ISSN 1346-5597

National Institute of Informatics

NII Technical Report

Uniqueness of D-normal Proofs

Makoto Tatsuta

NII-2006-011E
Sept. 2006

URL: http://www.nii.ac.jp/ Tokyo 101-8430, JAPAN +81-3-4212-2000 FAX: +81-3-3556-1916

Uniqueness of D-normal Proofs

Makoto Tatsuta

National Institute of Informatics
2-1-2 Hitotsubashi, Tokyo 101-8430, JAPAN
e-mail: tatsuta@nii.ac.jp

Abstract

This paper presents the notion of D-normal proofs, which is defined syntactically and gives one of the weakest condition for uniqueness of normal proofs. This paper proves the following results: (1) $\beta\eta D$-normal proofs of a formula are unique. (2) A β-normal proof of a PNN-formula is D-normal. (3) A β-normal proof of a minimal formula in BCK logic is D-normal. These results give other proofs of uniqueness of $\beta\eta$-normal proofs of a PNN-formula, and uniqueness of $\beta\eta$-normal proofs of a minimal formula in BCK logic.

1 Introduction

Number of normal proofs has been studied widely [2]. In this paper we will present the notion of D-normal proofs and discuss uniqueness of normal proofs using this notion.

We will present the notion of D-normal proofs, which is defined syntactically and gives sufficient condition for uniqueness of normal proofs. Several sufficient conditions for uniqueness of normal proofs such as the one-two property [2], balanced formulas [5], minimal formulas in BCK logic [3], provability with non-prime contraction [1] and the PNN condition [2] have been proposed. D-normality is one of the weakest condition in those conditions.

We will prove that D-normality condition is properly weaker than the PNN condition. The PNN condition was proposed recently and one of the weakest condition at that time. This gives another proof of uniqueness $\beta\eta$-normal proofs of a PNN-formula.

The uniqueness of $\beta\eta$-normal proofs of a minimal formula in BCK logic was a problem proposed by [4] and solved independently at almost the same time by Hirokawa [3] and the author [6]. The author used the notion of D-normal proofs for that purpose and proved that β-normal proofs of a minimal formula in BCK logic is D-normal.

Section 2 presents the notion of D-normal proofs and proves uniqueness of $\beta\eta D$-normal proofs. Section 3 proves that a β-normal proof of a PNN-formula is D-normal and gives another proof of uniqueness of $\beta\eta$-normal proofs of a PNN-formula. Section 4 states that a β-normal proof of a minimal formula in BCK logic is D-normal and proves uniqueness of $\beta\eta$-normal proofs of a minimal formula in BCK logic.

2 D-normal proofs

We study constructive propositional logic with only implicational formulas in natural deduction style. Formulas are constructed by $\to$ from propositional variables. The inference rules are as follows.

$$\dfrac{\begin{array}{c} l \\ A \\ \vdots \\ B \end{array}}{A \to B}\ (\to I)l \qquad \dfrac{A \to B \quad A}{B}\ (\to E)$$

For $(\to E)$, A is called a *minor premise*.

Definition 2.1. (Proof)

(1) A formula A itself is a *proof*.

[1]This paper was written in 1999 in order to prepare the conference talk, whose proceedings are [8], and provides detailed descriptions of proofs given in the talk.

(2) If π is a proof, then

$$\dfrac{\begin{matrix} \overset{l}{A} \\ \vdots\ \pi \\ B \end{matrix}}{A \to B}\ (\to I)\ l$$

is a *proof*, where B is the lowermost formula of π, l is a label which is not used in π, and

$$\begin{matrix} \overset{l}{A} \\ \vdots\ \pi \\ B \end{matrix}$$

is a proof obtained from π by replacing some uppermost occurrences of the formula A with $\overset{l}{A}$.

(3) If π_1 and π_2 are proofs,

$$\dfrac{\begin{matrix} \vdots\ \pi_1 \\ A \to B \end{matrix} \quad \begin{matrix} \vdots\ \pi_2 \\ A \end{matrix}}{B}\ (\to E)$$

is a *proof*, where $A \to B$ is the lowermost formula of π_1 and A is the lowermost formula of π_2.

We often write a proof without the inference rule names $(\to I)$ and $(\to E)$.

The lowermost formula of a proof π is the *conclusion* of π and denoted by $\mathrm{Concl}(\pi)$. Uppermost formulas without labels of a proof π are *assumptions* of π and the set of assumptions of π is denoted by $\mathrm{Ass}(\pi)$. Uppermost formulas with labels of a proof π are *discharged assumptions* of π. A proof π is called *closed* if π has no assumptions. A proof π is called a *proof of a formula* A if $\mathrm{Concl}(\pi) = A$ and π is closed.

A proof π is a *β-normal proof*, if π does not include the following part for any A and B.

$$\begin{matrix} \dfrac{\dfrac{\begin{matrix} \vdots \\ B \end{matrix}}{A \to B}\ (\to I) \quad \begin{matrix} \vdots \\ A \end{matrix}}{B}\ (\to E) \\ \vdots \end{matrix}$$

A proof π is a *η-normal proof*, if π does not include the following part for any A and B.

$$\begin{matrix} \dfrac{\dfrac{\begin{matrix} \vdots \\ A \to B \end{matrix} \quad \overset{l}{A}}{B}\ (\to E)}{A \to B}\ (\to I) l \\ \vdots \end{matrix}$$

A proof π is a *$\beta\eta$-normal proof* if π is a β-normal proof and an η-normal proof.

For a formula A, $\mathrm{core}(A)$ is a variable having the rightmost variable occurrence in A.

Definition 2.2. (D-normal proof)

A proof π is a *D-normal proof* if the following condition holds.

- If there exists a form

$$\begin{matrix} \dfrac{\begin{matrix} \overset{k}{A} \\ \vdots \\ C \end{matrix}}{B \to C}\ (\to I) l \\ \vdots \end{matrix}$$

  in π and $\mathrm{core}(A) = \mathrm{core}(B)$, then $k = l$ holds.

A proof π is a *$\beta\eta D$-normal proof* if π is a $\beta\eta$-normal proof and a *D-normal proof.*

Theorem 2.3.

If a formula has $\beta\eta D$-normal proofs π_1 and π_2, then $\pi_1 = \pi_2$.

A formula is a *D-normal formula*, if every $\beta\eta$-normal proof of A is a D-normal proof.

Remark that $\beta\eta$-normal proofs of a D-normal formula are the same.

The rest of this section proves Theorem 2.3.

Definition 2.4.

The *length* $|\pi|$ of a proof π is defined as follows.

1. If π is a formula, then $|\pi| = 0$.

2. If π is

$$\frac{\begin{array}{c}\vdots\ \pi_1\\ B\end{array}}{A \to B}\ (\to I)$$

then $|\pi| = |\pi_1| + 1$.

3. If π is

$$\frac{\begin{array}{c}\vdots\ \pi_1\\ A \to B\end{array}\quad \begin{array}{c}\vdots\ \pi_2\\ A\end{array}}{B}\ (\to E)$$

then $|\pi| = |\pi_1| + |\pi_2| + 1$.

Note that $|\pi|$ represents the number of inference rules used in π. For a part Σ of a proof π, $|\Sigma|$ is defined as the number of interence rules in Σ.

dis(π) denotes the set of discharged assumptions of the proof π. leaf(π) denotes Ass(π) + dis(π).

For an occurrence of a formula A in a proof π, we use the notation A^1 to denote that we think about the specific occurrence of A in π.

For a proof π and formula occurrences A^1, B^1 in π, a *thread from* A^1 *to* B^1 is a sequence of formula occurrences in π satisfying the followings.

- The sequence begins with A^1 and ends with B^1.
- Every formula occurrence in the sequence except the last is an upper formula occurrence of an inference, and is immediately followed by the lower formula occurrence of this inference.

For a formula A in a β-normal proof such that the inference rule for A is not $(\to I)$, the main formula of A (denoted by MainFormula(A)) is defined in the following way.

1. If A is an assumption or a discharged assumption, MainFormula(A) is A itself.

2. If A is infered by $(\to E)$ and the part above A of π is

$$\frac{\begin{array}{c}\vdots\ \pi_1\\ B \to A\end{array}\quad \begin{array}{c}\vdots\\ B\end{array}}{A}$$

then MainFormula(A) = MainFormula($B \to A$). Note that the last inference of π_1 is not $(\to I)$ since π is a β-normal proof.

The *main thread* of a proof π is the thread from MainFormula(Concl(π)) to Concl(π). We denote the set of formula occurrences in the main thread of π by thread(π).

A thread is a path in a proof tree.

Definition 2.5. (Initial Subproof)

A subproof Π of a proof π is an *initial subproof* of π if the followings hold:

- Their conclusions are the same.
- If $\overset{l}{A^1}$ is in π and Π does not include $\overset{l}{A^1}$, Π does not include A^1.

We write $\Pi \subset \pi$ to denote that Π is an initial subproof of π.

For a part Σ of a proof π, $\Sigma \in \pi - \Pi$ iff Σ is the part above A^1 of π for some A^1 in assumptions of Π.

Definition 2.6. (L-unique Proof)

A β-normal proof π is an L-unique proof if the following condition holds.

- If there exists a form

$$\begin{array}{c} \vdots\; \pi_1 \\ A \\ \vdots \\ \dfrac{C}{B \to C}\;(\to I)l \\ \vdots \end{array}$$

  in π and

  $B = D_1 \to \cdots \to D_n \to A \qquad (n \geq 0)$

  then the last inference of π_1 is not $(\to I)$ and

  $\mathrm{MainFormula}(A) = \overset{l}{B}.$

Definition 2.7. (Strongly η-normal Proof)

A proof π is a *strongly η-normal proof* if π does not include the following form.

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{cc} \vdots & \vdots \\ A \to B & A \end{array}}{B}\;(\to E)}{A \to B}\;(\to I) \\ \vdots \end{array}$$

Lemma 2.8.

If a closed βD-normal proof π includes

$$\begin{array}{c} \vdots\; \pi_1 \\ A \\ \vdots \\ \dfrac{C}{B \to C}\;l \\ \vdots \end{array}$$

and

$B = D_1 \to \cdots \to D_n \to A \qquad (n \geq 0)$

and π_1 is a strongly η-normal proof, then the last inference of π_1 is not $(\to I)$ and

$\mathrm{MainFormula}(A) = \overset{l}{B}.$

Proof 2.9.

Suppose that the assumptions of this lemma hold. Since π is a β-normal proof, there exist formulas E, X_1, …, X_p $\quad (p \geq 0)$, Y_1, …, Y_q $\quad (q \geq 0)$ such that

$A = X_1 \to \cdots \to X_p \to E$

and π_1 is

$$\dfrac{\dfrac{\begin{array}{c}\vdots\end{array}}{X_2 \to \cdots \to X_p \to E}}{X_1 \to \cdots \to X_p \to E}
\quad\text{with}\quad
\begin{array}{c}
\dfrac{\dfrac{\overset{k}{Y_1 \to \cdots \to Y_q \to E} \quad \overset{\vdots}{Y_1}}{Y_2 \to \cdots \to Y_q \to E} \quad \overset{\vdots}{Y_2}}{\vdots} \\
\dfrac{\dfrac{\overline{Y_q \to E} \qquad \overset{\vdots}{Y_q}}{E}}{X_p \to E} \\
\vdots \\
\overline{X_2 \to \cdots \to X_p \to E} \\
\overline{X_1 \to \cdots \to X_p \to E}
\end{array}$$

Since π is a D-normal proof and $\mathrm{core}(Y_1 \to \cdots \to Y_q \to E) = \mathrm{core}(E) = \mathrm{core}(A) = \mathrm{core}(B)$, we have $B = Y_1 \to \cdots \to Y_q \to E$, $k = l$.

From $Y_1 \to \ldots \to Y_q \to E = B = D_1 \to \ldots D_n \to A = D_1 \to \ldots \to D_n \to X_1 \to \ldots \to X_p \to E$, we have $X_i = Y_{q-p+i} \quad (1 \le i \le p)$, $A = Y_{q-p+1} \to \cdots \to Y_q \to E$ and π_1 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{l}{Y_1 \to \cdots \to Y_q \to E} \quad \overset{\vdots}{Y_1}}{Y_2 \to \cdots \to Y_q \to E} \quad \overset{\vdots}{Y_2}}{\vdots}}{Y_q \to E} \quad \overset{\vdots}{Y_q}}{E}}{Y_q \to E}}{\vdots}}{Y_{q-p+2} \to \cdots \to Y_q \to E}}{Y_{q-p+1} \to \cdots \to Y_q \to E}$$

Since π_1 is a strongly η-normal proof, we have $p = 0$, $A = E$, $B = Y_1 \to \cdots \to Y_q \to A$ and π_1 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{l}{Y_1 \to \cdots \to Y_q \to A} \quad \overset{\vdots}{Y_1}}{Y_2 \to \cdots \to Y_q \to A} \quad \overset{\vdots}{Y_2}}{\vdots}}{Y_q \to A} \quad \overset{\vdots}{Y_q}}{A}$$

Therefore the last inference of π_1 is not $(\to I)$ and

$\mathrm{MainFormula}(A) = \overset{l}{B}$.

□

Lemma 2.10.

A closed $\beta\eta D$-normal proof is a strongly η-normal proof.

Proof 2.11.

Suppose that a proof π is a $\beta\eta D$-normal proof and is not a strongly η-normal proof. In π, there is the following form such that π_1 is a strongly η-normal proof.

$$\begin{array}{c} \dfrac{\dfrac{\overset{\vdots}{A \to B} \quad \overset{\vdots\ \pi_1}{A}}{B}}{A^1 \to B^1}\, l \\ \vdots \end{array}$$

From Lemma 2.8, π_1 is

$$\overset{l}{A}$$

and $|\pi_1| = 0$. This contradicts the fact that π is an η-normal proof. □

Proposition 2.12.

A closed $\beta\eta D$-normal proof is an L-unique proof.

Proof 2.13.

It is proved immediately from Lemma 2.8 and Lemma 2.10. □

Lemma 2.14.

If a formula A has closed L-unique proofs π_1 and π_2, and for some initial subproof Π of π_1, $\Pi \subset \pi_1$ and $\Pi \subset \pi_2$ hold, then $\pi_1 = \pi_2$.

Proof 2.15.

We may suppose that $|\pi_1| - |\Pi| \geq |\pi_2| - |\Pi|$.

This lemma is proved by induction on $|\pi_1| - |\Pi|$.

Case 1. $|\pi_1| - |\Pi| = 0$.

$|\pi_1| - |\Pi| = |\pi_2| - |\Pi| = 0$ holds. Then $\pi_1 = \Pi = \pi_2$.

Case 2. $|\pi_1| - |\Pi| > 0$.

There exists a part $\Sigma_1 \in \pi_1 - \Pi$ such that $|\Sigma_1| > 0$. Then there exists a part $\Sigma_2 \in \pi_2 - \Pi$ such that $\mathrm{Concl}(\Sigma_1) = \mathrm{Concl}(\Sigma_2)$ in Π.

Case 2.1. The last inference of Σ_1 *is* $(\to I)$.

Σ_1 is

$$\begin{array}{c}\vdots\\ \dfrac{B}{A\to B}\;k\end{array}$$

Case 2.1.1. $|\Sigma_2| = 0$.

We show this case is impossible. Σ_2 is $\overset{l}{A\to B}$. Therefore in π_2, $A\to B$ is discharged by the inference $(\to I)l$ in the thread from $A\to B$ to $\mathrm{Concl}(\pi_2)$. Then in π_1, $A\to B$ is discharged by the inference $(\to I)l$ in the thread from $A\to B$ to $\mathrm{Concl}(\pi_1)$. Since π_1 is L-unique, the inference rule for $A\to B$ is not $(\to I)$ and we get contradiction.

Case 2.1.2. The last inference of Σ_2 *is* $(\to I)$.

By renaming discharging labels in π_2, Σ_2 is

$$\begin{array}{c}\vdots\\ \dfrac{B}{A\to B}\;k\end{array}$$

Let Π' be the following initial subproof of π_1:

$$\begin{array}{c}\dfrac{B}{A\to B}\;k\\ \vdots\,\Pi\end{array}$$

Then $\Pi' \subset \pi_1$, $\Pi' \subset \pi_2$ and $|\pi_1| - |\Pi| > |\pi_1| - |\Pi'| \geq |\pi_2| - |\Pi'|$ hold. By induction hypothesis, we have $\pi_1 = \pi_2$.

Case 2.1.3. The last inference of Σ_2 *is* $(\to E)$.

We show this case is impossible. Let

$\mathrm{MainFormula}(A\to B) = X_1 \to \cdots \to X_p \to A \to B. \qquad (p \geq 0)$

Σ_2 is

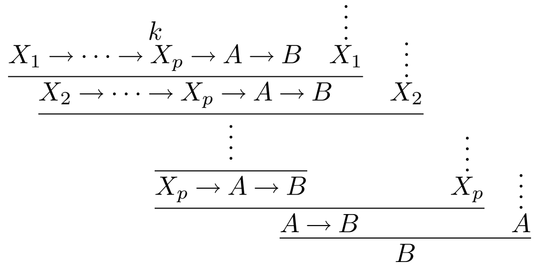
$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\overset{l}{X_1\to\cdots\to X_p\to A\to B}\quad \begin{array}{c}\vdots\\ X_1\end{array}}{X_2\to\cdots\to X_p\to A\to B}\quad \begin{array}{c}\vdots\\ X_2\end{array}}{}\\ \vdots\end{array}}{X_p\to A\to B}\quad \begin{array}{c}\vdots\\ X_p\end{array}}{A\to B}$$

In π_2, $X_1 \to \cdots \to X_p \to A \to B$ is discharged in the thread from $A\to B$ to $\mathrm{Concl}(\pi_2)$ by the inference $(\to I)l$. Therefore in π_1, $X_1 \to \cdots \to X_p \to A \to B$ is discharged in the thread from $A\to B$ to $\mathrm{Concl}(\pi_1)$ by the inference $(\to I)l$.

Since π_1 is L-unique, the last inference of Σ_1 is not $(\to I)$ and we get contradiction.

Case 2.2. The last inference of Σ_1 *is* $(\to E)$.

Σ_1 is



where $p \geq 0$.

Case 2.2.1. $|\Sigma_2| = 0$.

Since π_2 is closed, Σ_2 is $\overset{l}{B}$.

In π_2, B is discharged in the thread from B to $\mathrm{Concl}(\pi_2)$ by the inference $(\to I)l$. Therefore in π_1, B is discharged in the thread from B to $\mathrm{Concl}(\pi_1)$ by the inference $(\to I)l$.

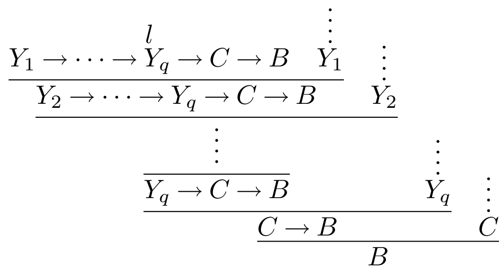
Since π_1 is L-unique, Σ_1 is $\overset{l}{B}$ and $|\Sigma_1| = 0$. This contradicts $|\Sigma_1| > 0$.

Case 2.2.2. The last inference of Σ_2 is $(\to I)$.

It is not the case by the same reason as Case 2.1.3.

Case 2.2.3. The last inference of Σ_2 is $(\to E)$.

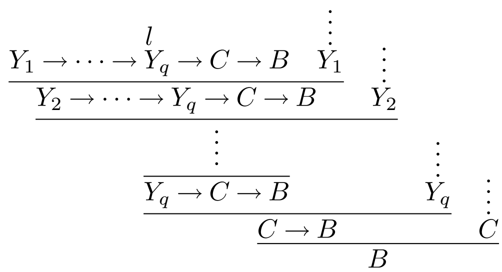
Σ_2 is



where $q \geq 0$.

In π_2, $Y_1 \to \cdots \to Y_q \to C \to B$ is discharged in the thread from B to $\mathrm{Concl}(\pi_2)$ by the inference $(\to I)l$. Therefore in π_1, $Y_1 \to \cdots \to Y_q \to C \to B$ is discharged in the thread from B to $\mathrm{Concl}(\pi_1)$ by the inference $(\to I)l$.

Since π_1 is L-unique, π_1 is



Therefore $A = C$. Let an initial subproof Π' of π_1 be given as follows:

$$\dfrac{A \to B \quad A}{B} \\ \vdots\ \Pi$$

Then $\Pi' \subset \pi_1$, $\Pi' \subset \pi_2$ and $|\Pi| - |\pi_1| > |\Pi'| - |\pi_1| \geq |\Pi'| - |\pi_2|$ hold. By induction hypothesis, we have $\pi_1 = \pi_2$. □

Proof 2.16. (of Theorem 2.3)

Suppose that a formula A has closed $\beta\eta D$-normal proofs π_1 and π_2. From Proposition 2.12, π_1 and π_2 are L-unique proofs. Let an initial subproof Π of π_1 be A.

From Lemma 2.14, since $\Pi \subset \pi_1$ and $\Pi \subset \pi_2$, we have $\pi_1 = \pi_2$. □

3 PNN condition

A formula A has *PNN-occurrences* of a variable B, if B has a positive occurrence and at least two negative occurrences in A. A formula A satisfies *PNN-condition*, if no variable has PNN-occurrences in A. A *PNN-formula* is a formula satisfying PNN-condition.

Theorem 3.1.

(1) A β-normal proof of a PNN-formula is a D-normal proof.
(2) A PNN-formula is a D-normal formula.

Remark that the converse of this theorem (1) does not hold. For example, the formula
$C \to (D \to B) \to (C \to B) \to (B \to A) \to A$
for distinct propositional variables A,B,C and D is a D-normal formula and has PNN-occurrences of B.

The following theorem is known [2]. By Theorem 2.3 and Theorem 3.1 immediately give another proof of this theorem.

Theorem 3.2.

$\beta\eta$-normal proofs of a formula satisfying the PNN-condition are unique.

The rest of this section proves these theorems.

Definition 3.3.

For a formula A, $\mathrm{Pos}(A)$ is the set of positive subformulas of A and $\mathrm{Neg}(A)$ is the set of negative subformulas of A. For a set S of formulas, $\mathrm{Pos}(S) = \cup_{A \in S}\mathrm{Pos}(A)$ and $\mathrm{Neg}(S) = \cup_{A \in S}\mathrm{Neg}(A)$.

A formula in a proof π is a minor premise in π if it is a minor premise of some inference rule $(\to E)$.

Proposition 3.4. (Signed Subformula Property for NJ)

(1) If a β-normal proof π has a discharged assumption $\overset{l}{A}$, the followings hold.
(a) $A \in Neg(Concl(\pi)) \cup Pos(Ass(\pi))$.
(b) If the last rule of π is not $(\to I)$, $A \in Pos(Ass(\pi))$.
(2) If a closed β-normal proof has a minor premise A, $A \in Pos(Concl(\pi))$ holds.

Proof 3.5.

(1) We prove this by induction on the proof π.

Case 1. π is a formula B. There is no $\overset{l}{A}$.

Case 2. π is

$$\dfrac{\begin{matrix}\vdots\ \pi_1 \\ B \to C\end{matrix} \quad \begin{matrix}\vdots\ \pi_2 \\ B\end{matrix}}{C}$$

If $\overset{l}{A}$ is in π_1, by induction hypothesis (b) for π_1, $A \in \mathrm{Pos}(\mathrm{Ass}(\pi_1))$. If $\overset{l}{A}$ is in π_2, by induction hypothesis (a) for π_2, $A \in \mathrm{Neg}(B) \cup \mathrm{Pos}(\mathrm{Ass}(\pi_2))$. Since we have $\mathrm{Neg}(B) \subset \mathrm{Pos}(B \to C) \subset \mathrm{Pos}(\mathrm{MainFormula}(B \to C)) \subset \mathrm{Pos}(\mathrm{Ass}(\pi_1))$, $A \in \mathrm{Pos}(\mathrm{Ass}(\pi))$ holds.

Case 3. π is

$$\dfrac{\begin{matrix}\overset{k}{B} \\ \vdots\ \pi_1 \\ C\end{matrix}}{B \to C}\ k$$

If $\overset{l}{A}$ is not $\overset{k}{B}$, by induction hypothesis (a) for π_1, $A \in \mathrm{Neg}(C) \cup \mathrm{Pos}(\mathrm{Ass}(\pi_1))$ holds and it is included in $\mathrm{Neg}(B \to C) \cup \mathrm{Pos}(\mathrm{Ass}(\pi))$.

If $\overset{l}{A}$ is $\overset{k}{B}$, $A \in \mathrm{Neg}(B \to C)$. Therefore $A \in \mathrm{Neg}(B \to C) \cup \mathrm{Pos}(\mathrm{Ass}(\pi))$.

(2) Suppose that the following part is in π:

$$\begin{matrix}\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \quad \begin{matrix}\vdots \\ A\end{matrix}}{B} \\ \vdots\end{matrix}$$

We have $A \in \mathrm{Neg}(A \to B) \subset \mathrm{Neg}(\mathrm{MainFormula}(A \to B))$. From (1), it is included in $\mathrm{Pos}(\mathrm{Concl}(\pi))$. □

$$\begin{array}{c} X \\ \| \\ Y \end{array}$$

Y denotes a proof whose conclusion is Y and whose assumptions include X and X and Y are in the same thread.

$X \supset_i Y$ denotes that $X = Z_1 \to \ldots Z_n \to Y$ for some $n \geq 0$, $Z_1, \ldots, Z_n$.

Lemma 3.6.

(1) If a β-normal proof of a PNN-formula has the following part, Z_3 is a minor premise, and $Z_3 \supset_i Y_3 \to Z_2$, we have $Y_3 \supset_i X_3 \to Y_2$.

$$\dfrac{Y_3 \to Z_2 \qquad \begin{array}{c} \dfrac{X_3 \to Y_2 \quad X_3}{Y_2} \\ \| \\ Y_3 \end{array}}{\begin{array}{c} Z_2 \\ \| \\ Z_3 \end{array}}$$

(2) If a β-normal proof of a PNN-formula has the following part, Y_3 is a minor premise, and $Y_1 = Y_3$, then we have $X_1 = X_3$.

$$\dfrac{\begin{array}{c} l_1 \\ Y_1 \\ \| \\ X_3 \to Y_2 \end{array} \quad \begin{array}{c} l_2 \\ X_1 \\ \| \\ X_3 \end{array}}{\begin{array}{c} Y_2 \\ \| \\ Y_3 \end{array}}$$

(3) If a β-normal proof π of a PNN-formula has the following part, X_3 is a minor premise, and $X_3 \to Y \in$ Pos(Concl(π)) for some Y, then we have $X_1 = X_3$.

$$\begin{array}{c} l \\ X_1 \\ \| \\ X_3 \end{array}$$

Proof 3.7.

(1) Let $Y = \text{core}(Y_3)$ and A be the conclusion of the proof. From Proposition 3.4 (2) for Z_3, we have $Z_3 \in \text{Pos}(A)$. Therefore $Y_3 \to Z_2 \in \text{Pos}(A)$. Let $Y_1 = \text{MainFormula}(Y_2)$. Let the part be

$$\dfrac{Y_3 \to Z_2 \quad n \qquad \begin{array}{c} \dfrac{\begin{array}{c} l \\ Y_1 \\ \| \\ X_3 \to Y_2 \end{array} \quad X_3}{Y_2} \\ \| \\ Y_3 \end{array}}{\begin{array}{c} Z_2 \\ \| \\ Z_3 \end{array}}$$

From Proposition 3.4 (2) for the inference rule ($\to I$) discharging the label l, $Y_1 \to B \in \text{Pos}(A)$ for some B. From Proposition 3.4 (2) for Y_3, $Y_3 \in \text{Pos}(A)$. By the PNN-condition for Y, $Y_1 = Y_3$. Since $Y_1 \supset_i X_3 \to Y_2$, we have $Y_3 \supset_i X_3 \to Y_2$. □

(2) Let $X = \text{core}(X_3)$ and A be the conclusion of the proof. From Proposition 3.4 (2) for Y_3, $Y_3 \in Pos(A)$. Since $Y_3 = Y_1 \supset_i X_3 \to Y_2$, $X_3 \to Y_2 \in \text{Pos}(A)$ holds. From Proposition 3.4 (2) for the inference rule ($\to I$) discharging the label l_2, $X_1 \to B \in \text{Pos}(A)$ holds for some B. From Proposition 3.4 (2) for X_3, $X_3 \in \text{Pos}(A)$ holds. By the PNN-condition for X, we have $X_1 = X_3$. □

(3) Let A be the conclusion of the proof. From Proposition 3.4 (2) for the inference rule ($\to I$) discharging the label l, $X_1 \to Z \in \text{Pos}(A)$ holds for some Z. From Proposition 3.4 (2) for X_3, $X_3 \in \text{Pos}(A)$ holds. By the PNN-condition for X, we have $X_1 = X_3$. □

Definition 3.8. (level)

For a proof π and an occurrence A of a formula in π, the level of A in π (denoted by $\text{level}_\pi(A)$) is defined as follows:

If π is A, $\text{level}_\pi(A) = 0$.

If π is

$$\dfrac{\begin{array}{c}\overset{l}{A}\\ \vdots\,\pi_1\\ B\end{array}}{A \to B}\,l$$

then $\mathrm{level}_\pi(A \to B) = 0$ and $\mathrm{level}_\pi(X) = \mathrm{level}_{\pi_1}(X)$ for $X \in \pi_1$.

If π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ A \to B\end{array} \quad \begin{array}{c}\vdots\,\pi_2\\ A\end{array}}{B}$$

then $\mathrm{level}_\pi(B) = 0$, $\mathrm{level}_\pi(X) = \mathrm{level}_{\pi_1}(X)$ for $X \in \pi_1$, and $\mathrm{level}_\pi(X) = \mathrm{level}_{\pi_2}(X) + 1$ for $X \in \pi_2$.

Proposition 3.9.

If a β-normal proof of a PNN-formula has discharged assumptions $\overset{l_1}{A_1}$ and $\overset{l_2}{A_2}$ and $\mathrm{core}(A_1) = \mathrm{core}(A_2)$, then we have $A_1 = A_2$.

Proof 3.10.

Let $A = \mathrm{core}(A_1)$ and B be the conclusion of the proof. From Proposition 3.4 (2) for the inference rules $(\to I)$ discharging the labels l_1 and l_2, $A_1 \to X_1 \in \mathrm{Pos}(B)$ and $A_2 \to X_2 \in \mathrm{Pos}(B)$ hold for some X_1 and X_2.

Either A_1 or A_2 is of level $n > 0$. We may suppose that A_1 is of level $n > 0$. Let A_3 be the lowermost formula of the thread including A_1. Since A_3 is a minor premise, from Proposition 3.4 (2), $A_3 \in Pos(B)$ holds.

By the PNN-condition for A, we have $A_1 = A_2$. □

Proposition 3.11.

A β-normal proof of a PNN-formula does not have the following part:

$$\begin{array}{cc}\overset{l_1}{A_0} & \overset{l_2}{A_1}\\ \| & \vdots\\ \multicolumn{2}{c}{B}\end{array}$$

where $\mathrm{core}(A_0) = \mathrm{core}(A_1)$ and $\overset{l_1}{A_0}$ and $\overset{l_2}{A_1}$ are distinct.

Proof 3.12.

Put $\overset{l_2}{A_1}$ be of level maximal. Let $A = \mathrm{core}(A_0)$ and B be the conclusion of the proof. By Proposition 3.9, we have $A_0 = A_1$. Let the part including l_1 and l_2 be as follows:

$$\dfrac{\begin{array}{c}\overset{l_1}{A_1}\\ \|\\ X_3^n \to A_2\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c}\overset{k_n}{X_1^n}\\ \|\\ X_3^{n-1} \to X_2^n\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c}\overset{k_4}{X_1^4}\\ \|\\ X_3^3 \to X_2^4\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c}\overset{k_3}{X_1^3}\\ \|\\ X_3^2 \to X_2^3\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c}\overset{k_2}{X_1^2}\\ \|\\ X_3^1 \to X_2^2\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c}\overset{k_1}{X_1^1}\\ \|\\ A_3 \to X_2^1\end{array} \quad \begin{array}{c}\overset{l_2}{A_1}\\ \|\\ A_3\end{array}}{X_2^1}\\ \|\\ X_3^1\end{array}}{X_2^2}\\ \|\\ X_3^2\end{array}}{X_2^3}\\ \|\\ X_3^3\end{array}}{X_2^4}\\ \vdots\\ X_3^{n-1}\end{array}}{X_2^n}\\ \|\\ X_3^n\end{array}}{A_2}$$

and π_1 be the part below X_2^1 of this and π_2 be the part above A_3 of this.

Case 1. $A_3 \not\supset_i X_3^n \to A_2$, $X_3^p \not\supset_i X_3^{p-1} \to X_2^p$ for $p = n, \dots, 2$, and $X_3^1 \not\supset_i A_3 \to X_2^1$.

From $A_3 \not\supset_i X_3^n \to A_2$, π_2 is as follows:

$$\begin{array}{c}\dfrac{\begin{array}{c} l_2\\ A_1\\ \|\\ X_3^n \to A_2\end{array} \quad \begin{array}{c} k'_n\\ X\\ \|\\ X_3^n\end{array}}{A_2}\\ \| \\ A_3\end{array}$$

By Proposition 3.9, $X = X_1^n$ holds. Therefore π_2 as as follows:

$$\begin{array}{c}\dfrac{\begin{array}{c} l_2\\ A_1\\ \|\\ X_3^n \to A_2\end{array} \quad \begin{array}{c} k'_n\\ X_1^n\\ \|\\ X_3^n\end{array}}{A_2}\\ \| \\ A_3\end{array}$$

From $X_3^n \not\supset_i X_3^{n-1} \to X_2^n$, the thread from $\overset{k'_n}{X_1^n}$ to X_3^n is as follows:

$$\begin{array}{c}\dfrac{\begin{array}{c} k'_n\\ X_1^n\\ \|\\ X_3^{n-1} \to X_2^n\end{array} \quad \begin{array}{c} k'_{n-1}\\ X\\ \|\\ X_2^{n-1}\end{array}}{X_2^n}\\ \| \\ X_3^n\end{array}$$

By Proposition 3.9, $X = X_1^{n-1}$ holds.

By repeating this discussion, we prove that π_2 has the following form:

$$\begin{array}{c}\dfrac{\begin{array}{c} l_2\\ A_1\\ \|\\ X_3^n \to A_2\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c} k'_n\\ X_1^n\\ \|\\ X_3^{n-1} \to X_2^n\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c} k'_4\\ X_1^4\\ \|\\ X_3^3 \to X_2^4\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c} k'_3\\ X_1^3\\ \|\\ X_3^2 \to X_2^3\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c} k'_2\\ X_1^2\\ \|\\ X_3^1 \to X_2^2\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c} k'_1\\ X_1^1\\ \|\\ A_3 \to X_2^1\end{array} \quad \begin{array}{c} l'_2\\ A_1\\ \|\\ A_3\end{array}}{X_2^1}\\ \|\\ X_3^1\end{array}}{X_2^2}\\ \|\\ X_3^2\end{array}}{X_2^3}\\ \|\\ X_3^3\end{array}}{X_2^4}\\ \vdots\\ X_3^{n-1}\end{array}}{X_2^n}\\ \|\\ X_3^n\end{array}}{A_2}\\ \| \\ A_3\end{array}$$

Hence π_2 has $\overset{l'_2}{A_1}$ and it contradicts the maximality of the level of $\overset{l_2}{A_1}$.

Case 2. $A_3 \supset_i X_3^n \to A_2$.

By Proposition 3.4 (2) for A_3, we have $X_3^n \to A_2 \in \mathrm{Pos}(B)$. By repeating Lemma 3.6 (2) from the thread

$\begin{array}{c} k_n \\ X_1^n \\ \| \\ X_3^n \end{array}$ to the thread $\begin{array}{c} k_1 \\ X_1^1 \\ \| \\ X_3^1 \end{array}$ and the thread $\begin{array}{c} l_2 \\ A_1 \\ \| \\ A_3 \end{array}$, we have $X_1^p = X_3^p$ for $p = n, \ldots, 1$ and $A_1 = A_3$. Therefore $X_3^p = X_1^p \supset_i X_3^{p-1} \to X_2^p$ holds for $p = n, \ldots, 2$, $X_3^1 = X_1^1 \supset_i A_3 \to X_2^1$ and $A_3 = A_1 \supset_i X_3^n \to A_2$. Hence we get $X_3^n \underset{\neq}{\supset_i} X_3^{n-1} \underset{\neq}{\supset_i} \ldots \underset{\neq}{\supset_i} X_3^1 \underset{\neq}{\supset_i} A_3 \underset{\neq}{\supset_i} X_3^n$ and contradiction.

Case 3. $A_3 \not\supset_i X_3^n \to A_2$ and $X_3^p \supset_i X_3^{p-1} \to X_2^p$ for some $2 \leq p \leq n$.

By repeating Lemma 3.6 (1) for $X_3^p \supset_i X_3^{p-1} \to X_2^p$, we have $X_3^q \supset_i X_3^{q-1} \to X_2^q$ for $q = p, \ldots, 2$ and $X_3^1 \supset_i A_3 \to X_2^1$. From $X_3^1 \supset_i A_3 \to X_2^1$, $A_3 \to X_2^1 \in \mathrm{Pos}(B)$ holds. By Lemma 3.6 (3) for the thread $\begin{array}{c} l_2 \\ A_1 \\ \| \\ A_3 \end{array}$ we have $A_1 = A_3$. Hence we get $A_3 = A_1 \supset_i X_3^n \to A_2$ and contradiction.

Case 4. $A_3 \not\supset_i X_3^n \to A_2$, $X_3^p \not\supset_i X_3^{p-1} \to X_2^p$ for $2 \leq p \leq n$, and $X_3^1 \supset_i A_3 \to X_2^1$.

By Proposition 3.4 (2) for X_3^1, we have $A_3 \to X_2^1 \in \mathrm{Pos}(B)$. By Lemma 3.6 (3) for the thread $\begin{array}{c} l_2 \\ A_1 \\ \| \\ A_3 \end{array}$ we have $A_1 = A_3$. Hence we get $A_3 = A_1 \supset_i X_3^n \to A_2$ and contradiction. □

Proof 3.13. (of Theorem 3.1)

(1) Suppose that we have a β-normal proof π of a PNN-formula C and π is not a D-normal proof.

By the definition of D-normality, π has the following part:

$$\dfrac{\begin{array}{c} (k) \\ A_0 \\ \vdots \\ B_1 \end{array}}{A_1 \to B_1}\ (l)$$

where $\mathrm{core}(A_0) = \mathrm{core}(A_1)$ and $k \neq l$.

Let $A_0 \to X$ be derived by the inference rule $(\to I)$ discharging the label k. By Proposition 3.4 (2), we have $A_0 \to X \in \mathrm{Pos}(C)$. By Proposition 3.4 (2), we have $A_1 \to B_1 \in \mathrm{Pos}(C)$. Let A_2 be the lowermost formula of the thread including $\overset{k}{A_0}$. Since A_2 is a minor premise, by Proposition 3.4 (2), we have $A_2 \in \mathrm{Pos}(C)$. By the PNN-condition for A, $A_0 \to X = A_1 \to B_1$ holds. Therefore $A_0 = A_1$ and $X = B_1$. Hence $\dfrac{X}{A_0 \to X}\ (k)$ is not in the thread including $\dfrac{B_1}{A_1 \to B_1}\ (l)$. Let B_2 and B_3 be the main formulas of $\dfrac{B_1}{A_1 \to B_1}\ (l)$ and $\dfrac{X}{A_0 \to X}\ (k)$ respectively. Then we have either

$$\begin{array}{c} l_1 \\ B_2 \\ \| \end{array} \begin{array}{c} l_2 \\ B_3 \\ \vdots \end{array} \atop B_4 \qquad \text{or} \qquad \begin{array}{c} l_1 \\ B_3 \\ \| \end{array} \begin{array}{c} l_2 \\ B_2 \\ \vdots \end{array} \atop B_4$$

and get contradiction by Proposition 3.11. □

(2) It is immediate from (1). □

Proof 3.14. (of Theorem 3.2)

Let π_1 and π_1 be $\beta\eta$-normal proofs of a PNN-formula A. By Theorem 3.1 (1), they are $\beta\eta D$-normal proofs. By Theorem 2.3, we have $\pi_1 = \pi_2$. □

4 BCK logic

BCK logic is the logic with only the restricted $(\to I)$ rule which can discharge at most one occurrence of an assumption. A formula is called *a minimal formula* if it is minimal in the variable substitution preorder of provable formulas.

Proposition 4.1.

In BCK logic, a β-normal proof of a minimal formula is a D-normal proof.

Combining Theorem 2.3 and this proposition, we have the following theorem.

Theorem 4.2.
In BCK logic, if a minimal formula A has $\beta\eta$-normal proofs π_1 and π_2, then $\pi_1 = \pi_2$.

References

[1] T. Aoto, Uniqueness of normal proofs in implicational implicational intuitionistic logic, *Journal of Logic, Language and Information*, to appear.

[2] T. Aoto, Number of Normal Proofs in Implicational Logics: Komori's Problem, Its Solution, and Beyond, Manuscript, 1999.

[3] S. Hirokawa, Principal types of BCK-lambda-terms, *Theoretical Computer Science,* **107** (1993) 253–276.

[4] Y. Komori, **BCK** Algebras and Lambda Calculus, In: *Proceedings of the 10th Symposium on Semigroups* (Josai Univ., Sakado, 1987) 5–11.

[5] G. E. Mints, Simple proof of the coherence theorem for cartesian closed categories, *Journal of Symbolic Logic.*

[6] M. Tatsuta, Uniqueness of Normal Proofs in Implicational Logic, Manuscript, 1988.

[7] M. Tatsuta, Uniqueness of normal proofs of minimal formulas, *Journal of Symbolic Logic* **58** (3) (1993) 789–799.

[8] M. Tatsuta, Uniqueness of D-normal Proofs, *Proceedings of 7th Asian Logic Conference* (1999) 41–42.