Computational Aspects of the Stochastic Finite Element Method

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The current scientific computation paradigm consists of mathematical models—often partial differential equations (PDEs)—describing certain physical phenomena under study whose solutions are approximated by numerical schemes carried out by computers. Among these components, great progress has resulted in the area of computer implementation due both to the rapid advance in computer speed and storage capacity as well as improvements in software aspects such as floating point standardization and programming methodology. Similarly, advances in numerical methods such as basic linear algebra libraries, discretization schemes and adaptivity make it possible to solve many nontrivial PDEs quickly and to as great an accuracy as desired.

An aspect of this general approach which deserves more attention is the fact that the data required by the model—various parameters such as the spatial distribution of material properties as well as source or boundary terms—are invariably assumed as known. In practice, however, such data is obtained from measurements or based on various assumptions, all subject to uncertainty. Indeed, it is quite possible for the effect of such uncertainty in the data to outweigh that of rounding or discretization errors. The idea of uncertainty quantification (UQ), i.e., quantifying the effects of uncertainty on the result of a computation, has received much interest of late. In stochastic approaches to UQ, the uncertain quantities are modeled as random variables, so that PDEs become stochastic PDEs (SPDEs). The most straightforward way of doing this is the Monte Carlo Method, in which many realizations of the random variables are generated, each leading to a deterministic problem, which is then solved using whatever methods are appropriate for the deterministic problem. A more ambitious approach is to solve the SPDE, the solution of which is a stochastic process, and to derive quantitative statements on the effect of data uncertainty from the distribution of this process.

Recently, a systematic approach for formulating and discretizing PDEs with random data known as the Stochastic Finite Element Method (SFEM) has become popular in the engineering community and subsequently analyzed by numerical analysts. The results of a SFEM approximation allows one to compute a large variety of statistical information via post processing, such as moments of the solution as well as the probability of certain events related to the PDE. The method is, however, computationally expensive, and it is one objective of this elementary introduction of the SFEM to identify the main tasks arising in the implementation of certain SFEM formulations. More specifically, we cover the following topics:

1. Elliptic Boundary Value Problems with Random Data
2. Background Material from Numerical Analysis and Statistics
3. The Stochastic Finite Element Method
4. Computational Aspects