (1) and (2) preserves the normal random characteristic for all random variables of these systems.

2.3. The Overall Functional Criterion

For each period k, the criterion C_k that represents the overall costs to run processes (1)-(2) is given as follows:

$$C_k = h_1.E\{x_1(k)^2\}+h_2.E\{x_2(k)^2\}+ c_1u_1(k)^2 +c_2E\{u_2(k)^2\}+c_3u_3(k)^2 \tag{3}$$

where h_1 and h_2 denote the holding prices for inventory maintenances of serviceable and returnable inventories units. The c_1, c_2 and c_3 are coefficients that denote prices related to manufacture, remanufacture and discard operations, respectively. The symbol E{.} denotes the expectation operator, which is here related to x_1, x_2 and u_2 that are random variables. At last, it is important to add that the total production cost is given by $Z_T = \sum_{k=1}^{T} C_k$.

2.4. Discrete-time Stochastic Problem

An optimal production-inventory sequential policy {(u_1(k), u_2(k), u_3(k)), with k = 0, 1, 2, …, T-1} can be provided from solution of a Linear-Quadratic Gaussian problem with chance constraints described as follows:

$$\begin{aligned}
\underset{u_1,u_2,u_3}{\text{Min}} \quad Z_T = h_1 E\left\{x_1(T)^2\right\} + h_2 E\left\{x_2(T)^2\right\} + \sum_{k=0}^{T-1} \Big\{ h_1 E\left\{x_1(k)^2\right\} \\
+ h_2 E\left\{x_2(k)^2\right\} + c_1 u_1(k)^2 + c_2 E\left\{u_2(k)^2\right\} + c_3 u_3(k)^2 \Big\} \\
\text{s.t.} \\
x_1(k+1) = x_1(k) + u_1(k) + u_2(k) - d(k) \\
x_2(k+1) = x_2(k) - u_2(k) - u_3(k) + r_\tau(k) \\
\text{Prob.}\left(0 \le x_1(k) \le \bar{x}_1\right) \ge 1 - \alpha \\
\text{Prob.}\left(0 \le x_2(k) \le \bar{x}_2\right) \ge 1 - \beta \\
\text{Prob.}\left(0 \le u_2(k) \le \bar{u}_2\right) \ge 1 - \theta \\
0 \le u_1(k) \le \bar{u}_1 ; u_3(k) \ge 0
\end{aligned} \tag{4}$$

where Prob.(.) denotes the probability operator related to chance-constraints. Parameters $\bar{x}_1, \bar{x}_2, \bar{u}_1$ and $\bar{u}_2$ denote upper boundaries of capacity for inventory and production units. The probabilistic index α denotes customer satisfaction related to the possibility of ready-delivery of products; index β represents the confidence degree of collected products from marketplace, and θ denotes chances of satisfying physical boundaries of remanufacturing (or refurbishing).

3. HOW TO SOLVE THE STOCHASTIC PROBLEM

Technically speaking, a closed-loop optimal solution for problem (4) with only two state variables (i.e., x_1 and x_2) can be provided by stochastic dynamic programming algorithm. However, numerical drawbacks appear as soon as the accuracy of solution is improved by reducing the level of discretization of each variable of the problem. As a result, time-consuming and storage capacity of machine processing increase enormously. Besides, if the number of state variables increases (i.e. large size problems), the complexity to solve the problem via this algorithm increases exponentially. For this reason, closed-loop optimal solutions are often discarded, and near-optimal solutions are preferred in practices (Bertesekas, 2005).

3.1. The Proposed Solution

Some characteristics, such as: linearity, convexity, and normal distribution of variables, allow simplifying the problem (4) reducing its complexity. Consequently, for instance, considering the Gaussian nature of systems (1) and (2), the evolution of decision variables of these systems under uncertainty of demand can be evaluated by their first and second statistic moments. As a result, the problem (4) can be disaggregated in two other linked problems, that is, Minimum Variance problem and Mean Value problem. These problems are discussed below.

3.2. Minimum Variance Problem

As mentioned in section 2.2, the open-loop evolution of serviceable inventory variance $V_{x_1}(k)$ increases over the time. Since it is assumed that remanufacturing rate $u_2(k)$ is proportionally associated to inventory x_2 by an adjust gain G (i.e., $u_2(k) = -G \cdot x_2(k)$), the variance $V_{u_2}(k)$ increases proportionally. An immediate consequence of this is that the chances-constraints of the problem (4) can have their respective lower and upper boundaries violated over a future period k of planning horizon.

In order to try to understand this statistic characteristic, consider the serviceable inventory chance-constrain, i.e., $\Pr ob.\left(0 \le x_1(k) \le \bar{x}_1\right) \ge 1-\alpha$; since $x_1(k) = \hat{x}_1(k) + \varepsilon_{x_1}(k)$, with $\varepsilon_{x_1}(k) \sim N(0, V_{x_1}(k))$, this probabilistic constraint can be rewritten as follows (Silva Filho and Ventura, 1999):

$$\Pr ob.\left(0 \le x_1(k) \le \bar{x}_1\right) = \frac{1}{\sqrt{2\pi V_{x_1}(k)}} \int_{-\hat{x}_1}^{\bar{x}_1 - \hat{x}_1} \exp\left(-\frac{1}{2}\frac{\varepsilon_{x_1}^2(k)}{V_{x_1}(k)}\right) \cdot \partial\varepsilon \;=\; \Phi_{x_1}\left(\frac{\bar{x}_1}{\sqrt{V_{x_1}(k)}}\right) \tag{5}$$

Note that the cumulative distribution of probability Φ_{x_1} grows over its domain. Similar characteristic is also observed to Φ_{u_2}. As a consequence, the chance-constraints associated to serviceable inventory and remanufactured variables can be violated for a given period k of the planning horizon T. In order to avoid such a drawback, we must maximize the chances of non-violation, that is:

$$\text{Max}\left\{\Pr ob.\left(0 \le x_1(k) \le \bar{x}_1\right) + \Pr ob.\left(0 \le u_2(k) \le \bar{u}_2\right)\right\} \ge 2 - (\alpha + \theta) \tag{6}$$

From (5) and with some handle approximation, we can rewrite (6) as follows:

$$\text{Min} \sum_{k=1}^{T-1} \frac{1}{\bar{x}_1^2} \left\{ V_{x_1}(k+1) + \left(\frac{\bar{x}_1^2}{\bar{u}_2^2}\right) \cdot V_{u_2}(k) \right\} \tag{7}$$

Therefore, the problem (7) means that to maximize chance-constraints problem, as given in (6), is equivalent to minimize a parametric variance problem given as follows (Clarke et al, 1975):

$$\begin{aligned} &\text{Min}\left\{V_{x_1}(k+1) + \rho \cdot V_{u_2}(k)\right\} \\ &\quad \text{s.t.} \\ &\qquad V_{x_1}(k+1) = (1-G)^2 \cdot V_{x_1}(k) + \sigma_d^2;\;\; V_{x_1}(0) = 0 \\ &\qquad V_{u_2}(k+1) = G^2 \cdot V_{x_1}(k) \end{aligned} \tag{8}$$

where ρ is the weighting parameter (i.e., the trade-off of variances) and $V_{x_1}(k) = \left\{ \sum_{i=1}^{k} (1-G)^{2*(i-1)} \right\} \cdot \sigma_d^2$,

and σ_d^2 represents the variance of demand that is finite over periods.

Note that the maximum growing of variances V_{x_1} and V_{u_2} occurs for k=T; see figure A.2 in the Appendix A. Thus, minimizing (8) during period T, a gain that relates x_1 and u_2 variables can be determined by (for more details, see equation (B.10) in Appendix B):

$$G = \frac{1}{1+\rho} \tag{9}$$

whereρ∈ (+∞, -1). Note the comparing (7) with (8), we can obtain an approximate value for the trade-off parameter ρ, which is given by $\rho^* = \frac{\bar{x}_1^2}{\bar{u}_2^2}$. Thus, we can finally determine the near optimal gain, that is,

$G^* = \frac{\bar{u}_2^2}{(\bar{x}_1^2 + \bar{u}_2^2)}$.

Appendix C exhibits some experiments related to the example, where we use the tradeoff parameter ρ to evaluate different solution to the problem (10). In that case we consider the value of ρ in the range from 10 to –0,5. For different values of ρ, 3D optimal trajectories related to inventory and productions variables are provided. The simulation can help to understand the influence of this parameter over the optimal solution of the problem. Additionally, optimal costs related to ρ are also provided. We can compare the value of ρ selected by our procedure with the one provided by simulation. The relationship between ρ and G is investigated.

3.3. Mean Value Problem

The *certainty equivalence* principle is used to simplify the problem (4); see Bertesekas (2005). In this way, mathematical expectation operator is applied to random variables, that is, $\hat{x}_1(k) = E\{x_1(k)\}$; $\hat{u}(k) = E\{u_2(k)\}$; and $\hat{d}(k) = E\{d(k)\}$. As a result, an equivalent deterministic problem, often called Mean Value problem is formulated as follows:

$$
\begin{aligned}
\underset{u_1,u_2,u_3}{\text{Min}} \quad & Z_T = h_1\hat{x}_1(T)^2 + h_2x_2(T)^2 + \sum_{k=0}^{T-1}\Big\{h_1\hat{x}_1(k)^2 + \\
& + h_2x_2(k)^2 + c_1u_1(k)^2 + c_1\hat{u}_2(k)^2 + c_3u_3(k)^2 \Big\} \\
\text{s.t.} \quad & \\
& \hat{x}_1(k+1) = \hat{x}_1(k) + u_1(k) + \hat{u}_2(k) - \hat{d}(k) \\
& \hat{x}_2(k+1) = \hat{x}_2(k) - \hat{u}_2(k) - u_3(k) + \eta\cdot\hat{d}(k-\tau) \\
& \underline{x}_\alpha(k) \le \hat{x}_1(k) \le \overline{x}_\alpha(k) \\
& \underline{x}_\beta(k) \le \hat{x}_2(k) \le \overline{x}_\beta(k) \\
& \underline{u}_\theta(k) \le \hat{u}_2(k) \le \overline{u}_\theta(k) \\
& 0 \le u_1(k) \le \overline{u}_1;\ u_3(k) \ge 0
\end{aligned} \tag{10}
$$

where $V_{x_2}(k+1) = V_{x_2}(k) + G^2 \cdot V_{x_1}(k)$, with $V_{x_2}(0)$=0, and as an immediate consequence we have that:

(a) $\underline{x}_\alpha(k) = \sqrt{\left\{\sum_{i=1}^{k}(1-G)^{2*(i-1)}\right\}}\sigma_d \cdot \Phi_{x_1}^{-1}(\alpha)$ denotes the serviceable safety stock;

(b) $\overline{x}_\alpha(k) = \overline{x}_1 - \sqrt{\left\{\sum_{i=1}^{k}(1-G)^{2*(i-1)}\right\}} \cdot \sigma_d \cdot \Phi_{x_1}^{-1}(\alpha)$ denotes upper bounds of serviceable inventory unit;

(c) $\underline{x}_\beta(k) = \sqrt{V_{x_2}(k)} \cdot \Phi_{x_2}^{-1}(\beta)$ denotes safety stock of returnable inventory unit;

(d) $\overline{x}_\beta(k) = \overline{x}_2 - \sqrt{V_{x_2}(k)} \cdot \Phi_{x_2}^{-1}(\beta)$ denotes upper bounds of storage in the returnable unit; and

(e) $\underline{u}_\theta(k) = k \cdot \sqrt{k} \cdot (1-G) \cdot G \cdot \sigma_d \cdot \Phi_{u_2}^{-1}(\theta)$

(f) $\overline{u}_\theta(k) = \overline{u}_2 - k \cdot \sqrt{k} \cdot (1-G) \cdot G \cdot \sigma_d \cdot \Phi_{u_2}^{-1}(\theta)$

3.4. Open-Loop No-updating with Adjust Term (OLN-AT)

Problem (10) is solved by any applicable method of mathematical programming. Thus, an optimal policy $\left(u_1^*(k)\ u_2^*(k)\ \ u_3^*(k), \text{with}\, k = 0,1,\cdots,T-1\right)$ is provided from an Open-Loop No-Updating with Adjust Term (OLN-AT) approach applied to this problem, see Figure 2.

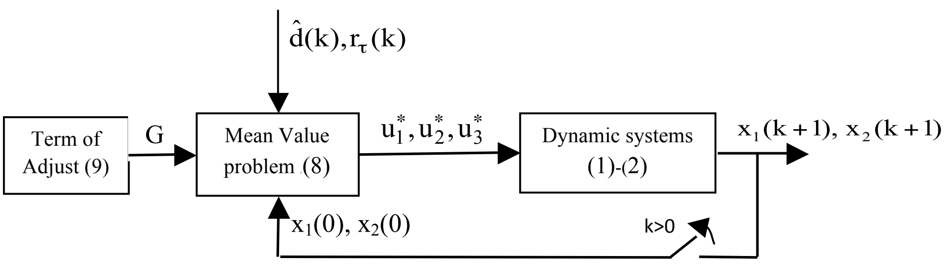

Figure.2. OLN-AT policy applied to Closed-loop Supply Chain

The optimal control policy provided by OLN-AT approach depends exclusively on the state of the system in the period k=0, that is, $\mathbf{x}_0$. Thus, even if, for other periods k>0, further information becomes available, these initially computed controls are enacted up to the end of the time horizons. It means that OLN-AT provides an optimal open-loop control sequence that is computed once for all periods of the planning horizon, taking into account only the initial state of serviceable and returnable units. In order to compensate the complete absence of information for periods k>0, an adjust term G is included in the problem (10). This adjust term influences the serviceable and returnable inventories (i.e., x_1 and x_2) providing, as a consequence, an impact over the remanufacturing rate (i.e., u_2).

4. USE CASE

Let us consider a use case, as for example, where a company manufactures a kind of product and distributes lots of it to specific customers. After two months, 70% of them are collected and storage into a returnable unit. In a second moment, they are checked and a decision is taken that involves maintaining part of products storage, remanufacturing other part or discard part of them. The company operates in a make-to-stock format, where inventories levels are created both in the forward channel and reverse channel of the supply chain, as illustrated in Figure 1. Besides, it is assumed that:

(a) The demand for this product is stationary, which means that the fluctuation of sales can be estimated with a good accuracy over time periods of the planning horizon.

(b) The cost of remanufacturing used-products, which involves disassembling, overhauling and replacement, is assumed lower than the cost of manufacturing new products. This means that used-products can be easily overhauled and replaced to a marketplace.

In the way of developing an annual production-inventory policy, based on manufacturing new products and remanufacturing used-products, a quadratic problem with chance-constraints, as formulated in (8), is considered. The data of this problem are listed in Tables 1 and 2.

Table 1: Average (mean) and standard deviation (std) of demand

Months	1	2	3	4	5	6	7	8	9	10
mean	66	58	62	54	67	51	51	54	36	72
std	$\sigma_d \approx 4.0$									

Table 2: Other data

Planning horizon: T=10 months
Time delay for returning τ = 2 months
Percentage of used-products collected from customers: η=70%
Initial inventory levels: $x_{01} = 40$ and $x_{02} = 30$
Inventory and production costs: h_1= \$1,7; h_2=\$1,5; c_1=\$1,3; c_2=\$1,2; and c_3=\$1
Customer satisfaction indexes: α = 95%; and β =50%
Capacity level: θ = 50%
Boundaries: $\underline{x}_1 = \underline{x}_2 = \underline{u}_1 = \underline{u}_2 = 0$; $\overline{x}_1 = \overline{x}_2 = 40$; $\overline{u}_1 = \overline{u}_2 = 80$

Considering that customer satisfaction level is α=0,95 and the variances tradeoff index is given by $\rho = x_1^2 / \overline{u}_2^2 = 0{,}25$ follows immediately that $\Phi_{x_1}^{-1}(0{,}95) \cong 1{,}65$ and G=0,8. As a result, lower and upper boundaries of serviceable inventory constraint (see (a) and (b)) are respectively described by:

$$\begin{cases} \underline{x}_{0,95}(k) = 13{,}2 \cdot \sqrt{\sum_{i=1}^{k} (1-G)^{2*(i-1)}} \\ \overline{x}_{0,95}(k) = \overline{x}_1 - 13{,}2 \cdot \sqrt{\sum_{i=1}^{k} (1-G)^{2*(i-1)}} \end{cases} \tag{11}$$

Note that β=θ=0,5 $\Rightarrow \Phi^{-1}(0{,}5) = 0$, then results that $\underline{x}_\beta(k) = \underline{u}_\theta(k) = 0$; $\overline{x}_\beta(k) = \overline{x}_2$; $\overline{u}_\theta(k) = \overline{u}_2$ for any period k; see Appendix C for more details.

4.1. Evaluating Two Scenarios

In the first scenario, adjust term is not considered (i.e., G=0). It means that there is no relation between variables $u_2(k)$ and $x_1(k)$. Thus, no adjust is applied to the chance-constraints of the problem (10). The second scenario considers a gain G that reduces the variances of serviceable inventory level and remanufacturing rate, and as a result reduces future uncertainties about demand, as discussed in following.

Scenario (1): OLN (without Adjust Term, i.e., G=0)

In this scenario, an optimal plan is provided by the solution of problem (10) without considering the gain G, that is, G=0. The inventory and production trajectories for forward and reverse channels are respectively exhibited in Figures 3 - 4.

Since no control is used to reduce the growth of the variance of the serviceable inventory $x_1(k)$ described by the system (1), a large safety stock is projected over periods to protect against the violation of constraint of the variable $x_1(k)$. In short, once variance of the serviceable variable $x_1(k)$ increases over period k, safety stock levels increase proportionally over the same period k. An immediate consequence of this is that the manufactured production rates and the remanufactured increase in order to fill up the appropriate levels of serviceable inventory; as can be seen comparing figures 3 and 4. Note from figure 4 that the rate of remanufactured $u_2(k)$ e is greater than the manufactured rate $u_1(k)$, what is due to the fact of assuming the cost for remanufacturing cheaper than cost for manufacturing new products.

For a more precise evaluation, we present in table 3 the monthly values of units of products held in the serviceable inventory (x_1) and units of collected products held in returnable inventory (x_2), as well as the monthly rates of unit of products manufactured (u_1), remanufactured (u_2) and discarded (u_3),respectively. The table also displays the system's monthly operating costs and total costs for each system unit. For example, in period 5 (k = 5), the monthly cost of the operation was $ 2,210.40 and the total cost of manufacturing was $ 6,939.40.

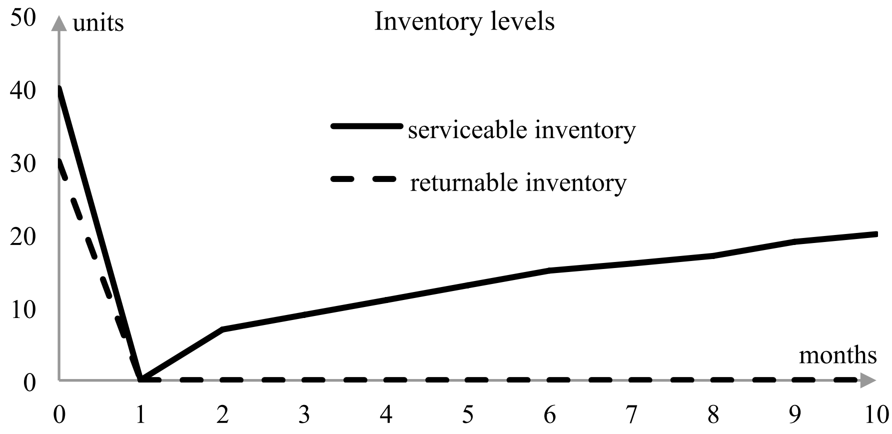

Figure 3. Optimal inventory levels (G=0)

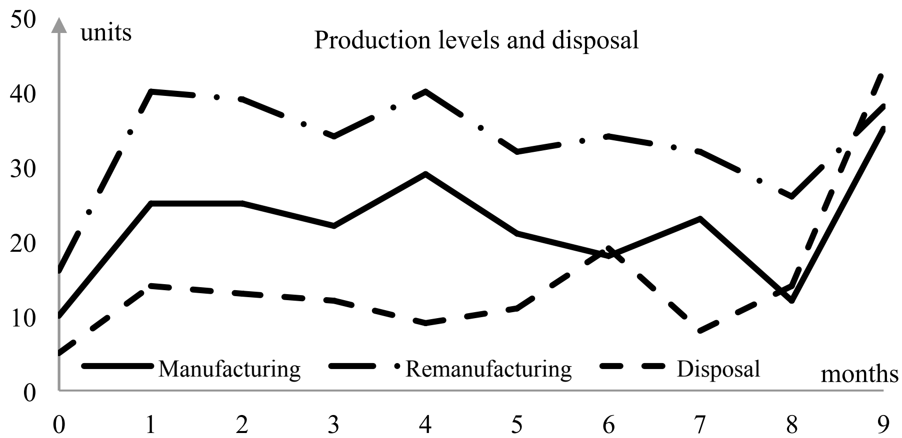

Figure. 4. Optimal production and disposal rates (G=0)

Table 3 – Optimal trajectories of inventory, production and discard variables and associated costs for G=0

Months	x_1	x_2	u_1	u_2	u_3	Cost per month
0	40	30	10	16	5	**4.532,20**
1	0	0	25	40	14	**2.928,50**
2	7	0	25	39	13	**2.890**
3	9	0	22	34	12	**2.298,10**
4	11	0	29	40	9	**3.300**
5	13	0	21	32	11	**2.210,40**
6	15	0	18	34	19	**2.551,90**
7	16	0	23	32	8	**2.415,70**
8	17	0	12	26	14	**1.685,70**
9	19	0	35	38	43	**5.788**
10	20	0	0	0	0	**680**
Cost per unit	**6.036,70**	**1.350**	**6.939,40**	**13.748,40**	**3.206**	**31.280,50**

Scenario (2): OLN-AT (with G=0,8)

Figures 5 and 6 illustrate an optimal open-loop inventory and production trajectories for forward and reverse channels, where the problem (10) is solved using now the adjust term given by gain G=0,8.

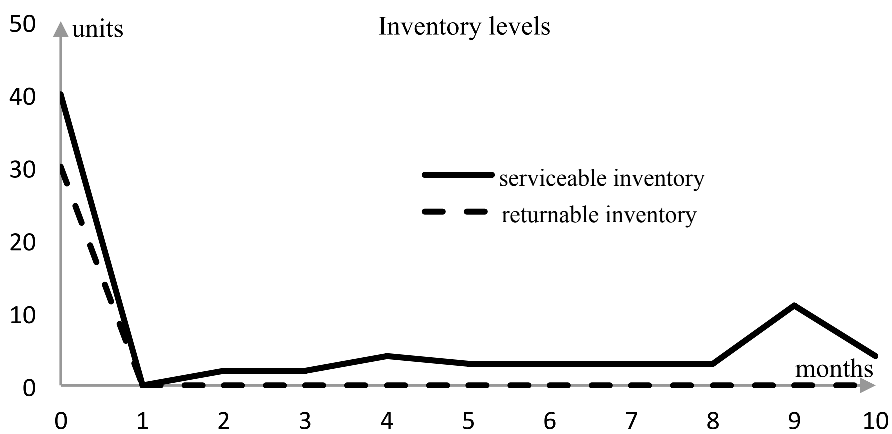

Figure 5. Optimal inventory levels (G=0,8)

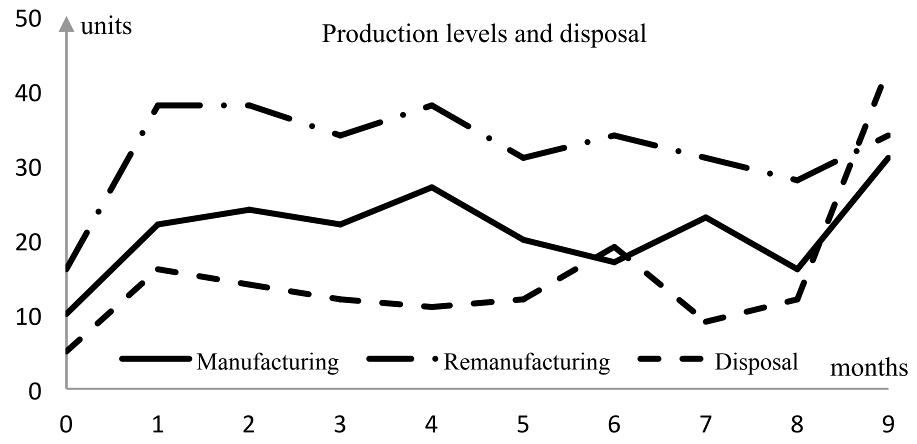

Figure 6. Optimal production and disposal rates (G=0,8)

In this scenario, we point out the influence of the gain G in the construction of safety stock of the serviceable inventory. In fact, comparing Figure 3 of the previous scenario to Figure 5 of the current scenario, it is possible to realize how safety stock of serviceable inventory is reduced by a gain G=0,8. There is an immediate consequence that is related to manufacture and remanufacture rates as can be observed from Figure 6 when compared to Figure 4. The conclusion is that control the evolution of the variances of serviceable inventory levels $x_1(k)$ in parallel with the control of remanufacturing rates $u_2(k)$ bring as main benefit the reduction of uncertainties about the demand fluctuation (i.e. unexpected orders reduction). This certainly helps to reduce the costs of the company's total production, see Table 3.

Table 4, given in sequence, exhibits the monthly values of units of products held in the serviceable inventory (x_1) and units of collected products held in returnable inventory (x_2), as well as the monthly rates of unit of products manufactured (u_1), remanufactured (u_2) and discarded (u_3),respectively.

Table 4 – Optimal trajectories of inventory, production and discard variables
and associated costs for G=0,8

Months	x1	x2	u1	u2	u3	Cost ($) per month
0	40	30	10	16	5	4.532,20
1	0	0	22	38	16	2.618,00
2	2	0	24	38	14	2.684,40
3	2	0	22	34	12	2.167,20
4	4	0	27	38	11	2.828,70
5	3	0	20	31	12	1.832,50
6	3	0	17	34	19	2.139,20
7	3	0	23	31	9	1.937,20
8	3	0	16	28	12	1.432,90
9	11	0	31	34	43	4.691,20
10	4	0	0	0	0	27,20
Cost ($) per unit	**3.054,90**	**1.350,00**	**6.250,40**	**12.914,40**	**3.321,00**	**26.890,70**

4.2. Comparing Scenarios Costs

Tables below show the result of the two evaluated scenarios in terms of costs for scenarios without gain G=0 and with gain G=0,8.

Initially, tables 5.a, 5.b, and 5.c compare the monthly costs associated with each operation of inventory and production. Table 5.c exhibit the difference between the costs of table 5.a and 5.b. As a result, positive values found in table 5.c shows how scenario with G=0 is more expensive than the scenario with G=0,8. Negative values show the inverse. Marked cells in table 5.c illustrated where scenario with G=0 is better than the scenario with G=0,8.

Table 5.a – Costs ($) with G=0

month	x1 ($)	x2 ($)	u1 ($)	u2 ($)	u3 ($)
0	2720	1350	130	307,2	25
1	0	0	812,5	1920	196
2	83,3	0	812,5	1.825,2	169
3	137,7	0	629,2	1.387,2	144
4	205,7	0	1.093,3	1.920	81
5	287,3	0	573,3	1.228,8	121
6	382,5	0	421,2	1.387,2	361
7	435,2	0	687,7	1.228,8	64
8	491,3	0	187,2	811,2	196
9	613,7	0	1.592,5	1.732,8	1.849
10	680	0	0	0	0
total	6.036,7	1.350	6.939,4	13.748,4	3.206

Table 5.b – Costs with G=0,8

month	x1 ($)	x2 ($)	u1 ($)	u2 ($)	u3 ($)
0	2.720	1350	130	307,2	25
1	0	0	629,2	1.732,8	256
2	6,8	0	748,8	1.732,8	196
3	6,8	0	629,2	1.387,2	144
4	27,2	0	947,7	1.732,8	121
5	15,3	0	520	1.153,2	144
6	15,3	0	375,7	1.387,2	361
7	15,3	0	687,7	1.153,2	81
8	15,3	0	332,8	940,8	144
9	205,7	0	1.249,3	1.387,2	1.849
10	27,2	0	0	0	0
total	30.54,9	1.350	6.250,4	1.2914,4	3.321

Table 5.c – Difference of the costs between scenario with G=0 and G=0,8

month	x1	x2	u1	u2	u3
0	0	0	0	0	0
1	0	0	183,3	187,2	-60
2	76,5	0	63,7	92,4	-27
3	130,9	0	0	0	0
4	178,5	0	145,6	187,2	-40
5	272	0	53,3	75,6	-23
6	367,2	0	45,5	0	0
7	419,9	0	0	75,6	-17
8	476	0	-145,6	-129,6	52
9	408	0	343,2	345,6	0
10	652,8	0	0	0	0
total	2981,8	0	689	834	-115

From the analysis of the three tables, we can conclude that the scenario with G=0,8 is more economic for the company than scenario with G=0. It is worth observing from the table 5.c that related with production (i.e., manufacturing and remanufacturing costs), the scenario with G=0 only had better cost performance in period k=8, for other periods scenarios G=0,8 was equal or superior in cost reduction. Besides, it is interesting to note that scenario without gain (i.e. G=0) provides less disposed products than the scenario with optimal gain. One possible reason for that is that the cost for discarding is lower than the cost for remanufacturing and thus all excess of collecting product after checking is sending to disposal instead remanufacturing.

Table 6 shows the monthly costs associated with the operation of systems (1) and (2). Note that scenario with G=0 provides always more cost than scenario with G=0,8. Positive values exhibited in the third column of Table 6 show how the scenario with G=0 increases the monthly cost of the system (1)-(2).

Table 6 – Monthly cost of operation of systems (1)-(2)

Months	**Monthly Cost with scenario G=0**	**Monthly Cost with scenario G=0,8**	**Difference between scenarios G=0 and G=0,8**
0	4.532,20	4.532,20	0,00
1	2.928,50	2.618,00	310,50
2	2.890	2.684,40	205,60
3	2.298,10	2.167,20	130,90
4	3.300	2.828,70	471,30
5	2.210,40	1.832,50	377,90
6	2.551,90	2.139,20	412,70
7	2.415,70	1.937,20	478,50
8	1.685,70	1.432,90	252,80
9	5.788	4.691,20	1.096,80
10	680	27,20	652,80
Total costs	**31.280,50**	**26.890,70**	4.389,80

From Table 7is possible to observe that OLU-AT allows the best use of the collected used-products.

Table 7. Total costs associated with two scenarios

Costs	**OLU scenario with G=0**	**OLU-AT scenario with G=0.8**	**Difference between scenarios**
Serviceable Inventory cost	6.036,70	3.054,90	2.981,80
Returnable inventory cost	1.350,00	1.350,00	0,00
Manufactured cost	6.939,40	6.250,40	689,00
Remanufactured cost	13.748,40	12.914,40	834,00
Disposal cost	3.206,00	3.321,00	-115,00
Total cost	**31.280,50**	**26.890,70**	4.389,80

Note from Table 7 that the cost of the serviceable unit is significantly reduced when we compare the application of OLU without gain G. Other costs, shown in this Table, are upper and lower but they do not affect significantly the result. The main conclusion is that the gain G reduces the uncertainties about the demand over serviceable inventory levels, which allows reducing costs and grows profit. Particularly, the total cost of scenarios 2 is 14% lower than the total cost of scenarios 1.

At last, note that is we observing before scenario 1 with G=0 provides an important reduction in a number of discarded products. As discussed before the reason for that is that the cost of discard is cheaper than the cost for remanufacturing. Consequently, once in scenario 1 the level of serviceable inventory is saved in reason of the uncertainty of demand, it is necessary to remanufacture more used-products and thus disposal fewer used-products.

4.3. The Green Company

Many reasons may justify the need for a company to be called green. Two very significant examples define how the company respects its customers and/or how it preserves the environment. Both situations have an economic impact on their supply chains. In our example, we consider the company wants to ensure economic sustainability through the remanufacturing of used products. In this case, the more it remanufactured and the less it discards is a good policy.

Analyzing the optimal solution with G gain, we observed that the company, though remanufacturing significantly, have a high rate of discarded products. Considering that this as an indicator of discarding, it contradicts the idea of a green company. The reason of this is that the cost of discarding(i.e., c_3 = 1)is lower than other costs incurred in the problem (see data in Table 2). Thus, the optimal solution of the problem always will try to use the disposal process in order to take advantage of this cost.

For the purpose of analysis, we will consider that for optimal solution generated with gain G = 0,8, we increase the cost of discarding from $ 1 to $ 3. In this case, figure 7 shows a reduction in the quantity of discarded products.

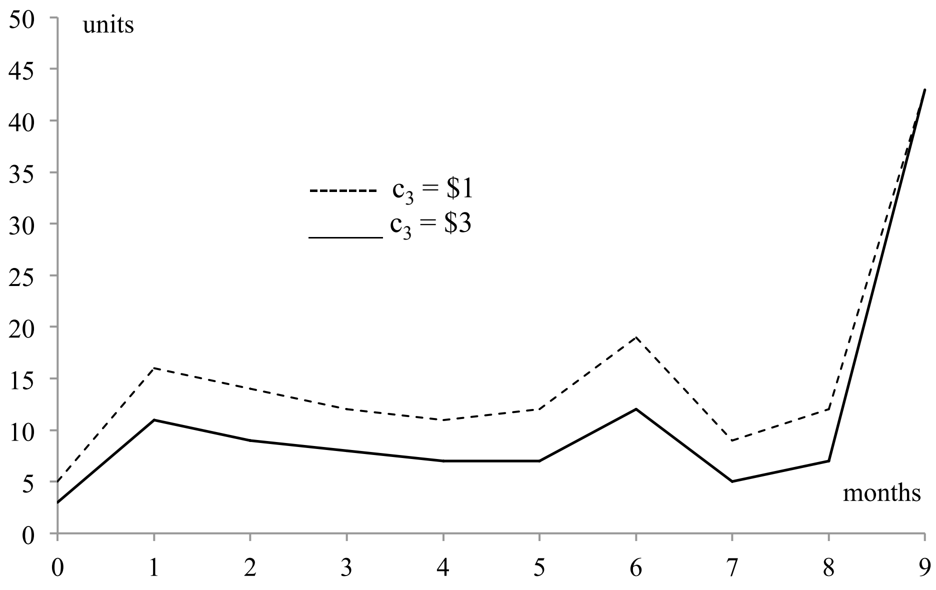

Figure 7. Comparing rates of discarding for different costs with G=0,8

The question that arises is how the optimal solution of the problem with gain G = 0,8 is affected by the increase in the cost of disposal. Table 8exhibits the optimal trajectories of inventory, productions and discard for the scenario with c_3=$3 and G=0,8.

Table 8 – Optimal trajectories of inventory, production and discard variables and associated costs with c_3 = \$3 and G=0,8

Months	x_1	x_2	u_1	u_2	u_3	Cost per month
0	40	30	8	18	3	\$ 4.569,00
1	0	0	16	43	11	\$ 2.914,60
2	1	0	20	43	9	\$ 2.983,50
3	2	0	18	38	8	\$ 2.352,80
4	4	0	23	42	7	\$ 2.978,70
5	3	0	15	36	7	\$ 2.010,00
6	3	0	10	41	12	\$ 2.594,50
7	3	0	19	35	5	\$ 2.029,60
8	3	0	13	33	7	\$ 1.688,80
9	13	0	30	33	43	\$ 8.311,10
10	4	0	0	0	0	\$ 27,20
Cost per unit	\$ 3.131,40	\$ 1.350,00	\$ 4.326,40	\$ 16.332,00	\$ 7.320,00	\$ 32.459,80

We can now compare Table 8 with Table 6. The difference between them gives important insights to the manager. Table 9 shows the result obtained from this difference. Note that negative values in cells of the table imply in the reduction of inventory levels, production rates or amount of used-products to be discarded, and also indicate costs reduction. In opposition, positive cells mean growing in these values.

Table 9 – The difference between values of Table 8 (c_3=\$3) and Table 6 (c_3=\$1) both with G=0,8.

Months	x_1	x_2	u_1	u_2	u_3	Costs/month
0	0	0	-2	2	-2	36,8
1	0	0	-6	5	-5	296,6
2	-1	0	-4	5	-5	299,1
3	0	0	-4	4	-4	185,6
4	0	0	-4	4	-4	150
5	0	0	-5	5	-5	177,5
6	0	0	-7	7	-7	455,3
7	0	0	-4	4	-4	92,4
8	0	0	-3	5	-5	255,9
9	2	0	-1	-1	0	3619,9
10	0	0	0	0	0	0
Total costs	76,5	0	-1924	3417,6	3999	5.569,1

For instance, a fast glance at Table 9 shows that there was a monthly reduction (negative value) of the quantity of used products disposed of for disposal (u_3), as already observed in Figure 7. There was an increase (positive value) in the amount of remanufactured products (u_2) with a reduction of new manufactured products (u_1). In terms of inventory movement (x_1 and x_2), little was observed. It is observed from this quick analysis that there was an increase in the company's costs of \$ 5,569.10. As a conclusion we can say that make a company green has a tangible price, that is, it can be computed.

Final comment: in terms of return to business we can conclude that the increase in the cost of disposal does not bring any economic advantage. On the contrary, it increases the total costs of the company. An important conclusion here is to strike a balance between remanufacturing and discarding that brings economic benefits and at the same time guarantees a green vision for the company and its supply chain.

5. CONCLUSION

A reverse logistics problem related to a closed-loop supply chain was discussed in this report. Such a supply chain is described by a forward production, inventory and distribution channel through which a single product flows toward the market and by a backward (i.e., reverse channel) through which used product returns after some period to be storage in a returnable inventory unit. These used products after an inspection can be remanufactured or disposed of. The overall system was modeled by discrete-time state equations and used as the basis for a production-inventory planning study. We defined a stochastic optimization problem whose objective is to minimize quadratic costs of inventory and production in the two channels of the supply chain taking into account chance-constraints in the inventory and production units. In order to facilitate the solution of the stochastic problem, an equivalent, but deterministic model was considered; as can be seen in (10). However, the solution of this problem is affected by the evolution of the variance of the serviceable inventory variable, which reduces the feasible boundaries of the constraints of the problem. As a result, infeasible solutions can be provided by our model. To overcome such a drawback effect, a minimum variance problem was considered. An optimal gain G was determined by relating the serviceable inventory variance of the direct channel to the variance of remanufactured. The objective of this gain was not only to smooth the growth of the variance values but also to connect the usable stock level with the remanufacturing rate. The solution of the problem using this gain G is shown to be effective in reducing costs and improving the performance of the overall closed loop system.

The efficient performance observed from the solution of the problem with the optimal gain G, led us to investigate a new scenario that takes into account a very common situation nowadays. This situation is based on the premise that how less we discard, more we take advantage of what has already been used and, as a consequence, we are contributing to the preservation of the environment. We can say that this is one of the premises among several that can characterize a green company. But a question arises, that is, what is the cost of this? Analyzing our initial optimal solution, given in Table 4, we verified that the cost for discarding is lower than the others costs. Thus, in order to investigate the influence of the cost, we create a new scenario where we raise the disposal cost three times. The solution presented and explained in the text through Tables 8 and 9, tell us that increasing the cost of disposing of products, in fact, decreases the number of products discarded. However, this increases simultaneously the total cost of the closed loop supply chain, because it also increases the quantities to be remanufactured. The main conclusion is that for a company to get a green certificate, it will have to raise its costs proportionally to the desired green grade.

Future studies should be proposed to extend this model to deal with multiple products and also to develop suboptimal approaches that are more efficient than open-loop approaches in their application to this type of problem. We also can evaluate of gain G in a more realistic solution of the problem (10) using an open-loop feedback control approach that uses a rolling horizon scheme for update the solution as soon as new information is captured from the marketplace. Figure 8 illustrate such an approach to be investigated.

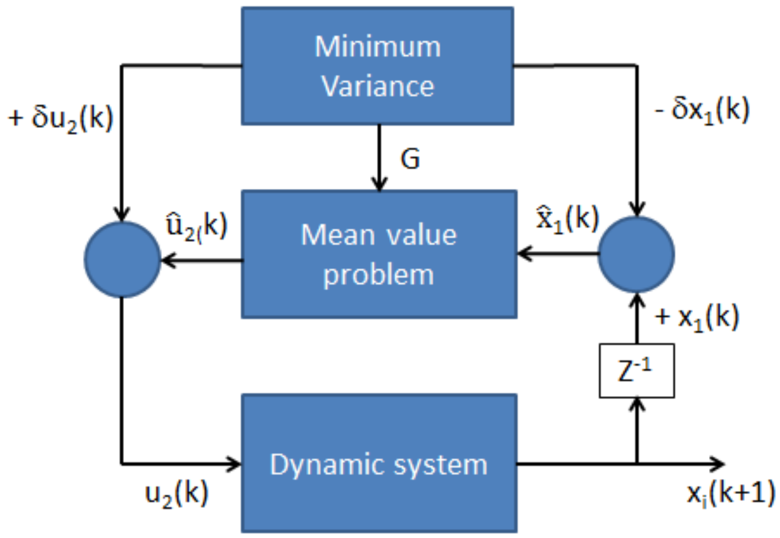

Figure 8. An open loop feedback approach applied to our model with gain G

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Appendix A – The probabilistic constraints

Consider the expression given in (5) and reproduced bellow:

$$\Pr ob.\left(0 \le x_1(k) \le \bar{x}_1\right) = \frac{1}{\sqrt{2\pi V_{x_1}(k)}} \int_{-\hat{x}_1}^{\bar{x}_1 - \hat{x}_1} \exp\left(-\frac{1}{2}\frac{\varepsilon_{x_1}^2(k)}{V_{x_1}(k)}\right) \cdot \partial\varepsilon_{x_1} = \Phi_{x_1}\left(\frac{\bar{x}_1}{\sqrt{V_{x_1}(k)}}\right) \qquad (A.1)$$

where $\sqrt{V_{x_1}}$ is the standard deviation; $\bar{x}_1$ is the upper bound; and Φ_{x_1} is the distribution function.

Let us interpret its meaning. First of all, it is important to note that we are assuming x1(k) is a normal variable. As a direct consequence of Chebycheff's inequality (see Papoulis and Pillai, 2002), it is possible to show that the maximum value of probability density function for these variables will be occur, exactly, when the mean values of these inventory variables (i.e., $\hat{x}_1(k)$) reach the centre of their constraints, that is, $\hat{x}_1(k) \approx \bar{x}_1/2$, where $\hat{x}_1$ is the mean value. Figure A.1exhibits this feature; see dark area:

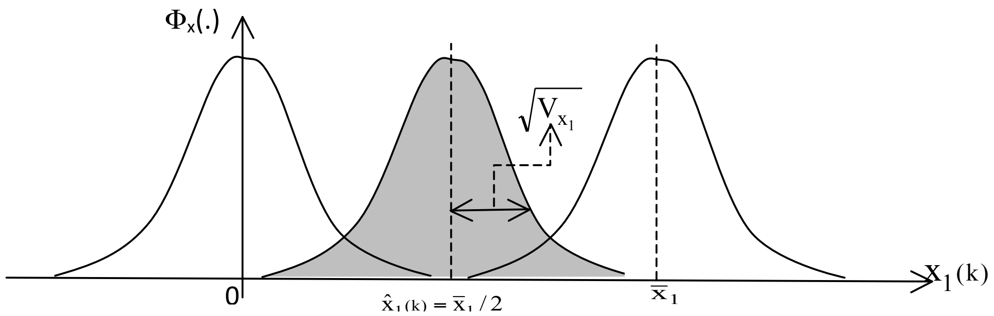

Figure A.1. Distribution of probability of a Gaussian random variable $x_1(k)$ under constraints

Thus, it is possible to show that (see Jazwinski (1970)):

$$\Pr ob(\hat{x}_1(k) - \bar{x}_1/2 \leq x_1(k) \leq \hat{x}_1(k) + \bar{x}_1/2) \quad \geq \frac{V_{x_1}(k)}{\bar{x}_1^2} \qquad (A.2)$$

The statistic construction of our report starts from the Chebycheff's inequality (A.2), but it is worth observing that there is course the evolution of the system over time, which follows a stochastic process. This evolution can be exhibited in a simplified way as follows:

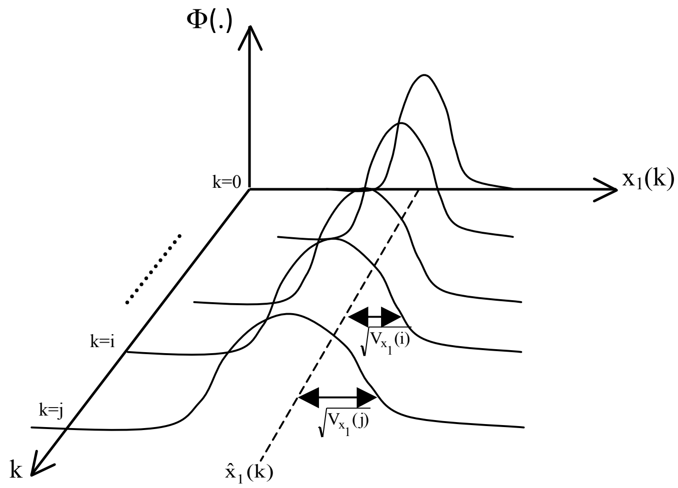

Figure A.2. Time evolution of distribution of probability of a Gaussian random variable $x_1(k)$

The problem here is the evolution of variable $x_1(k)$ over periods k. Note from the figure above that in measure of period k increases (e.g., j>i), the variance also increases ($V_{x_1}(j) > V_{x_1}(i)$). This characteristic causes instability to system (1)-(2) and a main consequence is the possibility of violation of lower and upper boundaries of $x_1(k)$, see figure A.3. Note that it is reason why we try to use probabilistic constraints. In fact, we try to reduce the chance of infeasibility. But, would it be possible to stabilize the growing of variance? A possibility is to use a minimum variance gain to reduce the growing of state and control variances. We discuss such an issue in the next appendix.

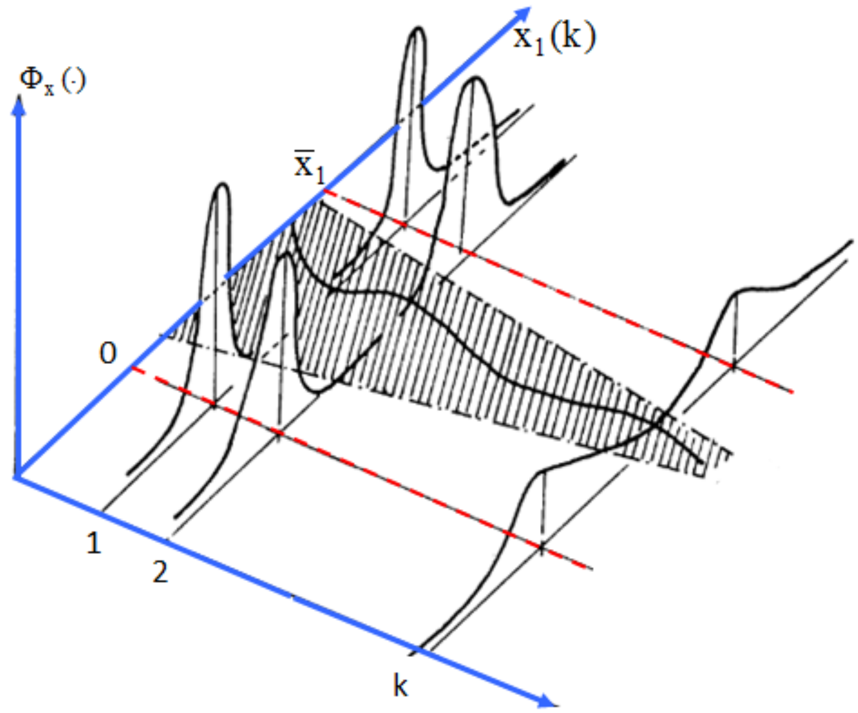

Figure A.3.

Appendix B – Minimum variance controller

In the report, we formulated a minimum variance problem from handling (5) and (6) expressions. But formally we can reach the same result as follows:

Let us consider the sub-system (1) that denotes the forward channel. We have assumed the $u_1(k)$ is a deterministic variable that can be used to guaranty a complement to serviceable unit (i.e., x_1). So, we can write the equation (1) considering only the stochastic component:

$\delta x_1(k+1) = \delta x_1(k)+\delta u_2(k)-\delta d(k)$ (B.1)

where $\delta x_1(k) = x_1(k)-\hat{x}_1(k)$; $\delta u_1(k) = u_1(k)-\hat{u}_1(k)=0$; $\delta u_2(k) = u_2(k)-\hat{u}_2(k)$; and $\delta d(k) = d(k)-\hat{d}(k)$.Note that the variables with a *hat mark* represent variables that assume average values. In this study, the variable u_1 is assumed deterministic (i.e. $u_1(k)=\hat{u}_1(k)$).

A typical problem of minimum variance is written as follows:

$$\underset{\delta u_2}{\text{Min}}\ J = E\left\{\delta x_1^2(k+1) + \rho \cdot \delta u_2^2(k)\right\}$$
$$\text{s.t.} \qquad \text{(B.2)}$$
$$\delta x_1(k+1) = \delta x_1(k) + \delta u_2(k) - \delta d(k)$$

Note that $\rho \in (+\infty, -1)$ denotes the trade-off parameter that allows the equilibrium balance between variance of state $x_1(k)$ and control $u_2(k)$. The next figure shows the behavior of ρ when variances of $\delta x_1(k)$ and $\delta u_2(k)$ evolves over the periods k.

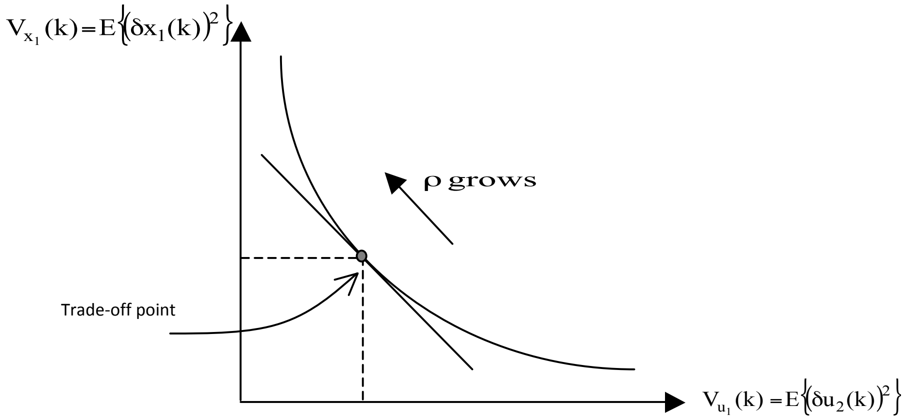

Figure B.1. $V_{x_1}(k)$ versus $V_{u_2}(k)$ parameterized by $\rho \in (+\infty, -1)$

In order to solve (B.2), we need to compute the prediction error. We start writing (B.1) as an input-output polynomial equation as follows, (see Clarke et al, 1975):

$\delta x_1(k+1) = B/A\delta u_2(k) - C/A\ \delta d(k)$ (B.3)

where A, B, and C are known polynomials with z^{-1} operator, that is: $A = 1-z^{-1}$; $B = +1$; and $C = +1$.

So now we can write the *one-step ahead* predictor $\delta\tilde{x}_1(\cdot)$ as follows:

$C\ \delta\tilde{x}_1(k+1) = F\ \delta x_1(k) + BE\ \delta u_2(k)$ (B.4)

From the identity $C=EA+z^{-1}F$, it is possible to verify that E=1 and F=1. As a result (B.4) can be rewritten as follows:

$$\delta\tilde{x}_1(k+1) = \delta x_1(k) + \delta u_2(k) \qquad \text{(B.5)}$$

Thus, the prediction error ε(k+1) can be calculated from the difference between (B.1) and (B.5), that is:

$$\varepsilon(k+1) = \delta x_1(k+1) - \delta\tilde{x}_1(k+1) = -\delta d(k) \qquad \text{(B.6)}$$

Let us include (B.6) into (B.2) to obtain:

$$J = E\left\{\left(\varepsilon(k+1) - \delta\tilde{x}_1(k+1)\right)^2 + \rho \cdot \delta u_2^2(k)\right\} \qquad \text{(B.7)}$$

Considering that ε(k+1) is not correlated to $\delta\tilde{x}_1(k+1)$, results then:

$$J = \delta\tilde{x}_1^2(k+1) + \rho \cdot \delta u_2^2(k) + E\left\{\varepsilon^2(k+1)\right\} \qquad \text{(B.8)}$$

Since ε(k+1) = - δd(k), we have $E\left\{\varepsilon^2(k+1)\right\} = \sigma_d^2$ and (B.7) is rewritten as follows:

$$J = \delta\tilde{x}_1^2(k+1) + \rho \cdot \delta u_2^2(k) + \sigma_d^2 =$$
$$= (\delta x_1(k) + \delta u_2(k))^2 + \rho \cdot \delta u_2^2(k) + \sigma_d^2 \qquad \text{(B.9)}$$

Taking the derivative of J with regard to δu_2, we obtain that

$$\delta u(k) = -1/(1+\rho)\cdot\delta x(k) = -G\cdot\delta x(k) \qquad \text{(B.10)}$$

Note that (B.10) is the control law that minimizes (B.2), being G=1/(1+ρ) the optimal gain.

Appendix C – Some scenarios with parameter ρ

I. Preliminaries

In this appendix we study the identification of parameter ρ based on scenarios (i.e. solving the problem for different values of ρ. Note that 3D trajectories related to inventory and productions variables are exhibited. The relationship between ρ and G are shows graphically and a table with cost for each value of ρ are also presented.

We consider the example given in section 4. It is important to understand that the optimization problem given in (10) is essentially deterministic and only the transformed chance-constraints are affected by the gain G (see in section 3, the constraints of problem (10)). The value of gain G depends on the weighting factor ρ. In addition, depending on the value of gain G, the upper and lower bounds tend to decrease and increase over period k, respectively. As a consequence, the feasible area created by this boundary shrinks, what can become unfeasible the problem (10). Next, we see an example of this based in section 4 of the report.

Let's consider, for instance, that lower and upper physical boundaries of variable $x_1(k)$ are respectively $\underline{x}_{min} = 20$ and $\overline{x}_{max} = 60$. As a result, the dynamic constraint $\underline{x}_\alpha(k) \le \hat{x}_1(k) \le \overline{x}_\alpha(k)$, as given in formulation of problem (10), can be calculated. For this example, we assume here that α=0.95 implying that $\Phi^{-1}_{x_1}(0.95) \cong 1.65$; and $\rho = 0.25 \Rightarrow G = 0.8$ exactly the same we have considered in section 4 of the report. As a result, $\underline{x}_{0.95}(k)$ and $\overline{x}_{0.95}(k)$ can be precisely calculated from the equations given below:

$$\begin{cases} \underline{x}_{0,95}(k) = 20 + 13{,}2 \cdot \sqrt{\sum_{i=1}^{k} (1-G)^{2*(i-1)}} \\ \overline{x}_{0,95}(k) = 40 - 13{,}2 \cdot \sqrt{\sum_{i=1}^{k} (1-G)^{2*(i-1)}} \end{cases}$$

Considering then a horizon of T=10 periods (i.e. k=0, 1, 2, …, 10), we have a graphic visual for the time evolution of these boundaries. Note that upper bound $\overline{x}_{0,95}(k)$ decreases over the periods, while simultaneously lower bound $\underline{x}_{0,95}(k)$ increases over the same periods. If happens that $\overline{x}_{0,95}(k) \le \underline{x}_{0,95}(k)$, problem (10) becomes infeasible.

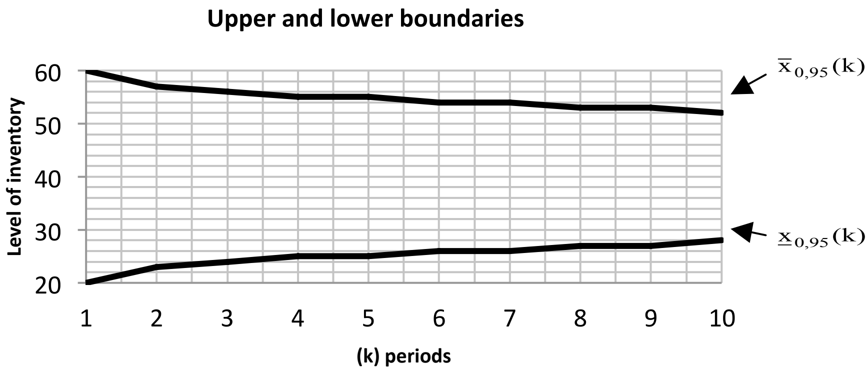

Figure C.1. Feasible area provided by the constraint $\underline{x}_{\alpha}(k) \le \hat{x}_1(k) \le \overline{x}_{\alpha}(k)$ shrinks over period k

<u>II. Creating scenarios to determine optimal ρ</u>

The objective is to find the optimal value of ρ∈(+∞, 1) by mean of scenarios. In fact, we can solve *several times* the problem (10), given in the report, for different values of ρ. Since the value of ρ=0,25 is supposed a good approximation of the optimal value of ρ, we can decide to use the range ρ∈ [10, -0,5] to search the best cost of the criterion of problem (10).
In following we show the results obtained in this study:

1º) Optimal trajectories of serviceable inventory, production and remanufacturing variables with ρ assuming the values: 10, 5, 1, 0.25, 0, -0.25, -0.5

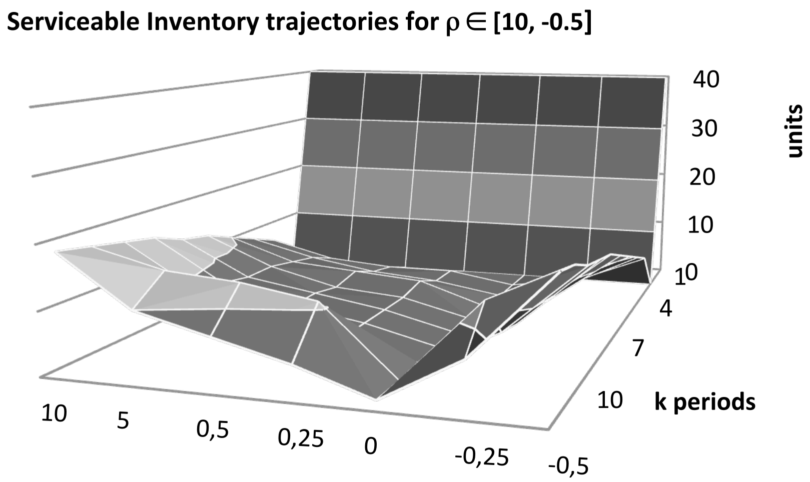

Figure C.2. Serviceable Inventory trajectories for ρ∈[10, -0.5]

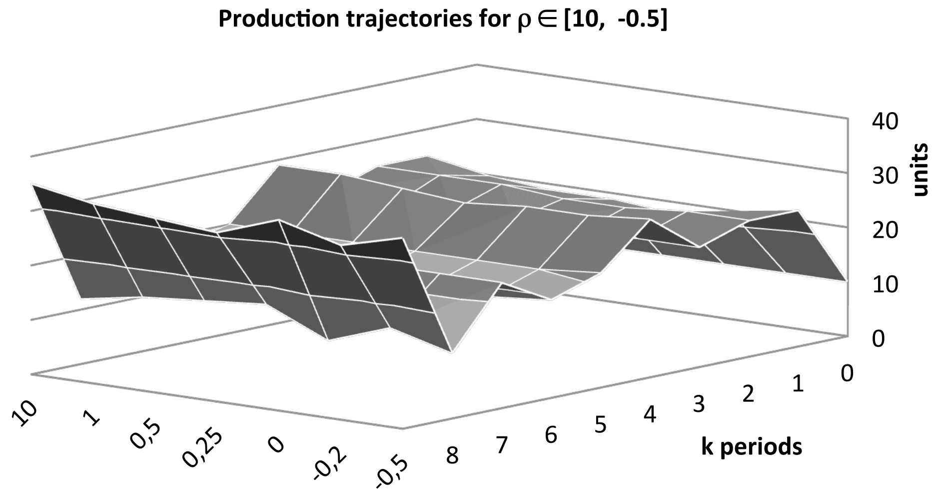

Figure C.3. Production trajectories for ρ∈[10, -0.5]

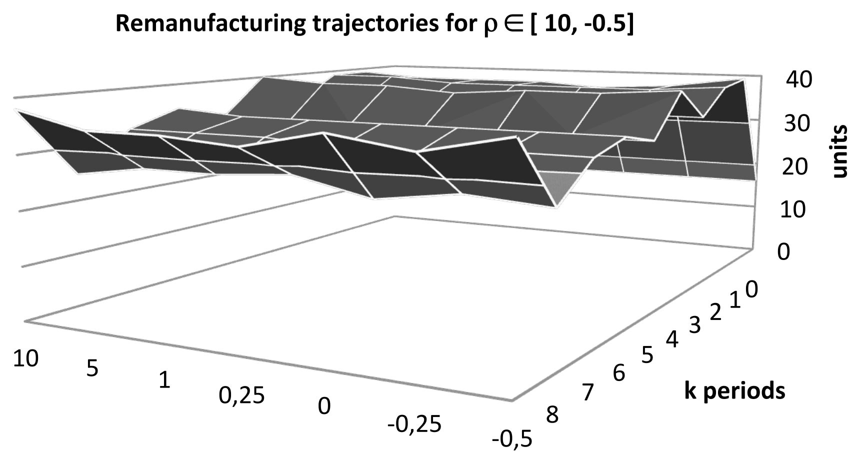

Figure C.4. Remanufacturing trajectories for ρ∈[10, -0.5]

We notice from figure C.2 that levels of inventory near to end-period (i.e., k=10) tend to decrease for values of ρ near 0. As all trajectories have almost similar shape, we can think that optimal ρ is close to 0. The trajectories of production and remanufacturing given by figures C3 e C4 are very similar in their shapes, but we can perceive that trajectories of figure C4 are more stable, what signalize to us that the remanufacturing process is more intense than the production process. We can observe this in the Table 3 of section 4 of the report.

Now we can determine the minimum cost by plotting the relation between costs and weighting factor ρ. Figure C.5 illustrates such a behavior and we can determine the minimum cost is \$ 26.788 for ρ =0.

Comparing our approximate value ρ=0.25 that provides a cost of \$ 26.890 with optimal scenario, what means a small difference between them of around 0.4%. Our conclusion is when we compared above costs with that one where we not consider the gain G (as showed in Table 3, section 4 of the report),we observe that difference is greater than 13%. That result is significant in our opinion.

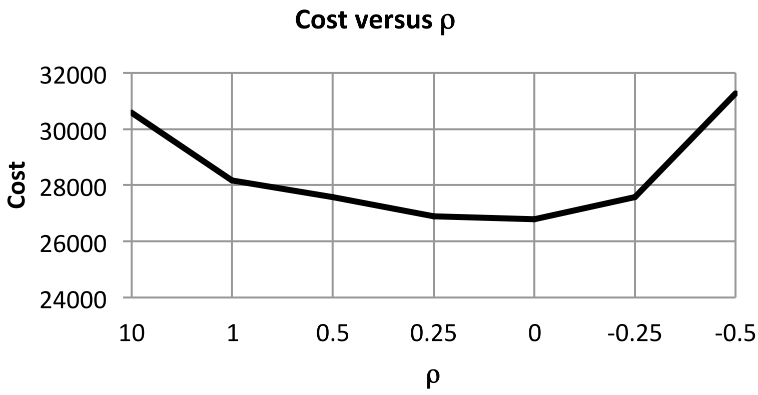

Figure C.5. Relation between cost of problem (10) versus ρ∈[10, -0.5]

Only a title of observation, we exhibit in figures C.6 and C.7 below, the plots of gain G versus weighting parameter ρand the feasible areas of constraint $\underline{x}_\alpha(k) \le \hat{x}_1(k) \le \overline{x}_\alpha(k)$ provided by ρ=0 and ρ=0,25.

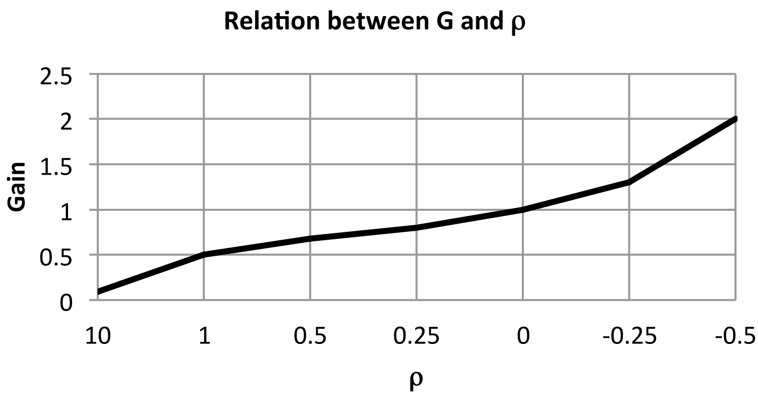

Figure C.6. Values of gain G when ρvaries in the range [10, -0.5]

Figure C.7 below shows the feasible areas for ρ=0,ρ=0.25, and ρ=10. Note that for ρ=0 we have that $\underline{x}_\alpha(k) = \underline{x}_1$ and $\overline{x}_\alpha(k) = \overline{x}_1$,while for ρ=0.25 and ρ =10, lower and upper bounds are computed from $\underline{x}_\alpha(k) = \sqrt{\left\{\sum_{i=1}^{k}(1-G)^{2*(i-1)}\right\}}\sigma_d \cdot \Phi^{-1}_{x_1}(\alpha)$ and $\overline{x}_\alpha(k) = \sqrt{\left\{\sum_{i=1}^{k}(1-G)^{2*(i-1)}\right\}}\sigma_d \cdot \Phi^{-1}_{x_1}(\alpha)$, respectively. At last, observe from figure C7that feasible areas of constraint decrease proportionally with the increasing ofρ. Thus, for $\rho > 10$, the problem (10) provides no feasible solution, because $\underline{x}_{\alpha}(k) > \overline{x}_{\alpha}(k)$.

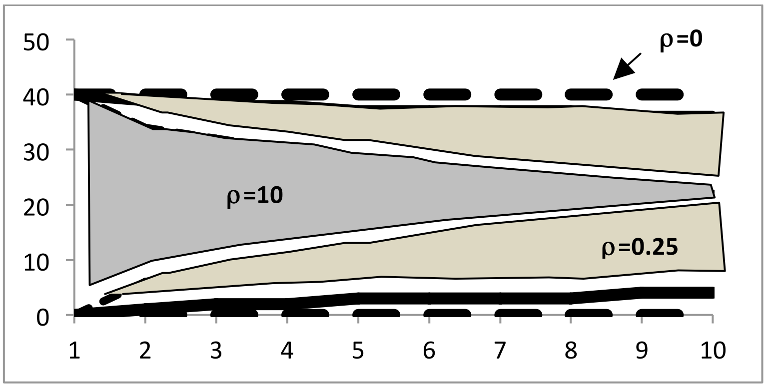

Figure C.7. Feasible areas of constraint $\underline{x}_{\alpha}(k) \leq \hat{x}_1(k) \leq \overline{x}_{\alpha}(k)$ for ρ=0, ρ=0.25 and ρ=10.

APPENDIX D   Part 1 - Mathematical Model by Excel

IFAC'2017 - LINEAR QUADRATIC PROBLEM VIA EXCEL/SOLVER

DATA:

Forecast	64	75	42	67	63	50	57	63	89	82	49	84	66	59	66	58	59	72	71	71	65	50	66	73
DEMAND =	64	75	42	67	63	50	57	63	89	82	49	84	66											

Estoques iniciais:	*x01*	*x02*
	40	30

Parâmetros de ajustes:			
k =	0,25	alfa =	0,9
G =	0,8		
%ret	0,8		

PARAMETROS	
Desv =	4
FI =	1,65

LIMITES	
xmin =	0
x1max=	40

x11	*x12*	*x13*	*x14*	*x15*	*x16*	*x17*	*x18*	*x19*	*x1,10*	*x21*	*x22*	*x23*	*x24*	*x25*	*x26*	*x27*	*x28*	*x29*	*x2,10*	*u10*	*u11*	*u12*	*u13*	*u14*	*u15*	*u16*	*u17*	*u18*	*u19*	*u20*	*u21*	*u22*	*u23*	*u24*	*u25*	*u26*	*u27*	*u28*	*u29*	*u30*	*u31*	*u32*	*u33*	*u34*	*u35*	*u36*	*u37*	*u38*	*u39*	LED	LDD
0	1,69	2	4,08	3	3	3	3	11,2	4	0	0	0	0	0	0	0	0	0	0	10,3	22,1	24,3	22,1	27,5	19,8	16,9	22,5	16,4	31,1	15,7	37,6	38	34	38,5	31,2	34,1	31,5	27,9	33,7	5,3	16,4	14	12	10,5	11,8	18,9	8,51	12,1	43		
1,7	1,7	1,7	1,7	1,7	1,7	1,7	1,7	1,7	1,7	1,5	1,5	1,5	1,5	1,5	1,5	1,5	1,5	1,5	1,5	1,3	1,3	1,3	1,3	1,3	1,3	1,3	1,3	1,3	1,3	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1	1	1	1	1	1	1	1	1	1		
-1																				1										1																				26	26
										1																				1										1										21	21
1	-1																				1										1																			58	58
										-1	1																				1										1									54	54
	1	-1																				1										1																		62	62
											-1	1																				1										1								52	52
		1	-1																				1										1																	54	54
		0										-1	1																				1										1							46	46
			1	-1																				1										1																67	67
			0										-1	1																				1										1						49	49
				1	-1																				1										1															51	51
				0										-1	1																				1										1					43	43
					1	-1																				1										1														51	51
					0										-1	1																				1										1				53	53
						1	-1																				1										1													54	54
						0										-1	1																				1										1			40	40
							1	-1																				1										1												36	36
							0										-1	1																				1										1		40	40
								1	-1																				1										1											72	72
								0										-1	1																														1	43	43
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			1																																															4,08	2
				1																																														3	3
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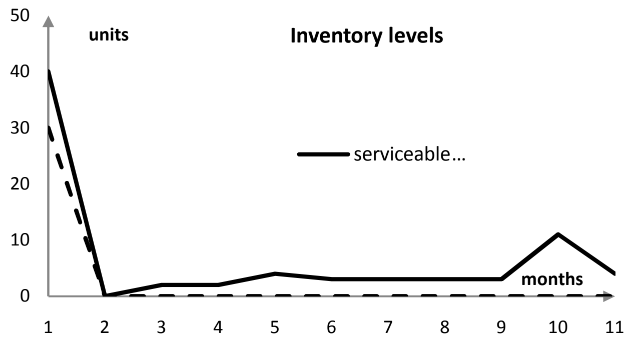
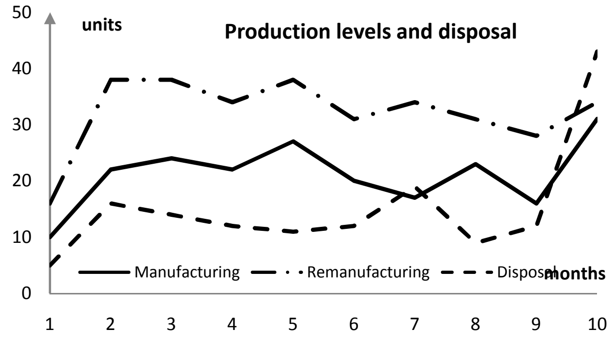
APPENDIX D Part 2 - Problem's solution (via Solver)

IFAC'2017 - QUADRATIC PROGRAMMING VIA EXCEL SOLVER	
Criterion:	
z =	$ 22.888,69

		Mês	X1	X2	U1	U2	U3
		0	40	30	10	16	5
		1	0	0	22	38	16
		2	2	0	24	38	14
		3	2	0	22	34	12
		4	4	0	27	38	11
		5	3	0	20	31	12
CUSTOS:		6	3	0	17	34	19
CD1 =	3054,90	7	3	0	23	31	9
CD2 =	1350,00	8	3	0	16	28	12
Manufac =	6250,40	9	11	0	31	34	43
Remanufac =	12914,40	10	4	0			
Disposal =	3321,00						
SOMA =	26890,70						

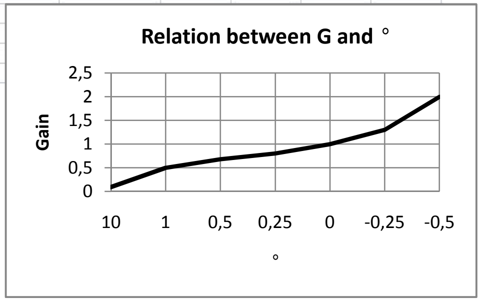


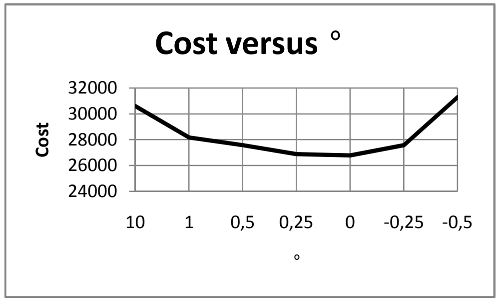


ρ	Gain G	Custo
10	0,091	30590,2
1	0,5	28174,1
0,5	0,68	27565,6
0,25	0,8	26890,7
0	1	26788
-0,25	1,3	27565,6
-0,5	2	31280,5

Nota: Para valores de ρ superiores a 3 e inferiores a -0.1, a solução do problema é impossível.

Pela simulação o ρ que produz o menor custo está no valor 0.

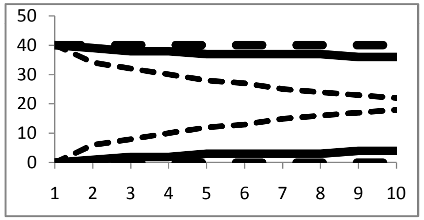


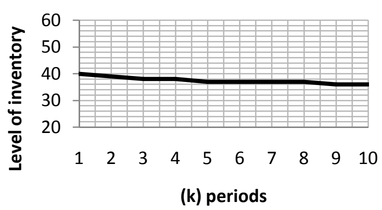


1o)	r = .	2 5
	xmin	xmax
	0	40
	1	39
	2	38
	2	38
	3	37
	3	37
	3	37
	3	37
	4	36
	4	36

2o)	r = 0	
	xmin	xmax
	0	40
	0	40
	0	40
	0	40
	0	40
	0	40
	0	40
	0	40
	0	40
	0	40

3o)	r = 1 0	
	xmin	xmax
	0	40
	6	34
	8	32
	10	30
	12	28
	13	27
	15	25
	16	24
	17	23
	18	22





Solução Malha-Fechada:								
			K	X1	X2	U1	U2	U3
			0	20	10	0	1	8