Chapter 3  Quantum Circuit Model of Quantum Computation

A standard way to implement the necessary unitary evolution in quantum algorithms is to use an universal gate set. In his chapter we will review this quantum circuit model of quantum computers.

3.1 Universal gate set


Arbitrary unitary operation on n-qubit system can be decomposed with

\[ \hat{U}(2^n) \rightarrow \hat{U}(2) + C - NOT \]

where \( \hat{U}(2) \) is a one-bit gate, \( C - NOT \) is a two-bit gate, and all other gates are universal gates.

3.2 Various quantum gates

3.2.1 \( m \)-Controlled-U gate \( \Lambda_m(U) \)

\[ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ u_{00} \\ u_{01} \\ u_{10} \\ u_{11} \end{pmatrix} \]

\[ \Lambda_m(U) |11 \cdots 1,0\rangle = u_{00} |11 \cdots 1,0\rangle + u_{10} |11 \cdots 1,1\rangle \]

\[ \Lambda_m(U) |11 \cdots 1,1\rangle = u_{01} |11 \cdots 1,0\rangle + u_{11} |11 \cdots 1,1\rangle \]

If \( \prod_k x_k = 1 \), \[ \Lambda_m(U) |x_1 x_2 \cdots x_m, y\rangle = |x_1 x_2 \cdots x_m, y\rangle \]

If \( \prod_k x_k = 0 \), \[ \Lambda_m(U) |x_1 x_2 \cdots x_m, y\rangle = |x_1 x_2 \cdots x_m, y\rangle \]
**3.2.2 one-bit gate: \( U(2) \) and \( SU(2) \)**

arbitrary \( 2 \times 2 \) unitary matrix \( U(2) = \Phi(\delta)R_z(\alpha)R_y(\theta)R_z(\beta) \)

special unitary matrix \( SU(2) = R_z(\alpha)R_y(\theta)R_z(\beta) \)

with \( \text{det}(M) = 1 \)

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**Example:**

Controlled-\( NOT \) (XOR)  
Controlled-\( Controlled-NOT \) (Toffoli)

\[ \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  
in the above.

[Diagram of a qubit with control and target gates, showing initial and final states, with rotations around z and y axes labeled with \( \alpha, \beta, \theta \).]
3.2.3 Two bit gate

\[
A = R_z(\alpha)R_y\left(\frac{\theta}{2}\right)
\]

\[
B = R_y\left(-\frac{\theta}{2}\right)R_z\left(-\frac{\alpha + \beta}{2}\right)
\]

\[
C = R_z\left(-\frac{\alpha - \beta}{2}\right)
\]

\[
A\sigma_x B\sigma_x C = R_z(\alpha)R_y\left(\frac{\theta}{2}\right)\sigma_x R_y\left(-\frac{\theta}{2}\right)R_z\left(-\frac{\alpha + \beta}{2}\right)\sigma_x R_z\left(-\frac{\alpha - \beta}{2}\right)
\]

\[
\hat{I} = \sigma_x \sigma_x
\]

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

π-rotation about x-axis = NOT gate

\[
\sigma_x R_y(\theta)\sigma_x = R_y(-\theta)
\]

\[
\sigma_x R_z(\alpha)\sigma_x = R_z(-\alpha)
\]

\[
= R_z(\alpha)R_y\left(\frac{\theta}{2}\right)R_y\left(\frac{\theta}{2}\right)R_z\left(\frac{\alpha + \beta}{2}\right)R_z\left(-\frac{\alpha - \beta}{2}\right)
\]

\[
= R_z(\alpha)R_y(\theta)R_z(\beta)
\]

\[
= W
\]

\[
\]

controlled-SU(2) gate
\[
x y \quad S \quad = \quad E
\]
controlled-phase shift-gate

\[
\begin{pmatrix}
1 & 0 \\
1 & e^{i\delta} \\
0 & e^{i\delta}
\end{pmatrix}
\]

\[
x y \quad U \quad = \quad A \oplus B \oplus C
\]
controlled-\(U\) gate

4 one-bit gates + 2 C-NOT gates

Applications of two-bit gate:

\[
x \oplus y \quad (\alpha |0\rangle + \beta |1\rangle)_x |0\rangle_y \\
\rightarrow \alpha |0\rangle_x |0\rangle_y + \beta |1\rangle_x |1\rangle_y
\]
entangler

\[
(x y \quad (\alpha |0\rangle + \beta |1\rangle)_x \otimes (\gamma |0\rangle + \delta |1\rangle)_y \\
\rightarrow (\gamma |0\rangle + \delta |1\rangle)_x \otimes (\alpha |0\rangle + \beta |1\rangle)_y
\]
swapping
3.2.4 Three bit gate

\[ V^2 = U \]

(1) \( V \) is applied to the target iff \( x_1 = 1 \)  
\( \iff \quad V^{x_1} \)

(2) \( V^+ \) is applied to the target iff \( x_1 \oplus x_2 = 1 \)  
\( \iff \quad (V^+)_{x_1 \oplus x_2} \)

(3) \( V \) is applied to the target iff \( x_2 = 1 \)  
\( \iff \quad V^{x_2} \)

\[
V^{x_1} (V^+)_{x_1 \oplus x_2} V^{x_2} = V^{x_1 + x_2 - x_1 \oplus x_2} = (V^2)^{x_1 \land x_2}
\]

\[
2x_1 \land x_2 = x_1 + x_2 - (x_1 \oplus x_2)
\]

controlled-controlled-\( U \) gate

1. The last one bit gate (C) in \( \Lambda_1(V) \) is cancelled with the first one bit gate \( (C^+) \) in \( \Lambda_1(V^+) \).

2. 

\[
\begin{array}{c}
12 - 2 \times 2 = 2 + 1 + 2 + 1 + 2 - 2
\end{array}
\]
3.2.5 n-bit gate

\( \Lambda_m (\sigma_x) \): \( m \)-controlled-NOT gate

\[
\begin{array}{c c c c c c c c c c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

- control bits
- work bits
- target bit

\( n \geq 5, \quad m \in \{3, \ldots, [n/2] \} \)

constructed by

\( 4 \times (m - 2) \Lambda_2 (\sigma_x) \):controlled-controlled-NOT gates

(example: \( n = 9, \ m = 5 \))

Iff \( x_1 = x_2 = x_3 = x_4 = x_5 = 1 \), one of the two 5-8-9 \( \Lambda_2 (\sigma_x) \) gates flipps the target 9.

(1) \( \Lambda_{n-2} (\sigma_x) \{ n > 5, \ m \in [2, \ldots, n - 3] \} \) can be constructed by

\( 2 \Lambda_m (\sigma_x) \) gates and \( 2 \Lambda_{n-m-1} (\sigma_x) \) gates

(example: \( n = 9, \ m = 5 \))
\[
\begin{align*}
2 \Lambda_m(\sigma_x) & \implies 2 \times 4 \times (m-2) \Lambda_2(\sigma_x) \\
2 \Lambda_{n-m-1}(\sigma_x) & \implies 2 \times 4 \times (n-m-3) \Lambda_2(\sigma_x) \\
& \quad \left\{ \begin{array}{l}
8 \times (n-5) \Lambda_2(\sigma_x) \\
\downarrow \Lambda_2(\sigma_x) \quad 4 \text{ one-bit gate} \\
2 \text{ C-NOT gate}
\end{array} \right.
\end{align*}
\]

32\((n - 5)\) one-bit gates
16 \((n - 5)\) C-NOT gates
\[
\downarrow \text{ scales } \sim O(n)
\]

One extra bit is required.

\[
(2) \quad \Lambda_{n-2}(U)
\]

\[
\begin{align*}
2 \Lambda_{n-2}(\sigma_x) & \implies 16 \times (n-5) \Lambda_2(\sigma_x) \\
& \quad \downarrow \\
& \quad \text{64\((n - 5)\) one-bit gates} \\
& \quad \text{32 \((n - 5)\) C-NOT gates} \\
\text{Controlled-}\text{-}U \text{ gate} & \implies 4 \text{ one-bit gate} \\
& \quad \downarrow \\
& \quad 2 \text{ C-NOT gate}
\end{align*}
\]

One initialized extra bit is required.
3.3 Implementation of Quantum Algorithms

3.3.1 Implementation of Deutsch-Josza algorithm

2-bit Boolean functions

\[ f(x) = x_1x_2 + x_1x_2 \]

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<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( f_c )</th>
<th>( f_{c1} )</th>
<th>( f_{c2} )</th>
<th>( f_{c3} )</th>
<th>( f_{c4} )</th>
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<td>1</td>
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Implementation of \( f_{x_1 \oplus x_2} = x_1x_2 + x_1x_2 \)

Canonical Sum of Product (CSOP) form of a Boolean function

\[ f(x) = \{x_1x_2, x_1x_2, x_1x_2, x_1x_2\} : n = 2 \text{ bit} \]

\[ f(x) = \sum_{y=0}^{2^n-1} a_y y, \quad a_y \in \{0, 1\} \]

Each term with \( a_y = 1 \) is implemented by \( \Lambda_n(\sigma_x) \) gate.
Construction of one term in \( f(x) \) requires at most,

\[ 2n \text{ NOT gates} + \text{one } \Lambda_n(\sigma_x) \text{ gate} \]

\[ 32 \ (n - 5) \quad \text{one-bit gates} \]
\[ 16 \ (n - 5) \quad \text{C-NOT gates} \]

There are \( 2^{n-1} \) terms in \( f(x) \) for balanced functions, which have \( a_y = 1 \) and so must be implemented. The other \( 2^{n-1} \) terms have \( a_y = 0 \), so need not be implemented.

\[ \sim O(n \ 2^{n-1}) \text{ one-bit gates and C-NOT gates} \]

This is a severe disadvantage of D-J algorithm.

### 3.3.2 Implementation of Grover algorithm

\[ |x\rangle \rightarrow (-1)^{f(x)} |x\rangle \]

\( n \)-bit Boolean function \( f(x) = y \)

There is only one term instead of \( 2^{n-1} \)

\( \Lambda_n(\sigma_x) \) gate

\[ \sim O(n) \text{ one-bit gates and C-NOT gates} \]

However, this oracle must be repeated \( \sim \sqrt{N(=2^{4^2})} \) times.
Physical Qubits
— Cavity QED Systems with Single-Electron-Doped Quantum Dots —

A post-microcavity with top and bottom DBRs and self-assembled InGaAs QDs


A simple planar microcavity with 2D lattice of site-controlled QDs


Appendix: Optically controlled quantum dot spin
Magnetic Spectrum of Charged Exciton (Trion) in InAs Quantum Dot — Artificial Three-Level Atom in Lambda Configuration —

Magnetic field in Voigt geometry

Trion $X^-$ electron spins in singlet spin is governed by heavy hole

$\delta_h = -\mu_B g_h B_{\text{ext}}$

$\delta_e = \mu_B g_e B_{\text{ext}}$

Electron $e^-$


Appendix: Optically controlled quantum dot spin
Ultra-fast Spin Rotation with Single Optical Pulse

• A single broadband optical pulse can implement an arbitrary one-bit gate with fidelity of 0.999.

\[ \text{If } \delta \ll \Omega_0, \Omega_1 \ll \Delta, \text{ an effective Rabi frequency} \]

\[ \Omega_{\text{eff}} = \frac{\Omega_0 \Omega_1^*}{2\Delta} \approx \frac{|\Omega(t)|^2}{2\Delta} \]

\[ \Omega(t) = \frac{\mu E(t)}{\hbar} \]

rotation angle \( \int \Omega_{\text{eff}} dt \) is proportional to pulse energy

• A system clock is provided by the pulse arrival time from the mode-locked laser.

Arbitrary single qubit gates SU(2) can be implemented in one-half of Larmor oscillation period.

**Experiment with an ensemble of donor spins:** K.M. Fu et al., Nature Physics 4, 780 (2008)

Appendix: Optically controlled quantum dot spin
Control optical pulse energy reduced by a cavity: 1/300
Optical pump pulse completely off during single qubit operation

Coherent Rabi oscillation experiment

<table>
<thead>
<tr>
<th>Rotation pulse power, $P_{RP}$ (mW)</th>
<th>Count rate ($10^4 s^{-1}$)</th>
</tr>
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<td>6</td>
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<tr>
<td>8</td>
<td>4</td>
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Ramsey Interference experiment

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<th>Time delay, $\tau$ (ps)</th>
<th>Count rate ($10^4 s^{-1}$)</th>
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</thead>
<tbody>
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<td>0</td>
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<tr>
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<td>1</td>
</tr>
<tr>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>150</td>
<td>3</td>
</tr>
</tbody>
</table>

Spin rotation during control pulse
(Larmor period 40 psec vs. pulse duration 4 psec)

Single qubit gate fidelity: $F=98\sim99\%$

Appendix: Optically controlled quantum dot spin
Two Qubit Gate in Dissipative Planar Microcavity
T. Ladd et al., arXiv:0910.4988 (quant-ph)

- Cavity resonance depends on qubit states
- Cavity field amplitude by detuned pulse depends on resonance position
  ⇒ Amplitude “path” of cavity internal field depends on qubit states

Unique mode spot size of 2D planar cavity

G. Björk et al., PRA 44, 669 (1991)

⇒ CZ gate for surface code creation in a massive parallel operation
⇒ Master equation simulations indicate fidelity > 99% with Q = 10^5

τ ~ 100 nsec (purely optical), τ ~ 100 psec (polaritonic)

Appendix: Optically controlled quantum dot spin
Primary Components of the Physical Layer

Appendix: Optically controlled quantum dot spin