Uniqueness of D-normal Proofs

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Abstract

This paper presents the notion of D-normal proofs, which is defined syntactically and gives one of the weakest conditions for uniqueness of normal proofs. This paper proves the following results: (1) \(\beta\eta D\)-normal proofs of a formula are unique. (2) A \(\beta\)-normal proof of a PNN-formula is D-normal. (3) A \(\beta\)-normal proof of a minimal formula in BCK logic is D-normal. These results give other proofs of uniqueness of \(\beta\eta\)-normal proofs of a PNN-formula, and uniqueness of \(\beta\eta\)-normal proofs of a minimal formula in BCK logic.

1 Introduction

Number of normal proofs has been studied widely [2]. In this paper we will present the notion of D-normal proofs and discuss uniqueness of normal proofs using this notion.

We will present the notion of D-normal proofs, which is defined syntactically and gives sufficient conditions for uniqueness of normal proofs. Several sufficient conditions for uniqueness of normal proofs such as the one-two property [2], balanced formulas [5], minimal formulas in BCK logic [3], provability with non-prime contraction [1] and the PNN condition [2] have been proposed. D-normality is one of the weakest conditions in those conditions.

We will prove that D-normality conditions properly weaker than the PNN condition. The PNN condition was proposed recently and one of the weakest condition at that time. This gives another proof of uniqueness \(\beta\eta\)-normal proofs of a PNN-formula.

The uniqueness of \(\beta\eta\)-normal proofs of a minimal formula in BCK logic was a problem proposed by [4] and solved independently at almost the same time by Hirokawa [3] and the author [6]. The author used the notion of D-normal proofs for that purpose and proved that \(\beta\)-normal proofs of a minimal formula in BCK logic is D-normal.

Section 2 presents the notion of D-normal proofs and proves uniqueness of \(\beta\eta D\)-normal proofs. Section 3 proves that a \(\beta\)-normal proof of a PNN-formula is D-normal and gives another proof of uniqueness of \(\beta\eta\)-normal proofs of a PNN-formula. Section 4 states that a \(\beta\)-normal proof of a minimal formula in BCK logic is D-normal and proves uniqueness of \(\beta\eta\)-normal proofs of a minimal formula in BCK logic.

2 D-normal proofs

We study constructive propositional logic with only implicational formulas in natural deduction style. Formulas are constructed by \(\rightarrow\) from propositional variables. The inference rules are as follows.

\[
\begin{align*}
\frac{\text{l}}{A} & \\
\vdots & \\
\frac{B}{A \rightarrow B} & (\rightarrow I) \\
\end{align*}
\]

\[
\frac{A \rightarrow B \quad A}{B} & (\rightarrow E)
\]

For (\(\rightarrow E\)), \(A\) is called a minor premise.

Definition 2.1. (Proof)

(1) A formula \(A\) itself is a proof.

\(^1\) This paper was written in 1999 in order to prepare the conference talk, whose proceedings are [8], and provides detailed descriptions of proofs given in the talk.
(2) If $\pi$ is a proof, then
\[
\begin{array}{c}
l \\
\vdots \\
\pi \\
\hline
A \rightarrow B \\
\end{array}
\]
\[
\frac{B}{A \rightarrow B} \quad (\rightarrow I) \quad l
\]
is a proof, where $B$ is the lowermost formula of $\pi$, $l$ is a label which is not used in $\pi$, and
\[
\begin{array}{c}
l \\
\vdots \\
\pi \\
\hline
A \\
\end{array}
\]
\[
\frac{B}{A \rightarrow B} \\
\]
is a proof obtained from $\pi$ by replacing some uppermost occurrences of the formula $A$ with $l$.

(3) If $\pi_1$ and $\pi_2$ are proofs,
\[
\begin{array}{c}
l \\
\vdots \\
\pi_1 \\
\vdots \\
\pi_2 \\
\hline
A \rightarrow B \\
\end{array}
\]
\[
\frac{A}{A \rightarrow B} \quad A \quad (\rightarrow E)
\]
is a proof, where $A \rightarrow B$ is the lowermost formula of $\pi_1$ and $A$ is the lowermost formula of $\pi_2$.

We often write a proof without the inference rule names ($\rightarrow I$) and ($\rightarrow E$).

The lowermost formula of a proof $\pi$ is the conclusion of $\pi$ and denoted by $\text{Concl}(\pi)$. Uppermost formulas without labels of a proof $\pi$ are assumptions of $\pi$ and the set of assumptions of $\pi$ is denoted by $\text{Ass}(\pi)$. Uppermost formulas with labels of a proof $\pi$ are discharged assumptions of $\pi$. A proof $\pi$ is called closed if $\pi$ has no assumptions. A proof $\pi$ is called a proof of a formula $A$ if $\text{Concl}(\pi) = A$ and $\pi$ is closed.

A proof $\pi$ is a $\beta$-normal proof, if $\pi$ does not include the following part for any $A$ and $B$.
\[
\begin{array}{c}
l \\
\vdots \\
\pi \\
\hline
A \rightarrow B \\
\end{array}
\]
\[
\frac{B}{A \rightarrow B} \quad (\rightarrow I) \quad l
\]
\[
\frac{A}{A \rightarrow B} \quad l
\]

A proof $\pi$ is a $\eta$-normal proof, if $\pi$ does not include the following part for any $A$ and $B$.
\[
\begin{array}{c}
l \\
\vdots \\
\pi \\
\hline
A \rightarrow B \\
\end{array}
\]
\[
\frac{A}{A \rightarrow B} \quad l
\]
\[
\frac{B}{A \rightarrow B} \quad (\rightarrow E)
\]

A proof $\pi$ is a $\beta\eta$-normal proof if $\pi$ is a $\beta$-normal proof and an $\eta$-normal proof.

For a formula $A$, $\text{core}(A)$ is a variable having the rightmost variable occurrence in $A$.

**Definition 2.2. (D-normal proof)**

A proof $\pi$ is a $D$-normal proof if the following condition holds.

- If there exists a form
\[
\begin{array}{c}
k \\
\vdots \\
\pi \\
\hline
A \\
\end{array}
\]
\[
\frac{C}{B \rightarrow C} \quad (\rightarrow I) \quad l
\]

in $\pi$ and $\text{core}(A) = \text{core}(B)$, then $k = l$ holds.

A proof $\pi$ is a $\beta\eta D$-normal proof if $\pi$ is a $\beta\eta$-normal proof and a $D$-normal proof.

**Theorem 2.3.**

If a formula has $\beta\eta D$-normal proofs $\pi_1$ and $\pi_2$, then $\pi_1 = \pi_2$. 
A formula is a $D$-normal formula, if every $\beta\eta$-normal proof of $A$ is a $D$-normal proof. Remark that $\beta\eta$-normal proofs of a $D$-normal formula are the same. The rest of this section proves Theorem 2.3.

**Definition 2.4.**
The length $|\pi|$ of a proof $\pi$ is defined as follows.

1. If $\pi$ is a formula, then $|\pi| = 0$.
2. If $\pi$ is

   \[
   \frac{\vdots \pi_1 \vdots}{A \rightarrow B} \rightarrow I
   \]

   then $|\pi| = |\pi_1| + 1$.
3. If $\pi$ is

   \[
   \frac{\vdots \pi_1 \vdots \pi_2}{A \rightarrow B \quad A} \rightarrow E
   \]

   then $|\pi| = |\pi_1| + |\pi_2| + 1$.

Note that $|\pi|$ represents the number of inference rules used in $\pi$. For a part $\Sigma$ of a proof $\pi$, $|\Sigma|$ is defined as the number of inference rules in $\Sigma$.

$\text{dis}(\pi)$ denotes the set of discharged assumptions of the proof $\pi$. $\text{leaf}(\pi)$ denotes $\text{Ass}(\pi) + \text{dis}(\pi)$.

For an occurrence of a formula $A$ in a proof $\pi$, we use the notation $A^1$ to denote that we think about the specific occurrence of $A$ in $\pi$.

For a proof $\pi$ and formula occurrences $A_1^1$, $B_1^1$ in $\pi$, a thread from $A_1^1$ to $B_1^1$ is a sequence of formula occurrences in $\pi$ satisfying the followings.

- The sequence begins with $A_1^1$ and ends with $B_1^1$.
- Every formula occurrence in the sequence except the last is an upper formula occurrence of an inference, and is immediately followed by the lower formula occurrence of this inference.

For a formula $A$ in a $\beta$-normal proof such that the inference rule for $A$ is not $(\rightarrow I)$, the main formula of $A$ (denoted by $\text{MainFormula}(A)$) is defined in the following way.

1. If $A$ is an assumption or a discharged assumption, $\text{MainFormula}(A)$ is $A$ itself.
2. If $A$ is infered by $(\rightarrow E)$ and the part above $A$ of $\pi$ is

   \[
   \frac{\vdots \pi_1 \vdots}{B \rightarrow A \quad \vdots \quad \frac{\vdots}{A}}
   \]

   then $\text{MainFormula}(A) = \text{MainFormula}(B \rightarrow A)$. Note that the last inference of $\pi_1$ is not $(\rightarrow I)$ since $\pi$ is a $\beta$-normal proof.

The main thread of a proof $\pi$ is the thread from $\text{MainFormula}(\text{Concl}(\pi))$ to $\text{Concl}(\pi)$. We denote the set of formula occurrences in the main thread of $\pi$ by $\text{thread}(\pi)$.

A thread is a path in a proof tree.

**Definition 2.5. (Initial Subproof)**

A subproof $\Pi$ of a proof $\pi$ is an initial subproof of $\pi$ if the followings hold:

- Their conclusions are the same.
- If $A_1^1$ is in $\pi$ and $\Pi$ does not include $A_1^1$, $\Pi$ does not include $A_1^1$.

We write $\Pi \subset \pi$ to denote that $\Pi$ is an initial subproof of $\pi$.

For a part $\Sigma$ of a proof $\pi$, $\Sigma \in \pi - \Pi$ iff $\Sigma$ is the part above $A_1^1$ of $\pi$ for some $A_1^1$ in assumptions of $\Pi$. 

3
Definition 2.6. (L-unique Proof)
A β-normal proof π is an L-unique proof if the following condition holds.

- If there exists a form
  \[ \vdots \ \cdot \cdot \cdot \ \pi_1 \]
  \[ \vdots \]
  \[ A \]
  \[ \vdots \]
  \[ B \rightarrow C \]
  \[ \rightarrow I \]
  \[ l \]
  \[ \vdots \]

in π and
\[ B = D_1 \rightarrow \cdots \rightarrow D_n \rightarrow A \quad (n \geq 0) \]
then the last inference of \( \pi_1 \) is not \( \rightarrow I \) and
\[ \text{MainFormula}(A) = l \]

Definition 2.7. (Strongly η-normal Proof)
A proof π is a strongly η-normal proof if π does not include the following form.

\[ \vdots \ \cdot \cdot \cdot \ \vdots \]
\[ A \rightarrow B \]
\[ A \]
\[ \rightarrow E \]
\[ \vdots \]
\[ A \rightarrow B \]
\[ \rightarrow I \]
\[ \vdots \]

Lemma 2.8.
If a closed βD-normal proof \( \pi \) includes

\[ \vdots \ \cdot \cdot \cdot \ \pi_1 \]
\[ \vdots \]
\[ A \]
\[ \vdots \]
\[ B \rightarrow C \]
\[ l \]

and
\[ B = D_1 \rightarrow \cdots \rightarrow D_n \rightarrow A \quad (n \geq 0) \]
and \( \pi_1 \) is a strongly η-normal proof, then the last inference of \( \pi_1 \) is not \( \rightarrow I \) and
\[ \text{MainFormula}(A) = l \]

Proof 2.9.
Suppose that the assumptions of this lemma hold. Since π is a β-normal proof, there exist formulas \( E, X_1, \ldots, X_p \) (\( p \geq 0 \)), \( Y_1, \ldots, Y_q \) (\( q \geq 0 \)) such that
\[ A = X_1 \rightarrow \cdots \rightarrow X_p \rightarrow E \]
and π1 is

\[ \vdots \ \cdot \cdot \cdot \ \vdots \]
\[ Y_1 \rightarrow \cdots \rightarrow Y_q \rightarrow E \]
\[ l \]
\[ Y_1 \]
\[ \vdots \]
\[ Y_q \rightarrow E \]
\[ l \]
\[ Y_q \]
\[ \vdots \]
\[ X_2 \rightarrow \cdots \rightarrow X_p \rightarrow E \]
\[ l \]
\[ X_2 \]
\[ \vdots \]
\[ X_1 \rightarrow \cdots \rightarrow X_p \rightarrow E \]
Since $\pi$ is a D-normal proof and $\text{core}(Y_1 \rightarrow \cdots \rightarrow Y_q \rightarrow E) = \text{core}(E) = \text{core}(A) = \text{core}(B)$, we have $B = Y_1 \rightarrow \cdots \rightarrow Y_q \rightarrow E$, $k = l$.

From $Y_1 \rightarrow \cdots \rightarrow Y_q \rightarrow E = B = D_1 \rightarrow \cdots \rightarrow D_n \rightarrow A = D_1 \rightarrow \cdots \rightarrow D_n \rightarrow X_1 \rightarrow \cdots \rightarrow X_p \rightarrow E$, we have $X_i = Y_{q-p+i}$ $(1 \leq i \leq p)$, $A = Y_{q-p+1} \rightarrow \cdots \rightarrow Y_q \rightarrow E$ and $\pi_1$ is

\[
\begin{array}{c}
Y_1 \rightarrow \cdots \rightarrow Y_q \rightarrow E \quad Y_1 \\
\vdots \\
Y_q \rightarrow E \quad Y_q \\
E \\
\vdots \\
Y_{q-p+2} \rightarrow \cdots \rightarrow Y_q \rightarrow E \\
Y_{q-p+1} \rightarrow \cdots \rightarrow Y_q \rightarrow E
\end{array}
\]

Since $\pi_1$ is a strongly $\eta$-normal proof, we have $p = 0$, $A = E$, $B = Y_1 \rightarrow \cdots \rightarrow Y_q \rightarrow A$ and $\pi_1$ is

\[
\begin{array}{c}
Y_1 \rightarrow \cdots \rightarrow Y_q \rightarrow A \quad Y_1 \\
\vdots \\
Y_q \rightarrow A \quad Y_q \\
A
\end{array}
\]

Therefore the last inference of $\pi_1$ is not $(-I)$ and

$\text{MainFormula}(A) = B'$.

\[\Box\]

Lemma 2.10.

A closed $\beta\eta D$-normal proof is a strongly $\eta$-normal proof.

Proof 2.11.

Suppose that a proof $\pi$ is a $\beta\eta D$-normal proof and is not a strongly $\eta$-normal proof. In $\pi$, there is the following form such that $\pi_1$ is a strongly $\eta$-normal proof.

\[
\begin{array}{c}
\vdots \\
\pi_1 \\
A \rightarrow B \quad A \\
B \\
A^{l} \rightarrow B^{l} \quad l \\
\vdots
\end{array}
\]

From Lemma 2.8, $\pi_1$ is

\[
\frac{\vdots}{A}
\]

and $|\pi_1| = 0$. This contradicts the fact that $\pi$ is an $\eta$-normal proof. $\Box$

Proposition 2.12.

A closed $\beta\eta D$-normal proof is an L-unique proof.

Proof 2.13.

It is proved immediately from Lemma 2.8 and Lemma 2.10. $\Box$

Lemma 2.14.

If a formula $A$ has closed L-unique proofs $\pi_1$ and $\pi_2$, and for some initial subproof $\Pi$ of $\pi_1$, $\Pi \subset \pi_1$ and $\Pi \subset \pi_2$ hold, then $\pi_1 = \pi_2$.
Proof 2.15.

We may suppose that $|\pi_1| - |\Pi| \geq |\pi_2| - |\Pi|$.
This lemma is proved by induction on $|\pi_1| - |\Pi|$.

Case 1. $|\pi_1| - |\Pi| = 0.$
$|\pi_1| - |\Pi| = |\pi_2| - |\Pi| = 0$ holds. Then $\pi_1 = \Pi = \pi_2$.

Case 2. $|\pi_1| - |\Pi| > 0.$
There exists a part $\Sigma_1 \in \pi_1 - \Pi$ such that $|\Sigma_1| > 0$. Then there exists a part $\Sigma_2 \in \pi_2 - \Pi$ such that $\text{Concl}(\Sigma_1) = \text{Concl}(\Sigma_2)$ in $\Pi$.

Case 2.1. The last inference of $\Sigma_1$ is $(\rightarrow I)$.
$\Sigma_1$ is

\[
\begin{array}{c}
\vdots \\
\frac{B}{A \rightarrow B} k
\end{array}
\]

Case 2.1.1. $|\Sigma_2| = 0.$
We show this case is impossible. $\Sigma_2$ is $A \rightarrow B$. Therefore in $\pi_2$, $A \rightarrow B$ is discharged by the inference $(\rightarrow I)l$ in the thread from $A \rightarrow B$ to Concl($\pi_2$). Then in $\pi_1$, $A \rightarrow B$ is discharged by the inference $(\rightarrow I)l$ in the thread from $A \rightarrow B$ to Concl($\pi_1$). Since $\pi_1$ is L-unique, the inference rule for $A \rightarrow B$ is not $(\rightarrow I)$ and we get contradiction.

Case 2.1.2. The last inference of $\Sigma_2$ is $(\rightarrow I)$.
By renaming discharging labels in $\pi_2$, $\Sigma_2$ is

\[
\begin{array}{c}
\vdots \\
\frac{B}{A \rightarrow B} k
\end{array}
\]

Let $\Pi'$ be the following initial subproof of $\pi_1$:

\[
\begin{array}{c}
\vdots \\
\frac{B}{A \rightarrow B} k
\end{array}
\]

Then $\Pi' \subseteq \pi_1$, $\Pi' \subseteq \pi_2$ and $|\pi_1| - |\Pi| > |\pi_1| - |\Pi'| \geq |\pi_2| - |\Pi'|$ hold. By induction hypothesis, we have $\pi_1 = \pi_2$.

Case 2.1.3. The last inference of $\Sigma_2$ is $(\rightarrow E)$.
We show this case is impossible. Let

\[\text{MainFormula}(A \rightarrow B) = X_1 \rightarrow \cdots \rightarrow X_p \rightarrow A \rightarrow B. \quad (p \geq 0)\]

$\Sigma_2$ is

\[
\begin{array}{c}
X_1 \rightarrow \cdots \rightarrow X_p \rightarrow A \rightarrow B \\
\vdots \\
X_2 \rightarrow \cdots \rightarrow X_p \rightarrow A \rightarrow B
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\frac{X_p \rightarrow A \rightarrow B}{A \rightarrow B}
\end{array}
\]

In $\pi_2$, $X_1 \rightarrow \cdots \rightarrow X_p \rightarrow A \rightarrow B$ is discharged in the thread from $A \rightarrow B$ to Concl($\pi_2$) by the inference $(\rightarrow I)l$. Therefore in $\pi_1$, $X_1 \rightarrow \cdots \rightarrow X_p \rightarrow A \rightarrow B$ is discharged in the thread from $A \rightarrow B$ to Concl($\pi_1$) by the inference $(\rightarrow I)l$.
Since $\pi_1$ is L-unique, the last inference of $\Sigma_1$ is not $(\rightarrow I)$ and we get contradiction.

Case 2.2. The last inference of $\Sigma_1$ is $(\rightarrow E)$.
$\Sigma_1$ is
where \( p \geq 0 \).

Case 2.2.1. \( |\Sigma_2| = 0 \).

Since \( \pi_2 \) is closed, \( \Sigma_2 \) is \( l \).

In \( \pi_2 \), \( B \) is discharged in the thread from \( B \) to Concl(\( \pi_2 \)) by the inference (\( \to \) l). Therefore in \( \pi_1 \), \( B \) is discharged in the thread from \( B \) to Concl(\( \pi_1 \)) by the inference (\( \to \) l).

Since \( \pi_1 \) is \( L \)-unique, \( \Sigma_1 \) is \( B \) and \( |\Sigma_1| = 0 \). This contradicts \( |\Sigma_1| > 0 \).

Case 2.2.2. The last inference of \( \Sigma_2 \) is (\( \to \) l).

It is not the case by the same reason as Case 2.1.3.

Case 2.2.3. The last inference of \( \Sigma_2 \) is (\( \to \) E).

\( \Sigma_2 \) is

\[
\begin{align*}
Y_1 \to \cdots \to Y_q \to C \to B & \quad Y_1 \quad \cdots \quad \vdots \\
Y_2 \to \cdots \to Y_q \to C \to B & \quad Y_2 \quad \cdots \quad \vdots \\
\vdots & \vdots \\
Y_q \to C \to B & \quad Y_q \quad \vdots \\
C \to B & \quad B
\end{align*}
\]

where \( q \geq 0 \).

In \( \pi_2 \), \( Y_1 \to \cdots \to Y_q \to C \to B \) is discharged in the thread from \( B \) to Concl(\( \pi_2 \)) by the inference (\( \to \) l). Therefore in \( \pi_1 \), \( Y_1 \to \cdots \to Y_q \to C \to B \) is discharged in the thread from \( B \) to Concl(\( \pi_1 \)) by the inference (\( \to \) l).

Since \( \pi_1 \) is \( L \)-unique, \( \pi_1 \) is

\[
\begin{align*}
Y_1 \to \cdots \to Y_q \to C \to B & \quad Y_1 \quad \cdots \quad \vdots \\
Y_2 \to \cdots \to Y_q \to C \to B & \quad Y_2 \quad \cdots \quad \vdots \\
\vdots & \vdots \\
Y_q \to C \to B & \quad Y_q \quad \vdots \\
C \to B & \quad B
\end{align*}
\]

Therefore \( A = C \). Let an initial subproof \( \Pi' \) of \( \pi_1 \) be given as follows:

\[
\begin{align*}
A \to B & \quad A \\
B & \quad \Pi
\end{align*}
\]

Then \( \Pi' \subset \pi_1 \), \( \Pi' \subset \pi_2 \) and \( |\Pi| - |\pi_1| > |\Pi'| - |\pi_1| \geq |\Pi'| - |\pi_2| \) hold. By induction hypothesis, we have \( \pi_1 = \pi_2 \).

\( \square \)

**Proof 2.16. (of Theorem 2.3)**
Suppose that a formula \( A \) has closed \( \beta \eta \)D-normal proofs \( \pi_1 \) and \( \pi_2 \). From Proposition 2.12, \( \pi_1 \) and \( \pi_2 \) are \( L \)-unique proofs. Let an initial subproof \( \Pi \) of \( \pi_1 \) be \( A \).

From Lemma 2.14, since \( \Pi \subset \pi_1 \) and \( \Pi \subset \pi_2 \), we have \( \pi_1 = \pi_2 \). \( \square \)
3 PNN condition

A formula $A$ has PNN-occurrences of a variable $B$, if $B$ has a positive occurrence and at least two negative occurrences in $A$. A formula $A$ satisfies PNN-condition, if no variable has PNN-occurrences in $A$. A PNN-formula is a formula satisfying PNN-condition.

Theorem 3.1.
(1) A $\beta$-normal proof of a PNN-formula is a D-normal proof.
(2) A PNN-formula is a D-normal formula.

Remark that the converse of this theorem (1) does not hold. For example, the formula

$$C \rightarrow (D \rightarrow B) \rightarrow (C \rightarrow B) \rightarrow (B \rightarrow A) \rightarrow A$$

for distinct propositional variables $A, B, C$ and $D$ is a D-normal formula and has PNN-occurrences of $B$.

The following theorem is known [2]. By Theorem 2.3 and Theorem 3.1 immediately give another proof of this theorem.

Theorem 3.2. $\beta\eta$-normal proofs of a formula satisfying the PNN-condition are unique.

The rest of this section proves these theorems.

Definition 3.3. For a formula $A$, $\text{Pos}(A)$ is the set of positive subformulas of $A$ and $\text{Neg}(A)$ is the set of negative subformulas of $A$. For a set $S$ of formulas, $\text{Pos}(S) = \bigcup_{A \in S} \text{Pos}(A)$ and $\text{Neg}(S) = \bigcup_{A \in S} \text{Neg}(A)$.

A formula in a proof $\pi$ is a minor premise in $\pi$ if it is a minor premise of some inference rule ($\rightarrow E$).

Proposition 3.4. (Signed Subformula Property for $NJ$)
(1) If a $\beta$-normal proof $\pi$ has a discharged assumption $A$, the followings hold.
   (a) $A \in \text{Neg(Concl(\pi))} \cup \text{Pos(Ass(\pi))}$. 
   (b) If the last rule of $\pi$ is not ($\rightarrow I$), $A \in \text{Pos(Ass(\pi))}$.

(2) If a closed $\beta$-normal proof has a minor premise $A$, $A \in \text{Pos(Concl(\pi))}$ holds.

Proof 3.5.
(1) We prove this by induction on the proof $\pi$.

Case 1. $\pi$ is a formula $B$. There is no $A$.

Case 2. $\pi$ is
$$\vdots \pi_1 \vdots \pi_2$$

$$B \rightarrow C \quad B$$

$$C$$

If $A$ is in $\pi_1$, by induction hypothesis (b) for $\pi_1$, $A \in \text{Pos(Ass(\pi_1))}$. If $A$ is in $\pi_2$, by induction hypothesis (a) for $\pi_2$, $A \in \text{Neg(B)} \cup \text{Pos(Ass(\pi_2))}$. Since we have $\text{Neg(B)} \subset \text{Pos(B \rightarrow C)} \subset \text{Pos(MainFormula(B \rightarrow C))}$, $A \in \text{Pos(Ass(\pi))}$ holds.

Case 3. $\pi$ is
$$\vdots \pi_1 \vdots$$

$$B$$

$$C$$

$$B \rightarrow C \quad B$$

$$k$$

If $A$ is not $B$, by induction hypothesis (a) for $\pi_1$, $A \in \text{Neg(C)} \cup \text{Pos(Ass(\pi_1))}$ holds and it is included in $\text{Neg(B \rightarrow C)} \cup \text{Pos(Ass(\pi))}$.

If $A$ is $B$, $A \in \text{Neg(B \rightarrow C)}$. Therefore $A \in \text{Neg(B \rightarrow C)} \cup \text{Pos(Ass(\pi))}$.

(2) Suppose that the following part is in $\pi$:

$$\vdots \vdots \vdots$$

$$A \rightarrow B \quad A$$

$$B$$

We have $A \in \text{Neg(A \rightarrow B)} \subset \text{Neg(MainFormula(A \rightarrow B))}$. From (1), it is included in $\text{Pos(Concl(\pi))}$. □
\[
\begin{align*}
X \parallel Y & \text{ denotes a proof whose conclusion is } Y \text{ and whose assumptions include } X \text{ and } Y \text{ are in the same thread.} \\
X \supset_i Y & \text{ denotes that } X = Z_1 \to \ldots Z_n \to Y \text{ for some } n \geq 0, Z_1, \ldots, Z_n. \\
\end{align*}
\]

**Lemma 3.6.**

1. If a \( \beta \)-normal proof of a PNN-formula has the following part, \( Z_3 \) is a minor premise, and \( Z_3 \supset Y \rightarrow Z_2 \), we have \( Y_3 \supset X_3 \rightarrow Y_2 \).

\[
\begin{align*}
\begin{array}{c}
X_3 \rightarrow Y_2 \quad X_3 \\
Y_2 \\
\end{array}
\end{align*}
\]

2. If a \( \beta \)-normal proof of a PNN-formula has the following part, \( Y_3 \) is a minor premise, and \( Y_1 = Y_3 \), then we have \( X_1 = X_3 \).

\[
\begin{align*}
\begin{array}{c}
Y_3 \\
X_3 \rightarrow Y_2 \\
Y_2 \\
\end{array}
\end{align*}
\]

3. If a \( \beta \)-normal proof \( \pi \) of a PNN-formula has the following part, \( X_3 \) is a minor premise, and \( X_3 \rightarrow Y \in \text{Pos}(\text{Concl}(\pi)) \) for some \( Y \), then we have \( X_1 = X_3 \).

\[
\begin{align*}
\begin{array}{c}
X_3 \\
\end{array}
\end{align*}
\]

**Proof 3.7.**

1. Let \( Y = \text{core}(Y_3) \) and \( A \) be the conclusion of the proof. From Proposition 3.4 (2) for \( Z_3 \), we have \( Z_3 \in \text{Pos}(A) \). Therefore \( Y_3 = Z_2 \in \text{Pos}(A) \). Let \( Y_1 = \text{MainFormula}(Y_2) \). Let the part be

\[
\begin{align*}
\begin{array}{c}
Y_1 \\
X_3 \rightarrow Y_2 \\
Y_2 \\
\end{array}
\end{align*}
\]

From Proposition 3.4 (2) for the inference rule \( (\rightarrow I) \) discharging the label \( l \), \( Y_1 \rightarrow B \in \text{Pos}(A) \) for some \( B \). From Proposition 3.4 (2) for \( Y_3, Y_3 \in \text{Pos}(A) \). By the PNN-condition for \( Y, Y_1 = Y_3 \). Since \( Y_1 \supset X_3 \rightarrow Y_2 \), we have \( Y_3 \supset X_3 \rightarrow Y_2 \).

2. Let \( X = \text{core}(X_3) \) and \( A \) be the conclusion of the proof. From Proposition 3.4 (2) for \( Y_3, Y_3 \in \text{Pos}(A) \). Since \( Y_3 = Y_1 \supset X_3 \rightarrow Y_2, X_3 \rightarrow Y_2 \in \text{Pos}(A) \) holds. From Proposition 3.4 (2) for the inference rule \( (\rightarrow I) \) discharging the label \( l_2 \), \( X_1 \rightarrow B \in \text{Pos}(A) \) holds for some \( B \). From Proposition 3.4 (2) for \( X_3, X_3 \in \text{Pos}(A) \) holds. By the PNN-condition for \( X \), we have \( X_1 = X_3 \).

3. Let \( A \) be the conclusion of the proof. From Proposition 3.4 (2) for the inference rule \( (\rightarrow I) \) discharging the label \( l \), \( X_1 \rightarrow Z \in \text{Pos}(A) \) holds for some \( Z \). From Proposition 3.4 (2) for \( X_3, X_3 \in \text{Pos}(A) \) holds. By the PNN-condition for \( X \), we have \( X_1 = X_3 \).

**Definition 3.8. (level)**

For a proof \( \pi \) and an occurrence \( A \) of a formula in \( \pi \), the level of \( A \) in \( \pi \) (denoted by \( \text{level}_\pi(A) \)) is defined as follows:

\[
\begin{align*}
\text{If } \pi \text{ is } A, \text{ then } \text{level}_\pi(A) = 0.
\end{align*}
\]
If $\pi$ is
\[
\begin{array}{c}
A \\
\vdots \quad \pi_1 \\
B \\
A \rightarrow B \\
\end{array}
\]
then $\operatorname{level}_\pi(A \rightarrow B) = 0$ and $\operatorname{level}_\pi(X) = \operatorname{level}_{\pi_1}(X)$ for $X \in \pi_1$.

If $\pi$ is
\[
\begin{array}{c}
\vdots \quad \pi_1 \quad \vdots \quad \pi_2 \\
A \rightarrow B \\
B \\
\end{array}
\]
then $\operatorname{level}_\pi(B) = 0$, $\operatorname{level}_\pi(X) = \operatorname{level}_{\pi_1}(X)$ for $X \in \pi_1$, and $\operatorname{level}_\pi(X) = \operatorname{level}_{\pi_2}(X) + 1$ for $X \in \pi_2$.

**Proposition 3.9.**

If a $\beta$-normal proof of a PNN-formula has discharged assumptions $l_1 A_1$ and $l_2 A_2$ and $\operatorname{core}(A_1) = \operatorname{core}(A_2)$, then we have $A_1 = A_2$.

**Proof 3.10.**

Let $A = \operatorname{core}(A_1)$ and $B$ be the conclusion of the proof. From Proposition 3.4 (2) for the inference rules ($\rightarrow I$) discharging the labels $l_1$ and $l_2$, $A_1 \rightarrow X_1 \in \operatorname{Pos}(B)$ and $A_2 \rightarrow X_2 \in \operatorname{Pos}(B)$ hold for some $X_1$ and $X_2$.

Either $A_1$ or $A_2$ is of level $n > 0$. We may suppose that $A_1$ is of level $n > 0$. Let $A_3$ be the lowermost formula of the thread including $A_1$. Since $A_3$ is a minor premise, from Proposition 3.4 (2), $A_3 \in \operatorname{Pos}(B)$ holds.

By the PNN-condition for $A$, we have $A_1 = A_2$. $\square$

**Proposition 3.11.**

A $\beta$-normal proof of a PNN-formula does not have the following part:
\[
\begin{array}{c}
l_1 \\
A_0 \\
\vdots \\
A_1 \\
B \\
\end{array}
\]
where $\operatorname{core}(A_0) = \operatorname{core}(A_1)$ and $A_0$ and $A_1$ are distinct.

**Proof 3.12.**

Put $A_1$ be of level maximal. Let $A = \operatorname{core}(A_0)$ and $B$ be the conclusion of the proof. By Proposition 3.9, we have $A_0 = A_1$. Let the part including $l_1$ and $l_2$ be as follows:

\[
\begin{array}{c}
l_1 \\
A_1 \\
\vdots \\
l_2 \\
A_2 \\
\end{array}
\]

where $\operatorname{core}(A_0) = \operatorname{core}(A_1)$ and $A_0$ and $A_1$ are distinct.
and $\pi_1$ be the part below $X_3^n$ of this and $\pi_2$ be the part above $A_3$ of this.

Case 1. $A_3 \not\supseteq_i X_3^n \rightarrow A_2$, $X_3^n \not\supseteq_i X_2^{n-1} \rightarrow X_2^n$ for $p = n, \ldots, 2$, and $X_3^n \not\supseteq_i A_3 \rightarrow X_2^n$.

From $A_3 \not\supseteq_i X_3^n \rightarrow A_2$, $\pi_2$ is as follows:

$$
\begin{array}{c|c|c}
l_2 & k'_n & X \\
A_1 & X_1^n & \parallel
\end{array}
\begin{array}{c|c|c}
X_3^n & A_2 & X_2^n \\
\parallel & \parallel & A_3
\end{array}
$$

By Proposition 3.9, $X = X_3^n$ holds. Therefore $\pi_2$ as as follows:

$$
\begin{array}{c|c|c}
l_2 & k'_n & X \\
A_1 & X_1^n & \parallel
\end{array}
\begin{array}{c|c|c}
X_3^n & A_2 & X_2^n \\
\parallel & \parallel & A_3
\end{array}
$$

From $X_3^n \not\supseteq_i X_3^n \rightarrow X_2^n$, the thread from $X_3^n$ to $X_2^n$ is as follows:

$$
\begin{array}{c|c|c}
k'_n & X_3^n & X \\
A_1 & X_1^n & \parallel
\end{array}
\begin{array}{c|c|c}
X_3^n & A_2 & X_2^n \\
\parallel & \parallel & A_3
\end{array}
$$

By Proposition 3.9, $X = X_3^n$ holds.

By repeating this discussion, we prove that $\pi_2$ has the following form:

$$
\begin{array}{c|c|c}
l_2 & k'_n & X \\
A_1 & X_1^n & \parallel
\end{array}
\begin{array}{c|c|c}
X_3^n & A_2 & X_2^n \\
\parallel & \parallel & A_3
\end{array}
$$

By Proposition 3.9, $X = X_3^n$ holds.

By repeating this discussion, we prove that $\pi_2$ has the following form:

$$
\begin{array}{c|c|c}
l_2 & k'_n & X \\
A_1 & X_1^n & \parallel
\end{array}
\begin{array}{c|c|c}
X_3^n & A_2 & X_2^n \\
\parallel & \parallel & A_3
\end{array}
$$

Hence $\pi_2$ has $A_1$, and it contradicts the maximality of the level of $A_1$.

Case 2. $A_3 \supseteq_i X_3^n \rightarrow A_2$.

By Proposition 3.4 (2) for $A_3$, we have $X_3^n \rightarrow A_2 \in \text{Pos}(B)$. By repeating Lemma 3.6 (2) from the thread
By Theorem 2.3, we have

\[ \forall X^n \exists A_1, \ldots, A_n \] and get contradiction by Proposition 3.11.

Case 3. \( A_3 \nvdash X^n \rightarrow A_2 \) and \( X^p \vdash X^q \rightarrow A_2 \) for some \( 2 \leq p \leq n \). By repeating Lemma 3.6 (1) for \( X^p \vdash X^q \rightarrow A_2 \), we have \( X^q \vdash X^q \rightarrow A_2 \) for \( q = p, \ldots, 2 \) and \( A_1 \).

Case 4. \( A_3 \nvdash X^n \rightarrow A_2 \), \( X^p \nvdash X^q \rightarrow A_2 \) for \( 2 \leq p \leq n \), and \( X^q \vdash X^q \rightarrow A_2 \).

By Proposition 3.4 (2) for \( X^q \), we have \( A_3 \rightarrow A_2 \) and \( A_3 \rightarrow A_2 \) and contradiction. \( \square \)

**Proof 3.13. (of Theorem 3.1)**

(1) Suppose that we have a \( \beta \)-normal proof \( \pi \) of a PNN-formula \( C \) and \( \pi \) is not a D-normal proof.

By the definition of D-normality, \( \pi \) has the following part:

\[
\begin{align*}
(k) & \\
A_0 & \\
\vdots & \\
B_1 & \\
\vdots & \\
A_1 & \\
& \rightarrow B_1 (l)
\end{align*}
\]

where \( \text{core}(A_0) = \text{core}(A_1) \) and \( k \neq l \).

Let \( A_0 \rightarrow X \) be derived by the inference rule (\( \rightarrow I \)) discharging the label \( k \). By Proposition 3.4 (2), we have \( A_0 \rightarrow X \in \text{Pos}(C) \). By Proposition 3.4 (2), we have \( A_1 \rightarrow B_1 \in \text{Pos}(C) \). Let \( A_2 \) be the lowermost formula of the thread including \( A_0 \). Since \( A_2 \) is a minor premise, by Proposition 3.4 (2), we have \( A_2 \in \text{Pos}(C) \). By the PNN-condition for \( A \), \( A_0 \rightarrow X = A_1 \rightarrow B_1 \) holds. Therefore \( A_0 = A_1 \) and \( X = B_1 \). Hence \( \frac{X}{A_0 \rightarrow X} (k) \) is not in the thread including \( \frac{A_1 \rightarrow B_1}{A_1} (l) \). Let \( B_2 \) and \( B_3 \) be the main formulas of \( \frac{A_1 \rightarrow B_1}{A_1} (l) \) and \( A_0 \rightarrow X (k) \) respectively. Then we have either

\[
\begin{align*}
L_1 & \quad L_2 & \quad \vdots \\
B_2 & \quad B_3 & \quad B_4 \\
& & \quad \vdots \\
& \quad & \quad B_4
\end{align*}
\]

and get contradiction by Proposition 3.11. \( \square \)

(2) It is immediate from (1). \( \square \)

**Proof 3.14. (of Theorem 3.2)**

Let \( \pi_1 \) and \( \pi_1 \) be \( \beta \eta \)-normal proofs of a PNN-formula \( A \). By Theorem 3.1 (1), they are \( \beta \eta D \)-normal proofs. By Theorem 2.3, we have \( \pi_1 = \pi_2 \). \( \square \)

## 4 BCK logic

BCK logic is the logic with only the restricted (\( \rightarrow I \)) rule which can discharge at most one occurrence of an assumption. A formula is called a minimal formula if it is minimal in the variable substitution preorder of provable formulas.

**Proposition 4.1.**

In BCK logic, a \( \beta \)-normal proof of a minimal formula is a D-normal proof.

Combining Theorem 2.3 and this proposition, we have the following theorem.
Theorem 4.2.
In BCK logic, if a minimal formula $A$ has $\beta\eta$-normal proofs $\pi_1$ and $\pi_2$, then $\pi_1 = \pi_2$.

References


