Complexity Theory for Map-Reduce

Communication and Computation Costs
Enumerating Triangles and Other Sample Graphs
Theory of Mapping Schemas
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Cost of Map-Reduce Computations

- \textit{Communication cost} = number of key-value pairs sent to reducers.
- \textit{Computation cost} = execution time at reducers.
  - Computation at mappers normally proportional to communication cost.
Costs – Observations

- These costs are what you pay for at EC2.
- Often, communication cost dominates.
- Communication cost typically grows with the number of Reduce tasks.
- But latency shrinks with the number of tasks, so there is a tradeoff to be made.
Why One Round?

◆ “Other things being equal,” it saves communication.

◆ But really: whatever you do with map-reduce, each round does something that you can study and perform as well as possible.
Finding All Instances of a Sample Graph

Communication Cost: Multiway Joins and Conjunctive Queries

Computation Cost: “Convertible Algorithms,” Graph Decompositions
Given a data graph, find all triples of nodes that form a triangle.

Use one round of map-reduce.

Data graph represented by relation $E(A,B)$.
- $A, B$ are nodes, and $A < B$ (some order).
- $(A,B)$ is an edge.
Partition Method (Suri-Vassilvitskii)

- Partition nodes into $b$ groups $S_1, \ldots, S_b$.
- Each reducer responsible for a set of three groups.
- Map to reducer $\{i, j, k\}$ all edges whose nodes are both in the union of $S_i, S_j, S_k$.
- Each reducer has a little graph – finds the triangles in that graph.
Partition Method – (2)

- An edge whose ends are in different groups is sent to (only) b-2 reducers.
- But an edge with both ends in the same group goes to \( \binom{b-1}{2} \) reducers.
- Communication cost (asymptotically) $3b/2$ per edge.
Convention

◆ Data graph has $n$ nodes and $m$ edges; sample graph has $p$ nodes.
  - $p = 3$ for triangle.
Our Approach

- Represent triangle-finding by a CQ
  \[ E(X,Y) \land E(X,Z) \land E(Y,Z) \land X < Y < Z. \]
- Use multiway join (Afrati & U, 2010).
- Hash nodes to b buckets.
- Reducer <-> list of buckets for X, Y, Z.
- **Trick**: < for nodes = bucket number.
  - Resolve ties by name of node.
Our Approach – (2)

- As a result, reducer \([i,j,k]\) gets data only if \(i < j < k\).
- Number of needed reducers = \(\binom{b+2}{3}\), or approximately \(b^3/6\).
- Each edge goes to exactly \(b\) reducers.
  - Which ones? \(\text{Sort}(\text{node1, node2, any})\).
- Communication cost \(bm\), vs. \(3bm/2\) (for the same number of reducers).
Generalization to All Sample Graphs

- For an arbitrary sample graph, we need one CQ for each order of the nodes.
  - \( p! \) CQ’s, in principle.
- But the sample graph may have a nontrivial automorphism group.
- **Example**: square has \( 4! = 24 \) orders but 8 automorphisms.
  - Rotate to 4 positions, flip or don’t.
Generalization – (2)

- We want only one CQ for each member of the quotient group (permutations/automorphisms).

**Example:** square

\[
\begin{align*}
E(W,X) \land E(X,Y) \land E(Y,Z) \land E(W,Z) & \land W<X<Y<Z \\
E(W,X) \land E(Y,X) \land E(Y,Z) \land E(W,Z) & \land W<Y<X<Z \\
E(W,X) \land E(X,Y) \land E(Z,Y) \land E(W,Z) & \land W<X<Z<Y
\end{align*}
\]
Generalization – (3)

- Implement with one reducer for each nondecreasing sequence of p integers in the range \([1, b]\) (number of buckets).
- That reducer gets all edges \((i, j)\) if \(i < j\) and buckets of \(i\) and \(j\) are both in that sequence of integers.
- This reducer implements each of the conjunctive queries on its data.
Generalization – (4)

- Asymptotically $b^p/p!$ reducers.
- Asymptotically beats generalized partition (reducer <-> set of $p$ blocks) by a small factor $1 + 1/(p-1)$. 
Convertible Algorithms

A serial algorithm is *convertible* (wrt a strategy for creating key-value pairs) if the total computation time of this algorithm at the reducers is of the same order as the serial algorithm.
Convertible Algorithms – (2)

Assuming random distribution of edges, a serial algorithm running in time $n^a m^b$ is convertible (with respect to partition or our scheme) iff $p \leq a + 2b$.

For triangles, $O(m^{3/2})$ is achievable and best possible, so convertible.

$3 \leq 0 + 2(3/2)$. 
Convertible Serial Algorithms

- There is an $O(m^{p/2})$ algorithm for many sample graphs.
  - Graphs with a Hamilton cycle.
  - Single edges.
  - Any combination of these.
    - Take union of graphs.
    - Throw in any additional edges you like.
Example
What If No Such Decomposition?

- If there are q isolated nodes after the best decomposition, then there is a serial algorithm with running time $O(n^q m^{(p-q)/2})$.
- All these algorithms are best possible (Noga Alon 1981).
  - They match the output size.
- All these algorithms are convertible.
Limited-Degree Data Graphs

◆ If there are no nodes of degree $\geq \sqrt{m}$, then for every connected sample graph there is a serial algorithm that runs in time $O(m^{p/2})$.

◆ Again – convertible.
Mapping Schemas

Definition

Examples: Triangles and Hamming Distance

A Lower Bound
Comments

- Ideas are very new, not published or even written up.
- Approach originated with Anish das Sarma.
- We have results for finding sample graphs, Hamming distance, and containment join.
- We welcome work in this area.
Definition of Mapping Schema

- Set of inputs (that may be present, depending on the input data).
  - Distinction: for triangles, every possible edge is an “input”; some will really be there in any data set.

- Set of outputs.

- For each output: a set of inputs that must be present for that output to be made.
Example: Mapping Schema for Triangles

- **Inputs** = edges = pairs of nodes.
- **Outputs** = triangles = sets of three input edges that must be present for that triangle to be present in the graph.
Example: Mapping Schema for Hamming Distance = 1

- Inputs = binary strings of length b.
- Outputs = pairs of inputs of Hamming distance 1.
Mapping-Schema Optimization Problem

- Use $p$ reducers.
- Each reducer assigned at most $q$ inputs.
- For each output, its set of inputs must be contained in the set of inputs assigned to at least one reducer.
- Find input-$\rightarrow$reducer assignment to minimize replication $= pq$ divided by the number of inputs.

$\bullet = \text{communication cost per input.}$
Lower Bound for HD =1

- **Theorem** (Semih Salihoglu): if a reducer gets $q$ inputs, the maximum number of output sets it can cover is $(q/2) \log_2 q$.

- Since there are $(b/2)2^b$ outputs: $p(q/2) \log_2 q \geq (b/2)2^b$.

- Replication $= pq/2^b \geq b/\log_2 q$. 
One reducer for each output

Splitting generalizes: replication = i ≥ 2; \( \log_2 q = \frac{b}{i} \).

Splitting: each string sent to one reducer for its first \( \frac{b}{2} \) bits, another for its last \( \frac{b}{2} \) bits.

All inputs to one reducer

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Communication/Computation Tradeoff
Research Program

1. Get upper/lower bounds on communication/reducer-size tradeoff for many different problems.

2. Relate structure of mapping schema to costs.
   - E.g., how does size of min-cuts relate to replication.