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On the Semantics of Classical First-Order Logic with Constructive Double Negation

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Abstract

Constructive negation in intuitionistic logic (called strong negation [7]) can be used to directly represent negative assertions, and for which its semantics [8, 1] is defined in Kripke models by two satisfaction relations ($\models_P$ and $\models_N$). However, the interpretation and satisfaction based on the conventional semantics do not fit in with the definition of negation in knowledge representation when considering a double negation of the form $\sim\neg$, for strong negation $\sim$ and classical negation $\neg$ (which we call constructive double negation). The problem is caused by the fact that the semantics makes the axiom $\sim\neg A \leftrightarrow A$ valid. By way of solution, this paper proposes an alternative semantics for constructive double negation $\sim\neg A$ by capturing the constructive meaning of the combinations of the two negations. In the semantics, we consider the constructive double negation $\sim\neg A$ as partial to the classical double negation $\neg\neg A$ and as exclusive to the classical negation $\neg A$. Technically, we introduce infinite satisfaction relations to interpret the partiality that is sequentially created by each constructive double negation (of the forms $\sim\neg A, \sim\neg\sim\neg A, \dots$).

1 Introduction

Various logic negations have been motivated and proposed in the fields of knowledge representation and logic programming. For example, negation as failure [2] in logic programming languages [6] is based on the closed world assumption (CWA), for which $\neg A$ is true if the proposition A is not provable, which is obviously different from negation in classical logic (called classical negation). Ordinary logic programming cannot assert negative facts since it prohibits negation occurring in the head of each Horn clause. To settle this weakness, Wagner [9, 10] proposed a logic programming language equipped with strong negation. This logic program considers both negation as failure (called weak negation) and strong negation where strong negative literals (the strong negation of atomic formulas) are allowed in the heads of Horn clauses.

Strong negation is considered a useful connective for directly expressing negative assertions that are not represented by other negations (classical negation, Heyting negation, etc.). The law of double negation holds for strong negation but the law of excluded middle does not. In contrast, both laws are valid for classical negation, and they are not valid for Heyting negation (in intuitionistic logic). Strong negation (or called constructive falsity) was first introduced by Nelson [7], and the semantics of first-order logic with strong negation, defined by Kripke models, was considered in [8]. Moreover, Akama [1] formalized an intuitionistic logic with Heyting negation and strong negation, and its model theory. There has been related research on negation in logic and linguistics (cf. [4]). Kaneiwa and Tojo [5] proposed an order-sorted logic with implicitly negative sorts where some lexical negations are treated as a strong negation of sorts.

However, in the existing semantics of strong negation, combinations of strong negation and classical negation (i.e., double negation) give rise to inapplicable meaning from the viewpoint of knowledge representation. Let $\sim$ be strong negation and $\neg$ be classical negation. Then, there is a semantic problem such that for characterizing the double negation of the form $\sim\neg A$ (which we will call *constructive double negation*), the axiom $\sim\neg A \leftrightarrow A$ is valid in the Kripke semantics [1]. Based on this we remark that $\sim\neg A \rightarrow A$ is adequate but $\sim\neg A \leftarrow A$ is undesirable in knowledge representation since A does not imply the strong negation $\sim\neg A$ for the classical negation $\neg A$.

In this paper, to eliminate the axiom $\sim\neg A \leftarrow A$ in the logic, we present an alternative semantics for the constructive double negation $\sim\neg A$. We modify the semantics of the intuitionistic logic with Heyting negation and strong negation [1] into a classical version of first-order logic with strong negation. In the semantics, strong negation $\sim A$ is partial to classical negation $\neg A$ and exclusive to positive assertion A. Analogously, constructive double negation $\sim\neg A$ is interpreted by the partiality to classical double negation $\neg\neg A$ and by the exclusivity to classical negation $\neg A$. Based on the partiality of each constructive double negation, satisfaction relations corresponding to the partial negations are infinitely introduced in the improved semantics. We present a complete logical system for the constructive double negation, i.e., any valid formula in the semantics can be derived from the axioms.

This paper is arranged as follows: As a classical variant of intuitionistic logic, Section 2 defines the syntax and semantics of first-order logic with strong negation and classical negation. In Section 3, we raise an objection to the semantics of Section 2, and then discuss negative assertions in knowledge representation, relevant to the meaning of strong negation, classical negation, and constructive double negation. Based on the specification, we define models, interpretations, and satisfaction relations that follow the property of constructive double negation. Section 3.3 establishes axioms and rules for the first-order logic with constructive double negation. In Section 4, we prove the completeness of the logical system.

2 First-order logic with strong negation

We define the syntax and semantics of first-order logic with strong negation and classical negation. This logic is characterized as a classical variant of intuitionistic logic with Heyting negation and strong negation [1].

2.1 Syntax

An alphabet of a first-order predicate language L contains the following symbols: $\mathcal{C}$ is the set of constant symbols, $\mathcal{F}$ is the set of function symbols, and $\mathcal{P}$ is the set of predicate symbols. $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\neg$ (classical negation), and $\sim$ (strong negation) are logical connectives; $\forall$, $\exists$ are universal and existential quantifiers, respectively; and $(,)$ are parentheses. In particular, $\mathcal{P}_n$ denotes the set of n-ary predicate symbols and $\mathcal{F}_n$ denotes the set of n-ary function symbols. V is the set of variable symbols.

In the usual way of first-order logic, terms are defined by the following:

Definition 2.1 (Terms) *The set TERM of terms is the smallest set defined by the following rules.*

1. *For every $x \in V$, $x \in TERM$.*
2. *For every $c \in \mathcal{C}$, $c \in TERM$.*
3. *If $f \in \mathcal{F}_n$ and $t_1, \ldots, t_n \in TERM$, then $f(t_1, \ldots, t_n) \in TERM$.*

A term is said to be ground if it contains no variables. In the following definition, formulas are constructed with strong negation ($\sim$) and classical negation ($\neg$).

Definition 2.2 (Formulas) *The set FORM of formulas is the smallest set defined by the following rules.*

1. *If $p \in \mathcal{P}_n$ and $t_1, \ldots, t_n \in TERM$, then $p(t_1, \ldots, t_n) \in FORM$.*
2. *If $A, B \in FORM$, then $\neg A$, $\sim A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $\forall x A$, $\exists x A$ $\in FORM$.*

Let A be a formula. It is said to be a closed formula if it contains no free variables. The forms $\neg\neg A$, $\sim\sim A$, $\sim\neg A$, and $\neg\sim A$ are called *double negation* of A, i.e., any two negation symbols preceding A. In particular, the form $\sim\neg A$ is said to be a *constructive double negation* of A and the form $\neg\sim A$ is said to be a *weak double negation* of A. Let C be a sequence of classical negation $\neg$ and strong negation $\sim$. Then, $(C)^n$ expresses a chain of length n of C. For example, $(\sim\neg)^2 A$ and $(\neg\sim)^0 A$ denote $\neg\sim\neg\sim A$ and A, respectively.

2.2 Semantics

In Kripke models, the conventional semantics of intuitionistic logic with Heyting negation and strong negation is defined by two satisfaction relations [1]. It gives us a means to define models, interpretations, and satisfaction relations in classical first-order logic with strong negation.

Definition 2.3 (Model for L with strong negation) *A model for a first-order language L with strong negation (called an L-model) is a tuple $M = (U, I_P, I_N)$, where U is a non-empty set and I_P, I_N are interpretation functions such that*

1. *for $c \in \mathcal{C}$, $I_P(c) \in U$ and $I_N(c) \in U$,*
2. *for $f \in \mathcal{F}_n$, $I_P(f) : U^n \to U$ and $I_N(f) : U^n \to U$,*
3. *for $p \in \mathcal{P}_n$, $I_P(p) \subseteq U^n$ and $I_N(p) \subseteq U^n$,*
4. *$I_P(c) = I_N(c)$,*
5. *$I_P(f) = I_N(f)$,*
6. *$I_P(p) \cap I_N(p) = \emptyset$.*

Note that the interpretation functions I_P and I_N are defined as the same mappings for function and constant symbols (in (4) and (5)), and each of these interpretation functions for predicate symbols is exclusive to the other (in (6)). A substitution θ is a mapping from variables $x_1, \ldots, x_n$ into terms $t_1, \ldots, t_n$ with $\theta(x_i) \neq x_i$, denoted by $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$. We introduce the set $\mathcal{C}_U$ of new constants (denoted d_i) for elements d_i in U where every new constant in $\mathcal{C}_U$ is interpreted by itself. The interpretations of terms are given in the following definition.

Definition 2.4 (Interpretations of terms) *Let $M = (U, I_P, I_N)$ be an L-model. The interpretations $[\![\ ]\!]_P$ and $[\![\ ]\!]_N$ of terms are defined by the following rules.*

1. *$[\![c]\!]_* = I_*(c)$ for $c \in \mathcal{C}$*
2. *$[\![d]\!]_* = d$ for $d \in \mathcal{C}_U$*
3. *$[\![f(t_1, \ldots, t_n)]\!]_* = I_*(f)([\![t_1]\!]_*, \ldots, [\![t_n]\!]_*)$*

where $* \in \{P, N\}$.

The satisfaction relation of an L-model M and closed formula A is defined as follows.

Definition 2.5 (Satisfaction with strong negation) *Let $M = (U, I_P, I_N)$ be an L-model and A be a closed formula. The satisfaction relations $M \models_P A$ and $M \models_N A$ are defined by the following rules.*

1. $M \models_P p(t_1, \ldots, t_n)$ *iff* $([\![t_1]\!]_P, \ldots, [\![t_n]\!]_P) \in I_P(p)$
2. $M \models_P \neg A$ *iff* $M \not\models_P A$
3. $M \models_P \sim A$ *iff* $M \models_N A$
4. $M \models_P A \wedge B$ *iff* $M \models_P A$ *and* $M \models_P A$
5. $M \models_P A \vee B$ *iff* $M \models_P A$ *or* $M \models_P A$
6. $M \models_P A \rightarrow B$ *iff* $M \not\models_P A$ *or* $M \models_P A$
7. $M \models_P \forall x A$ *iff for all* $d \in U$, $M \models_P A\{x/d\}$
8. $M \models_P \exists x A$ *iff for some* $d \in U$, $M \models_P A\{x/d\}$
9. $M \models_N p(t_1, \ldots, t_n)$ *iff* $([\![t_1]\!]_N, \ldots, [\![t_n]\!]_N) \in I_N(p)$
10. $M \models_N \neg A$ *iff* $M \models_P A$
11. $M \models_N \sim A$ *iff* $M \models_P A$
12. $M \models_N A \wedge B$ *iff* $M \models_N A$ *or* $M \models_N A$
13. $M \models_N A \vee B$ *iff* $M \models_N A$ *and* $M \models_N A$
14. $M \models_N A \rightarrow B$ *iff* $M \models_P A$ *and* $M \models_N A$
15. $M \models_N \forall x A$ *iff for some* $d \in U$, $M \models_N A\{x/d\}$
16. $M \models_N \exists x A$ *iff for all* $d \in U$, $M \models_N A\{x/d\}$

Let A be an (open) formula. $M \models_P A$ if $M \models_P \forall A$ where $\forall A$ is the universal closure of A. A is L-satisfiable if $M \models_P A$ for some L-model M, and it is L-unsatisfiable otherwise. A is L-valid if $M \models_P A$ for all L-models M.

In the next section, we will consider that the abovementioned definitions do not adequately describe the meaning of strong negation and classical negation in knowledge representation. The important point is that statement (10) in Definition 2.5 yields a disagreement with our intuitive interpretation, since it makes $\sim\neg A$ and A equivalent. We address the issue by considering the meaning of simple, constructive and weak double negations $\sim\sim A$, $\neg\neg A$, $\sim\neg A$, and $\neg\sim A$ (composed by strong negation and classical negation). Subsequently, we will define an alternative semantics reflecting on the interpretation of constructive double negation.

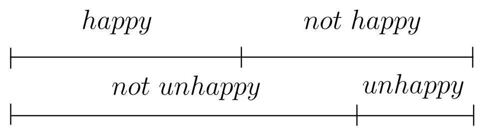


Figure 1: A relationship among *happy*, *not happy*, and *unhappy*

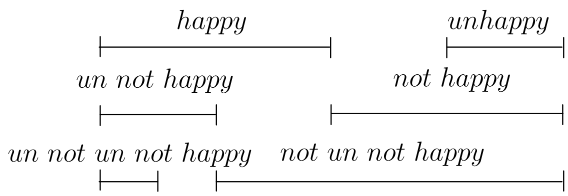


Figure 2: Combinations of the negative particle *not* and the negative prefix *un*

3 Constructive double negation

The purpose of this section is to establish the semantics of first-order logic with constructive double negation, and its corresponding axioms and rules for strong negation and classical negation.

3.1 Negation in knowledge representation

There exists a difference between strong negation and classical negation. Semantically, classical negation can be regarded as the complement of assertions, whereas strong negation is more specific and partial than classical negation. The use of strong negation in knowledge representation and logic programming was motivated by the adequacy for expressing explicit negative assertions. The partial negation is informally specified by the fact that it holds in a more narrow or specific scope than classical negation. We have defined an interpretation of strong negation in Section 2; however, this is only a classical variant of Akama's model theory, i.e., it is not based on the philosophical observation of negation.

We will reanalyze whether the semantics presented in Section 2.2 corresponds to a partial or specific negation in explicit negative assertions. As an example, let us consider the sentence "John is happy." In order to succinctly represent it, we employ the simple expression *happy*. Moreover, two types of negation of the sentence are: "John is not happy" and "John is unhappy." These are also simply expressed by *not happy* and *unhappy* (as if *not* and *un* are logical connectives). In Figure 1, we visually describe the scope as a width for which each of the sentences holds. In semantics, *happy* and *not happy* are mutually complementary and exclusive, and in contrast, *happy* and *unhappy* are exclusive to each other but not comple-

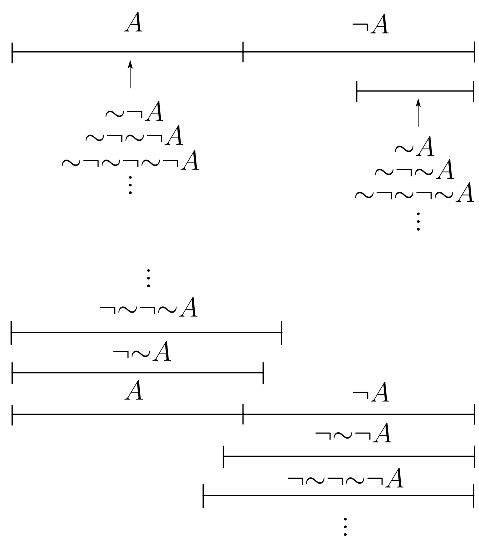


Figure 3: Strong negation and classical negation based on (Akama, 1988)

mentary. That is, there exists an assertion exclusive to both assertions *happy* and *unhappy*, such as, "John is neither happy nor unhappy." Additionally, *unhappy* and *not unhappy* are mutually complementary and exclusive.

Furthermore, we take into account the complicated combinations[1] of *not* and *un* as in Figure 2. Similar to the relationship between *happy* and *unhappy*, if we attach *un* to the expression *not happy*, then *not happy* and *un not happy* are mutually exclusive but *un not happy* is partial to *not not happy* (i.e., *un not happy* and *not happy* are not complementary). By attaching *not* to *un not happy*, *un not happy* and *not un not happy* are mutually complementary and exclusive. Iteratively, putting *un* to *not un not happy* leads to the fact that *not un not happy* and *un not un not happy* are mutually exclusive but *un not un not happy* is partial to *not not un not happy*. These operations can be infinitely iterated. Such properties in negative combinations are consistent with the meaning of negation in knowledge representation since it does not lose the partiality, complementarity and exclusivity of *un* and *not*.

The following definition regards the negative particle *not* as a classical

[1]Some combinations (e.g. *un not happy*) do not exist in natural languages. In our approach, we analyze logic, capturing the roles as negative connectives of *not* and *un*, and the generality of logic allows complex combinations of connectives. For example, temporal logic treats temporal connectives, some combinations of which do not correspond to any natural language sentences, e.g., **FFPP**A where **F** (future) and **P** (past) are temporal connectives.

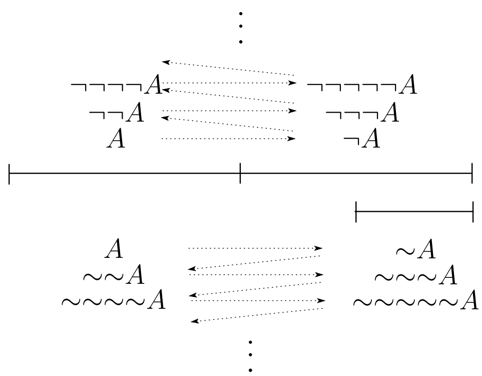


Figure 4: The meaning of double negation $\neg\neg$, $\sim\sim$

negation and the negative prefix *un* as a strong negation.

$$\begin{aligned} \neg happy &=_{def} not\ happy \\ \sim happy &=_{def} unhappy \end{aligned}$$

By illustrating the scope of the meaning of each negative expression, Figure 3 informally shows the combinations of strong negation and classical negation based on the semantics of Section 2. Consequently, the interpretation and satisfaction are partially inconsistent with the idea for *not* and *un*. In the case of attaching constructive double negation $\sim\neg$ to the formula A any number of times, the formulas $(\sim\neg)^n A$ become equivalent to A. For example, it results in the following axiom.

$$\sim\neg happy \quad \leftrightarrow \quad happy\ (?)$$

In addition, if the form $\sim\neg$ is added to $\sim A$ any number of times, then the formulas $(\sim\neg)^n \sim A$ is equivalent to $\sim A$. As shown in Figures 1 and 2, we desire the specification wherein $\sim\neg happy$ and *happy* (i.e., *un not happy* and *happy*) are not equivalent. This semantic disagreement motivates us to propose an alternative semantics for the constructive double negation $(\sim\neg)^n A$. Namely, statement (10) in Definition 2.5 has to be improved using the specification.

Moreover, the formula A belongs to the scope of the formulas $(\neg\sim)^n A$ of weak double negation $\neg\sim$ (as in Figure 3). This case is consistent with the interpretation of *not* and *un*. For example, we have the following:

$$happy \to \neg\sim happy,$$

which means that *happy* implies *not unhappy*. Therefore, we need to preserve the validity of $A \to \neg\sim A$ in our semantics. In addition to this, the simple double negations $\neg\neg$, $\sim\sim$ of the same negations must be true, as

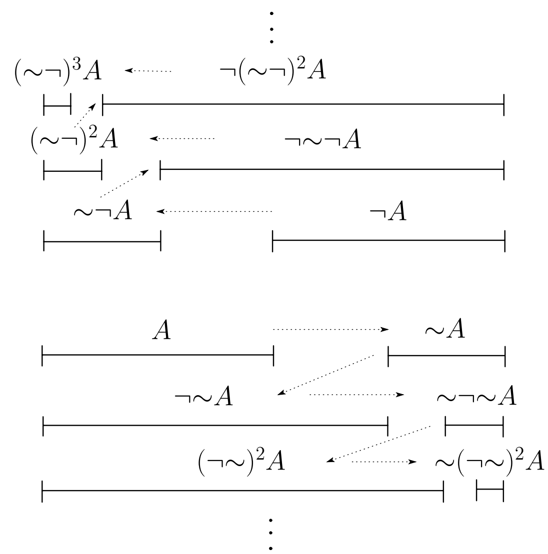
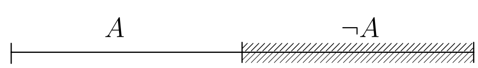


Figure 5: The suitable meaning of strong negation and classical negation

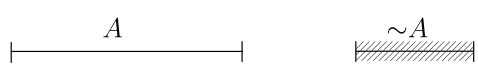
described in Figure 4, i.e., the formulas $\neg\neg A$, $\sim\sim A$ are equivalent to A. For example, the following axioms hold.

$$\begin{array}{rcl} \neg\neg \mathit{happy} & \leftrightarrow & \mathit{happy} \\ \sim\sim \mathit{happy} & \leftrightarrow & \mathit{happy} \end{array}$$

Suitable meaning of the constructive double negation $\sim\neg$ using the description of negation in figures remains to be specified. Let us intuitively fix the properties of classical negation $\neg A$ and strong negation $\sim A$. The following describes the classical negation $\neg A$ to be true in the entire scope for which the positive formula A does not hold, i.e., $\neg A$ and A are mutually complementary.



As shown below, the strong negation $\sim A$ is true in part of the scope complementary to the positive formula A.



In other words, it is partial to the classical negation $\neg A$. In the meaning of these figures, $\sim A \rightarrow \neg A$ holds but $\neg A \rightarrow \sim A$ does not hold.

Based on the description of negation, the interaction of strong negation and classical negation infinitely constructs the scopes for which constructive and weak double negations $(\sim\neg)^n A$, $\sim(\neg\sim)^n A$, $(\neg\sim)^n A$, and $\neg(\sim\neg)^n A$ hold

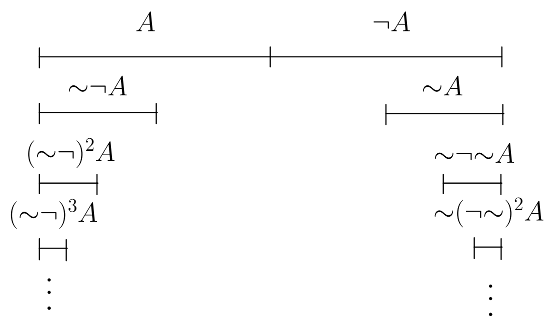


Figure 6: The suitable meaning of constructive double negation

(as in Figure 5). The arrows in Figure 5 explain the constructing processes wherein a formula A leads to the strong negation $\sim A$, and then its complement is the classical negation $\neg\sim A$ of $\sim A$, and so on. In addition, the classical negation $\neg A$ derives its strong negation $\sim\neg A$, and then its complement $\neg\sim\neg A$ can be obtained. As a result, the scopes of $(\sim\neg)^n A$ and $\sim(\neg\sim)^n A$ decrease by a good extent, as in Figure 6, where the constructive double negation $\sim\neg A$ is exclusive to $\neg A$ and partial to $\neg\neg A$.

The properties of the weak and simple double negations $\neg\sim$, $\neg\neg$, and $\sim\sim$ in the semantics of Section 2 are preserved in the descriptive specification of negation, which is employed to define the semantics of first-order logic with constructive double negation.

3.2 Semantics for constructive double negation

We define models for first-order logic with strong negation, classical negation, and constructive double negation.

Definition 3.1 (Model for L with constructive double negation) *A model M^+ for language L with constructive double negation (called an L^+-model) is a tuple $(U, \{I_{(\sim\neg)^i} \mid i \in \omega\}, \{I_{\sim(\neg\sim)^i} \mid i \in \omega\})$ where U is a non-empty set and $I_{(\sim\neg)^i}, I_{\sim(\neg\sim)^i}$ are interpretation functions such that*

1. *for $c \in \mathcal{C}$, $I_{(\sim\neg)^i}(c) \in U$ and $I_{\sim(\neg\sim)^i}(c) \in U$,*
2. *for $f \in \mathcal{F}_n$, $I_{(\sim\neg)^i}(f) : U^n \to U$ and $I_{\sim(\neg\sim)^i}(f) : U^n \to U$,*
3. *for $p \in \mathcal{P}_n$, $I_{(\sim\neg)^i}(p) \subseteq U^n$ and $I_{\sim(\neg\sim)^i}(p) \subseteq U^n$,*
4. *$I_{(\sim\neg)^i}(c) = I_{\sim(\neg\sim)^j}(c)$,*
5. *$I_{(\sim\neg)^i}(f) = I_{\sim(\neg\sim)^j}(f)$,*
6. *$I(p) \cap I_{\sim}(p) = \emptyset$,*

Table 1: Satisfaction for the double negations $\neg\sim$ and $\sim\neg$

0	1	2	$\cdots$	n
$M \models A$	$M \models \sim\neg A$	$M \models (\sim\neg)^2 A$	$\cdots$	$M \models (\sim\neg)^n A$
	$\Updownarrow$	$\Updownarrow$		$\Updownarrow$
	$M \models_{\sim\neg} A$	$M \models_{\sim\neg} \sim\neg A$	$\cdots$	$M \models_{\sim\neg} (\sim\neg)^{n-1} A$
		$\Updownarrow$		$\Updownarrow$
		$M \models_{(\sim\neg)^2} A$	$\cdots$	$M \models_{(\sim\neg)^2} (\sim\neg)^{n-2} A$
		$\Updownarrow$		$\Updownarrow$
		$\cdots$	$\cdots$	$\cdots$
				$\Updownarrow$
				$M \models_{(\sim\neg)^n} A$
$M \models \sim A$	$M \models \sim\neg\sim A$	$M \models \sim(\neg\sim)^2 A$	$\cdots$	$M \models \sim(\neg\sim)^n A$
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$		$\Updownarrow$
$M \models_{\sim} A$	$M \models_{\sim} \neg\sim A$	$M \models_{\sim} (\neg\sim)^2 A$	$\cdots$	$M \models_{\sim} (\neg\sim)^n A$
	$\Updownarrow$	$\Updownarrow$		$\Updownarrow$
	$M \models_{\sim\neg\sim} A$	$M \models_{\sim\neg\sim} \neg\sim A$	$\cdots$	$M \models_{\sim\neg\sim} (\neg\sim)^{n-1} A$
		$\Updownarrow$		$\Updownarrow$
		$\cdots$	$\cdots$	$\cdots$
				$\Updownarrow$
				$M \models_{\sim(\neg\sim)^n} A$

7. $\overline{I_{\sim(\neg\sim)^i}(p)} \cap I_{\sim(\neg\sim)^{i+1}}(p) = \emptyset$,

8. $I_{(\sim\neg)^{i+1}}(p) \cap \overline{I_{(\sim\neg)^i}(p)} = \emptyset$,

where $i, j \in \omega$.

The interpretation functions $I_{(\sim\neg)^i}$ and $I_{\sim(\neg\sim)^i}$ for all $i \in \omega$ interpret the constructive double negations $(\sim\neg)^i A$, $\sim(\neg\sim)^i A$ of any formula A. Constant and function symbols are identically defined in the interpretation functions (in (4) and (5)), but predicate symbols are exclusive in the pairs of $\langle I(p), I_{\sim}(p)\rangle$, $\langle \overline{I_{\sim(\neg\sim)^i}(p)}, I_{\sim(\neg\sim)^{i+1}}(p)\rangle$, and $\langle I_{(\sim\neg)^{i+1}}(p), \overline{I_{(\sim\neg)^i}(p)}\rangle$ (in (6)-(8)). Given an L^+-model $M^+ = (U, \{I_{(\sim\neg)^i} \mid i \in \omega\}, \{I_{\sim(\neg\sim)^i} \mid i \in \omega\})$, the interpretations $[\![\ ]\!]_{(\sim\neg)^i}$ and $[\![\ ]\!]_{(\neg\sim)^i}$ of terms are defined as those provided in Definition 2.4. Using the interpretation, we define the satisfaction relation of an L^+-model and closed formula A as follows.

Definition 3.2 (Satisfaction with constructive double negation) *Let $M^+ = (U, \{I_{(\sim\neg)^i} \mid i \in \omega\}, \{I_{\sim(\neg\sim)^i} \mid i \in \omega\})$ be an L^+-model and A be a closed formula. The satisfaction relations $M^+ \models_{(\sim\neg)^i} A$ and $M^+ \models_{\sim(\sim\neg)^i} A$ are defined by the following rules.*

1. $M^+ \models_{(\sim\neg)^i} p(t_1, \ldots, t_n)$ *iff* $([\![t_1]\!]_{(\sim\neg)^i}, \ldots, [\![t_n]\!]_{(\sim\neg)^i}) \in I_{(\sim\neg)^i}(p)$

2. $M^+ \models_{(\sim\neg)^i} \neg A$ *iff* $M^+ \not\models_{(\sim\neg)^i} A$ $(i = 0)$

3. $M^+ \models_{(\sim\neg)^i} \neg A$ *iff* $M^+ \models_{\sim(\neg\sim)^{i-1}} A$ $(i > 0)$

4. $M^+ \models_{(\sim\neg)^i} \sim A$ *iff* $M^+ \models_{\sim(\neg\sim)^i} A$

5. $M^+ \models_{(\sim\neg)^i} A \wedge B$ *iff* $M^+ \models_{(\sim\neg)^i} A$ *and* $M^+ \models_{(\sim\neg)^i} A$

6. $M^+ \models_{(\sim\neg)^i} A \vee B$ *iff* $M^+ \models_{(\sim\neg)^i} A$ *or* $M^+ \models_{(\sim\neg)^i} A$

7. $M^+ \models_{(\sim\neg)^i} A \rightarrow B$ *iff* $M^+ \not\models_{(\sim\neg)^i} A$ *or* $M^+ \models_{(\sim\neg)^i} A$

8. $M^+ \models_{(\sim\neg)^i} \forall x A$ *iff for all* $d \in U$, $M^+ \models_{(\sim\neg)^i} A\{x/d\}$

9. $M^+ \models_{(\sim\neg)^i} \exists x A$ *iff for some* $d \in U$, $M^+ \models_{(\sim\neg)^i} A\{x/d\}$

10. $M^+ \models_{\sim(\neg\sim)^i} p(t_1, \ldots, t_n)$ *iff* $([\![t_1]\!]_{\sim(\neg\sim)^i}, \ldots, [\![t_n]\!]_{\sim(\neg\sim)^i}) \in I_{\sim(\neg\sim)^i}(p)$

11. $M^+ \models_{\sim(\neg\sim)^i} \neg A$ *iff* $M^+ \models_{(\sim\neg)^{i+1}} A$

12. $M^+ \models_{\sim(\neg\sim)^i} \sim A$ *iff* $M^+ \models_{(\sim\neg)^i} A$

13. $M^+ \models_{\sim(\neg\sim)^i} A \wedge B$ *iff* $M^+ \models_{\sim(\neg\sim)^i} A$ *or* $M^+ \models_{\sim(\neg\sim)^i} A$

14. $M^+ \models_{\sim(\neg\sim)^i} A \vee B$ *iff* $M^+ \models_{\sim(\neg\sim)^i} A$ *and* $M^+ \models_{\sim(\neg\sim)^i} A$

15. $M^+ \models_{\sim(\neg\sim)^i} A \rightarrow B$ *iff* $M^+ \models_{(\sim\neg)^{i+1}} A$ *and* $M^+ \models_{\sim(\neg\sim)^i} A$

16. $M^+ \models_{\sim(\neg\sim)^i} \forall x A$ *iff for some* $d \in U$, $M^+ \models_{\sim(\neg\sim)^i} A\{x/d\}$

17. $M^+ \models_{\sim(\neg\sim)^i} \exists x A$ *iff for all* $d \in U$, $M^+ \models_{\sim(\neg\sim)^i} A\{x/d\}$

Satisfaction and validity on $\models$ (or $\models_{(\sim\neg)^0}$) are defined in the same way as $\models_P$. As shown in Table 1, the satisfaction of formulas of the forms $(\sim\neg)^i A$ and $\sim(\neg\sim)^i A$ is defined by the infinite satisfaction relations $\models_{(\neg\sim)^i}$, $\models_{\sim(\neg\sim)^i}$ with $i \in \omega$. In contrast, the two satisfaction relations $\models$, $\models_\sim$ define the satisfaction of formulas of the forms $\neg\neg A$ and $\sim\sim A$ since we have that $M^+ \models \neg\neg A \Leftrightarrow M^+ \models A$ and $M^+ \models \sim\sim A \Leftrightarrow M^+ \models A$.

Lemma 3.1 *Let A be a closed formula and let $\mathcal{M}(A) = \{M^+ \mid M^+ \models A\}$ and co-$\mathcal{M}(A) = \{M^+ \mid M^+ \not\models A\}$. The following statements hold.*

1. $\mathcal{M}(A) \cap \mathcal{M}(\sim A) = \emptyset$

2. *co-*$\mathcal{M}(\sim(\neg\sim)^i A) \cap \mathcal{M}(\sim(\neg\sim)^{i+1} A) = \emptyset$

3. $\mathcal{M}((\sim\neg)^{i+1} A) \cap$ *co-*$\mathcal{M}((\sim\neg)^i A) = \emptyset$

Proof. This lemma is shown by induction on the structure of a closed formula A. If A is an atomic formula, then by (6)-(8) in Definition 3.1, the three statements are true. In the following, we will show the statements for the cases $A = \neg F$ and $A = \sim F$.

($A = \neg F$). (1) $M^+ \models \neg F$ if and only if $M^+ \not\models F$. By the induction hypothesis, if $M^+ \not\models F$, then $M^+ \not\models \sim\neg F$ (by statement (3)). Hence, we have $M^+ \not\models \sim\neg F$. (2) Let $M^+ \not\models \sim(\neg\sim)^i\neg F$. By the induction hypothesis, if $M^+ \not\models (\sim\neg)^{i+1}F$, then $M^+ \not\models (\sim\neg)^{i+2}F$ (by statement (3)). (3) $M^+ \models (\sim\neg)^{i+1}\neg F$ iff $M^+ \models_{(\sim\neg)^{i+1}} \neg F$ iff $M^+ \models_{\sim(\neg\sim)^i} F$. By the induction hypothesis, if $M^+ \models \sim(\neg\sim)^i F$ ($i = 0$), then $M^+ \not\models F$ (by statement (1)). So $M^+ \models (\sim\neg)^i\neg F$ ($i = 0$). If $M^+ \models \sim(\neg\sim)^i F$ ($i > 0$), then $M^+ \models \sim(\neg\sim)^{i-1}F$ (by statement (2)). Moreover, $M^+ \models_{\sim(\neg\sim)^{i-1}} F$ iff $M^+ \models_{(\sim\neg)^i} \neg F$. Hence, $M^+ \models (\sim\neg)^i\neg F$ ($i > 0$).

($A = \sim F$). (1) Let $M^+ \models \sim F$. By the induction hypothesis, if $M^+ \models \sim F$, then $M^+ \not\models F$ (by statement (1)). Then, $M^+ \not\models_\sim \sim F$. So $M^+ \not\models \sim\sim F$ is derived. (2) $M^+ \not\models \sim(\neg\sim)^i\sim F$ iff $M^+ \not\models_{\sim(\neg\sim)^i} \sim F$ iff $M^+ \not\models_{(\sim\neg)^i} F$ iff $M^+ \not\models (\sim\neg)^i F$. By the induction hypothesis, if $M^+ \not\models (\sim\neg)^i F$, then $M^+ \not\models (\sim\neg)^{i+1}F$ (by statement (3)). So $M^+ \not\models_{(\sim\neg)^{i+1}} F$ iff $M^+ \not\models_{\sim(\neg\sim)^{i+1}} \sim F$. Then we have $M^+ \not\models \sim(\neg\sim)^{i+1}\sim F$. (3) Let $M^+ \models (\sim\neg)^{i+1}\sim F$. By the induction hypothesis, if $M^+ \models \sim(\neg\sim)^{i+1}F$, then $M^+ \not\models \sim(\neg\sim)^i F$ (by statement (2)). It follows that $M^+ \not\models (\sim\neg)^i\sim F$.

Similarly, we can prove the statements for the other cases $F_1 \wedge F_2$, $F_1 \vee F_2$, $F_1 \to F_2$, $\forall x F$ and $\exists x F$.

3.3 Axioms and rules

Axioms and rules for first-order logic with classical negation, strong negation and constructive double negation are arranged as below.

1. all tautologies in classical propositional logic
2. $\vdash \forall x A(x) \to A(t)$
3. $\vdash A(t) \to \exists x A(x)$
4. if $\vdash A$ and $\vdash A \to B$, then $\vdash B$
5. if $\vdash A$, then $\vdash \forall x A(x)$
6. $\vdash \sim A \to (A \to B)$
7. $\vdash (\sim\neg)^i\sim(A \to B) \leftrightarrow (\sim\neg)^i(A \wedge \sim B)$
8. $\vdash (\sim\neg)^i\sim(A \wedge B) \leftrightarrow (\sim\neg)^i(\sim A \vee \sim B)$
9. $\vdash (\sim\neg)^i\sim(A \vee B) \leftrightarrow (\sim\neg)^i(\sim A \wedge \sim B)$

10. $\vdash \sim(\neg\sim)^i\neg(A \to B) \leftrightarrow \sim(\neg\sim)^i(A \wedge \neg B)$

11. $\vdash \sim(\neg\sim)^i\neg(A \wedge B) \leftrightarrow \sim(\neg\sim)^i(\neg A \vee \neg B)$

12. $\vdash \sim(\neg\sim)^i\neg(A \vee B) \leftrightarrow \sim(\neg\sim)^i(\neg A \wedge \neg B)$

13. $\vdash (\sim\neg)^i(A \leftrightarrow \sim\sim A)$

14. $\vdash \sim(\neg\sim)^i(A \leftrightarrow \neg\neg A)$

15. $\vdash \sim\forall x A(x) \leftrightarrow \exists x{\sim}A(x)$

16. $\vdash \sim\exists x A(x) \leftrightarrow \forall x{\sim}A(x)$

17. $\vdash A \to \neg\sim A$

18. $\vdash \sim\neg A \to A$

Axioms and rules (1)-(5) are valid in classical first-order logic, and axioms (6)-(18) are additionally introduced with respect to strong negation and constructive double negation.

4 Completeness

Let T be a set of closed formulas in a language L (called a theory in L). T is inconsistent if $T \vdash A$ and $T \vdash \neg A$, and it is consistent otherwise. T is complete if T is consistent and $T \vdash A$ or $T \vdash \neg A$ for any closed formula A. T contains witnesses if for every existential formula $\exists x A[x]$ in T, we have $\exists x A[x] \to A[c] \in T$.

Proposition 4.1 *Axioms (1)-(3),(6)-(18) for first-order logic with constructive double negation are valid.*

Proof. The validity of axioms (6), (17) can be shown by statement (1) in Lemma 3.1. Axiom (18) is valid by statement (3) in Lemma 3.1. Also it can be proved for axioms (1)-(3), (7)-(16).

Theorem 4.1 (Soundness) *Let T be a consistent theory and A be a closed formula. If $T \vdash A$, then $T \models A$.*

Proof. By Proposition 4.1, this is shown by induction on the length n of a derivation of A from T.

Theorem 4.2 *Let T be a theory and A be a formula. If $T \vdash A$, then there exists a finite subset T' of T such that $T' \vdash A$.*

Corollary 4.1 *For every $i \in \omega$, let T_i be a theory which is consistent. If for $i < j$, $T_i \subseteq T_j$, then the theory $T = \bigcup_{n\in\omega} T_n$ is consistent.*

Theorem 4.3 (Deduction theorem) *Let A be a closed formula. If $T \cup \{A\} \vdash B$, then $T \vdash A \to B$.*

Proofs of Theorem 4.2, Corollary 4.1 and Theorem 4.3 can be found in [3].

Corollary 4.2 *Let A be a closed formula. $T \vdash A$ if and only if $T \cup \{\neg A\}$ is inconsistent.*

Proof. By Theorem 4.3, we can show this.

In order to prove the completeness of our logical system, we will show the model existence theorem for a complete theory T.

Theorem 4.4 (Model existence theorem) *If T is a complete theory that contains witnesses, then T has an L^+-model.*

Proof. We construct a canonical model $M_{\mathcal{C}}^+ = (\mathit{TERM}_0, \{I_{(\sim\neg)^i} \mid i \in \omega\}, \{I_{\sim(\neg\sim)^i} \mid i \in \omega\})$ for L with constructive double negation as follows:

1. TERM_0 is the set of ground terms in L.
2. $I_{(\sim\neg)^i}(c) = I_{\sim(\neg\sim)^i}(c) = c$
3. $I_{(\sim\neg)^i}(f)(t_1, \ldots, t_n) = I_{\sim(\neg\sim)^i}(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$
4. $(t_1, \ldots, t_n) \in I_{(\sim\neg)^i}(p)$ iff $p(t_1, \ldots, t_n) \in \Delta_{(\sim\neg)^i}(T)$
5. $(t_1, \ldots, t_n) \in I_{\sim(\neg\sim)^i}(p)$ iff $p(t_1, \ldots, t_n) \in \Delta_{\sim(\neg\sim)^i}(T)$

where $t_1, \ldots, t_n \in \mathit{TERM}_0$, $\Delta_{(\sim\neg)^i}(T) = \{A \mid T \vdash (\sim\neg)^i A\}$ and $\Delta_{\sim(\neg\sim)^i}(T) = \{A \mid T \vdash \sim(\neg\sim)^i A\}$.

Let A be a closed formula in T. We will show the following claim:

$$(i) A \in \Delta_{(\sim\neg)^i}(T) \quad \textit{iff} \quad M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} A$$
$$(ii) A \in \Delta_{\sim(\neg\sim)^i}(T) \quad \textit{iff} \quad M_{\mathcal{C}}^+ \models_{\sim(\neg\sim)^i} A$$

($A = F_1 \wedge F_2$). (i) Let $F_1 \wedge F_2 \in \Delta_{(\sim\neg)^i}(T)$. We have $T \vdash (\sim\neg)^i(F_1 \wedge F_2)$. By the tautology $A \wedge B \to A$ and axioms (7)-(12), $T \vdash (\sim\neg)^i F_1$ and $T \vdash (\sim\neg)^i F_2$. By the induction hypothesis, $M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} F_1$ and $M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} F_2$. Hence, $M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} F_1 \wedge F_2$. The other direction can be proved by the tautology $A \to (B \to (A \wedge B))$ and Axioms (7)-(12).

(ii) Suppose we have $F_1 \wedge F_2 \in \Delta_{\sim(\neg\sim)^i}(T)$. Then, $T \vdash \sim(\neg\sim)^i(F_1 \wedge F_2)$ if and only if $T \vdash \sim(\neg\sim)^i F_1 \vee \sim(\neg\sim)^i F_2$. If $T \not\vdash \sim(\neg\sim)^i F_1$, then $T \vdash \neg\sim(\neg\sim)^i F_1$ (because T is complete). Then, the tautology $(A \vee B) \to (\neg A \to B)$ derives $T \vdash \sim(\neg\sim)^i F_2$. By the induction hypothesis, $M_{\mathcal{C}}^+ \models_{\sim(\neg\sim)^i} F_2$. Therefore, $M_{\mathcal{C}}^+ \models_{\sim(\neg\sim)^i} F_1 \wedge F_2$. The other direction is easier.

($A = \sim F_1$). (i) $\sim F_1 \in \Delta_{(\sim\neg)^i}(T)$ iff $T \vdash \sim(\neg\sim)^i F_1$ iff $F_1 \in \Delta_{\sim(\neg\sim)^i}$ iff $M_{\mathcal{C}}^+ \models_{\sim(\neg\sim)^i} F_1$ (by the induction hypothesis) iff $M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} \sim F_1$. (ii) $\sim F_1 \in \Delta_{\sim(\neg\sim)^i}(T)$ iff $T \vdash (\sim\neg)^i \sim\sim F_1$ iff $F_1 \in \Delta_{(\sim\neg)^i}$ (by Axiom (13)) iff $M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} F_1$ (by the induction hypothesis) iff $M_{\mathcal{C}}^+ \models_{\sim(\neg\sim)^i} \sim F_1$.

($A = \neg F_1$). (i) we treat the cases $i = 0$ and $i > 0$. Let $i = 0$. $\neg F_1 \in \Delta(T)$ iff $T \vdash \neg F_1$ iff $T \not\vdash F_1$ (by the fact that T is complete) iff $M_{\mathcal{C}}^+ \not\models F_1$ (by the induction hypothesis) iff $M_{\mathcal{C}}^+ \models \neg F_1$. Let $i > 0$. $\neg F_1 \in \Delta_{(\sim\neg)^i}(T)$ iff $T \vdash (\sim\neg)^i \neg F_1$ iff $T \vdash \sim(\neg\sim)^{i-1} F_1$ (by Axiom (14)) iff $M_{\mathcal{C}}^+ \models_{\sim(\neg\sim)^{i-1}} F_1$ (by the induction hypothesis) iff $M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} \neg F_1$. (ii) $\neg F_1 \in \Delta_{\sim(\neg\sim)^i}(T)$ iff $T \vdash (\sim\neg)^{i+1} F_1$ iff $M_{\mathcal{C}}^+ \models_{(\sim\neg)^{i+1}} F_1$ (by the induction hypothesis) iff $M_{\mathcal{C}}^+ \models_{\sim(\neg\sim)^i} \neg F_1$.

($A = \forall x F_1[x]$). (i) Let $\forall x F_1 \in \Delta_{(\sim\neg)^i}(T)$. So $T \vdash (\sim\neg)^i \forall x F_1$. By Axioms (1),(2),(15),(16), we have $T \vdash (\sim\neg)^i F_1[t]$ for any $t \in \mathit{TERM}_0$. By the induction hypothesis, for any $t \in \mathit{TERM}_0$, $M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} F_1[t]$. Then, $M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} \forall x F_1$.

In the other direction, suppose $\forall x F_1 \notin \Delta_{(\sim\neg)^i}(T)$. Since T is complete, $T \vdash \neg(\sim\neg)^i \forall x F_1$. By Axioms (1), (15), (16), $T \vdash \exists x \neg(\sim\neg)^i F_1$. Because T contains witnesses, $T \vdash \neg(\sim\neg)^i F_1[c]$ is obtained. Then, $T \not\vdash (\sim\neg)^i F_1[c]$. By the induction hypothesis, $M_{\mathcal{C}}^+ \not\models_{(\sim\neg)^i} F_1[c]$. Hence, $M_{\mathcal{C}}^+ \not\models_{(\sim\neg)^i} \forall x F_1$.

(ii) Let $\forall x F_1 \in \Delta_{\sim(\neg\sim)^i}(T)$. We have $T \vdash \sim(\neg\sim)^i \forall x F_1$. Axioms (1), (15), (16) and witnesses of T derive $T \vdash \sim(\neg\sim)^i F_1[c]$. By the induction hypothesis, $M_{\mathcal{C}}^+ \models_{\sim(\sim\neg)^i} F_1[c]$. Hence, $M_{\mathcal{C}}^+ \models_{\sim(\neg\sim)^i} \forall x F_1[x]$. Also, the other direction can be proved.

Similar to these, the statements in the other cases $F_1 \vee F_2$, $F_1 \to F_2$ and $\exists x F_1$ can be derived. Then, we obtain the conclusion that $T \vdash F$ iff $M_{\mathcal{C}}^+ \models F$.

Moreover, to prove that the canonical model is an L^+-model for L with constructive double negation, the sets $I(p) \cap I_{\sim}(p)$, $\overline{I_{\sim(\neg\sim)^i}(p)} \cap I_{\sim(\neg\sim)^{i+1}}(p)$ and $I_{(\sim\neg)^{i+1}}(p) \cap \overline{I_{(\sim\neg)^i}(p)}$ must be empty (by Definition 3.1).

Let $\vec{t} \in I(p)$. Then, $T \vdash p(\vec{t})$ and, by Axiom (17), $T \vdash p(\vec{t}) \to \neg\sim p(\vec{t})$. So, since $T \vdash \neg\sim p(\vec{t})$, we have $M_{\mathcal{C}}^+ \models \neg\sim p(\vec{t})$. Hence, $M_{\mathcal{C}}^+ \not\models_{\sim} p(\vec{t})$.

Let $\vec{t} \notin I_{\sim(\neg\sim)^i}(p)$. By definition, $T \not\vdash \sim(\neg\sim)^i p(\vec{t})$. Because T is complete, $T \vdash \neg\sim(\neg\sim)^i p(\vec{t})$. By Axiom (17), $T \vdash (\neg\sim)^{i+1} p(\vec{t}) \to (\neg\sim)^{i+2} p(\vec{t})$. So we have $T \vdash (\neg\sim)^{i+2} p(\vec{t})$. Then, $M_{\mathcal{C}}^+ \not\models \sim(\neg\sim)^{i+1} p(\vec{t})$. Thus, $M_{\mathcal{C}}^+ \not\models_{\sim(\neg\sim)^{i+1}} p(\vec{t})$.

Let $\vec{t} \in I_{(\sim\neg)^{i+1}}(p)$. By definition, $T \vdash (\sim\neg)^{i+1} p(\vec{t})$. By Axiom (18), $T \vdash (\sim\neg)^{i+1} p(\vec{t}) \to (\sim\neg)^i p(\vec{t})$. Then, $T \vdash (\sim\neg)^i p(\vec{t})$. So, we have $M_{\mathcal{C}}^+ \models (\sim\neg)^i p(\vec{t})$. Therefore, $M_{\mathcal{C}}^+ \models_{(\sim\neg)^i} p(\vec{t})$.

Lemma 4.1 *Let T be a consistent theory in L. Then, there exists a complete theory that contains witnesses.*

Proof. Let $F_0, F_1, \dots$ be an enumeration of the closed formulas in L^* where $L^* = L \cup \{c_0, c_1, \dots\}$, and let $Cons(F_i)$ denote the set of constants occurring in F_i. The following procedure generates a theory T_n:

1. $T_0 = T$ and $L_0 = L$.
2. $T_{n+1} = T_n \cup \{\neg F_n\}$ and $L_{n+1} = L_n \cup Cons(F_n)$ if $T_n \cup \{F_n\}$ is inconsistent.
3. $T_{n+1} = T_n \cup \{F_n\}$ and $L_{n+1} = L_n \cup Cons(F_n)$ if $T_n \cup \{F_n\}$ is consistent and F_n is not of the form $\exists x G[x]$.
4. $T_{n+1} = T_n \cup \{F_n, G[c]\}$ with $c \in L^* - L_n$ and $L_{n+1} = L_n \cup Cons(F_n) \cup \{c\}$ if $T_n \cup \{F_n\}$ is consistent and F_n is of the form $\exists x G[x]$.

If $T_n \cup \{F_n\}$ is inconsistent, then $T_n \vdash \neg F_n$ in L_{n+1} (by Corollary 4.2). Since T_n is consistent and $T_n \not\vdash F$ in L_{n+1}, $T_{n+1} = T_n \cup \{\neg F_n\}$ is consistent. In the case $T_{n+1} = T_n \cup \{\exists x G[x], G[c]\}$, suppose T_{n+1} is inconsistent. By Corollary 4.2, we have $T_n \cup \{\exists x G[x]\} \vdash \neg G[c]$ in L_{n+1}. In [3], if $T \vdash \neg F[c]$ in $L \cup \{c\}$, then $T \vdash \neg \exists x F[x]$ in L. Hence, $T_n \cup \{\exists x G[x]\} \vdash \neg \exists x G[x]$ in L_n. This is contradictory. So T_{n+1} is consistent.

We can construct the following sets:

$$T^* = \bigcup_{n \in \omega} T_n \textit{ and } L^* = \bigcup_{n \in \omega} L_n$$

Suppose that the theory T^* is inconsistent. We have $T^* \vdash F$ and $T^* \vdash \neg F$. By Corollary 4.1, there exists a theory T_i that is inconsistent. This is contradictory. Hence, T^* is consistent.

To prove that T^* is complete, let A be a closed formula in the language L^*. Then, there exists $F_n = A$ such that either $F_n \in T_{n+1}$ or $\neg F_n \in T_{n+1}$ in the construction of T_n.

Next we will show T^* contains witnesses. Let $\exists x G[x]$ be a closed formula in the language L^*. Then, there exists $F_n = \exists x G[x]$, and by the construction of T_n, either $\neg F_n \in T_{n+1}$ or $F_n, G[c] \in T_{n+1}$. Thus, we have $T_n \vdash \exists x G[x] \to G[c]$, and therefore $T_n \not\vdash \neg(\exists x G[x] \to G[c])$ in L_n. By Corollary 4.2, $T_n \cup \{\exists x G[x] \to G[c]\}$ is consistent. Hence, $\exists x G[x] \to G[c] \in T^*$.

Theorem 4.5 *Every consistent theory has an L^+-model.*

Proof. By Lemma 4.1 and Theorem 4.4, this can be proved.

Theorem 4.6 (Completeness) *Let T be a consistent theory and A be a closed formula. If $T \models A$, then $T \vdash A$.*

Proof. Assume we have $T \not\vdash A$. Then, by Corollary 4.2, $T \cup \{\neg A\}$ is consistent. By Theorem 4.5, $T \cup \{\neg A\}$ has an L^+-model. Therefore, we obtain $T \not\models A$.

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